Depinning of Geometric Flows

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Outline

I will examine three examples of perimeter energies with a periodic microgeometry defined on the plane. After scaling, such energies all Γ-converge to the same crystalline perimeter, whose evolution is described by motion by crystalline curvature.

Nevertheless the limit motion in the three case is influenced by pinning effects due to the presence of local minima that are not detected (or partially detected) by the Γ-limit.

Those effects and the resulting equations are different in the three examples.

Setting: flat flow of crystalline perimeter energies

(Simplest) **crystalline perimeter energy** in $\mathbb{R}^2$

$$F(A) = \int_{\partial A} \|\nu\|_1 d\mathcal{H}^1$$

$\|\nu\|_1 = |\nu_1| + |\nu_2|$, ν = normal to ∂A (Wulff shape = coordinate square)

Almgren-Taylor-Wang scheme (ATW): flat flow A defined as:
- given initial datum A_0, at fixed time-scale τ define A_k by minimizing iteratively

$$A \mapsto F(A) + \frac{1}{\tau} D(A, A_{k-1}) \qquad (1)$$

$D(A, A')$ = “dissipation” $\sim L^2$-distance of ∂A friom $\partial A'$
- define time-continuous piecewise-constant interpolation: $A^\tau(t) = A_{\lfloor t/\tau \rfloor}$
- compute time-continuous limit $A(t) = \lim_{\tau \to 0} A^\tau(t)$

“The flat flow A of the crystalline perimeter is motion by crystalline curvature” (Almgren-Taylor)

Limit equation for A

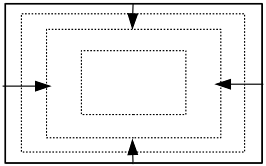
$$v = \kappa$$

κ = crystalline curvature

Example: A side of a coordinate rectangle of length L has curvature

$$\kappa = \frac{2}{L}$$

Such a rectangle contracts homothetically to its center in finite time.



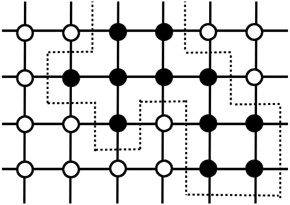
Example 1: evolution of spin systems /pinning by discreteness (original personal motivation)

Simplest geometry: square lattice $\mathbb{Z}^2$, nearest-neighbour interactions, $u_i \in \{-1,+1\}$ spin variable ($i \in \mathbb{Z}^2$), energy

$$E(u) = -\sum_{(i,j)} u_i u_j \sim \sum_{(i,j)} (u_i - u_j)^2$$

Scaling: $E_\varepsilon(u) = \sum_{(i,j)} \varepsilon (u_i - u_j)^2$ ($i \in \varepsilon\mathbb{Z}^2$)

A spin function $u : \varepsilon\mathbb{Z}^2 \to \{\pm 1\}$ is identified with its p.c. interpolation $\sim$ set $\{u = 1\}$



Continuous limit: we have

$$E_\varepsilon(u) \xrightarrow{\Gamma} F(u) = \int_{\partial\{u=1\}} \|\nu\|_1 d\mathcal{H}^1$$

We will identify a function $u \in \{\pm 1\}$ with the set $A = \{u = 1\}$
$\Rightarrow$ the limit is the “crystalline perimeter of u”

Q. How is spin-type evolution related with motion by crystalline curvature?

General remarks

(Pinning by discreteness) "Almgren-Taylor-Wang evolution" is **always pinned at fixed ε**: minimize iteratively

$$E_\varepsilon(u) + \frac{1}{\tau} D_\varepsilon(u, u_k) \qquad (2)$$

(D_ε = "ATW dissipation" for discrete sets).
If $u \neq u_k$ then we have $\frac{1}{\tau} D_\varepsilon(u, u_k) \geq \frac{\varepsilon^2}{\tau} >> E_\varepsilon(u) - E(u_k)$.
Hence, for τ small enough $u_{k+1} = u_k$, so that $u_k = u_0$ for all k.

$\Rightarrow$ **need to define a ε/τ-dependent evolution**

Definition (for arbitrary perimeter energies F_ε): a minimizing movement (M.M.) along the energies F_ε at time scale $\tau = \tau(\varepsilon)$ (from u_0^ε) is any time-continuous $u(t)$ constructed as

- $u_k = u_k^{\varepsilon,\tau}$ minimizes iteratively (1) with $u_0^{\varepsilon,\tau} = u_0^\varepsilon$
- (piecewise-constant extension) $u^\varepsilon(t) = u_{\lfloor t/\tau \rfloor}^{\varepsilon,\tau}$
- take the limit as $\varepsilon \to 0$ (up to subsequences)

Note: if $F_\varepsilon = F$ then u is the ATW motion $\sim$ flat flow

A general result (extreme minimizing movements)

(for abstract equi-coercive perimeter energies E_ε with $E_\varepsilon \to F$)

• there exists a scale $\tau_* = \tau_*(\varepsilon)$ such that if $\tau << \tau_*$ then any M.M. along F_ε coincides with a **limit of ATW motions** at fixed ε

• there exists a scale $\tau^* = \tau^*(\varepsilon)$ such that if $\tau >> \tau^*$ then any M.M. along F_ε coincides with an **ATW motion of the limit** F

Q. (if the two extreme motions are different) determine the **critical scalings**, and the set of all possible M.M.

Pinning of Spin System

• **critical scaling** $\tau_* = \tau^* = \varepsilon$

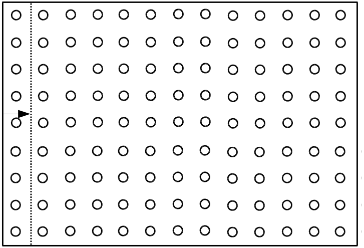
• (pinning) if $\tau << \varepsilon$ then the motion is trivial for all initial data

• (effective motion) if $\tau/\varepsilon \to \gamma$ then the motion is given by a discrete motion by crystalline curvature

$$v = \frac{1}{\gamma}\lfloor \gamma\kappa \rfloor$$

(κ = crystalline curvature) (B-Gelli-Novaga ARMA 2008)

Microscopic mechanism: barriers from local minima



Notes

1) the right-hand side is a discontinuous function $\Rightarrow$ general need to enlarge the possible class of geometric motions.

2) Even in the simplest case of a rectangle as an intial datum u_0 this is a system of ODE with non-uniqueness phenomena

3) The details of the motion depend on the patterns of local minima of F_ε and not only on the Γ-limit (B-Scilla IFB 2013)

4) The limit equation may depend on γ in a more complex way

$$v = \frac{1}{\gamma} f_{\rm hom}(\gamma\kappa)$$

with f_γ a *homogenized velocity* (B-Scilla IFB 2013)

5) We may not have a unique effective motion: the limit equation

$$v = f^{\gamma}_{\rm hom}(\kappa)$$

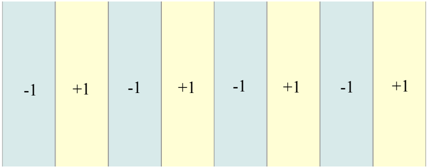
may really depend on γ, not only through a scaling (Scilla, Adv. Math. Sci. Appl. 2013)

Example 2. Homogenization of crystalline perimeter with a layered forcing term/ Pinning by homogenization of barriers

(B-Malusa-Novaga, in progress)

Setting: usual crystalline perimeter on subsets of $\mathbb{R}^2$;
zero-mean 1-periodic forcing term

$$g(x,y) = g(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1/2 \\ -1 & \text{if } 1/2 \leq x < 1 \end{cases}$$



$$F_\varepsilon(A) = \int_{\partial A} \|\nu\|_1 \, d\mathcal{H}^1 + \int_A g\left(\frac{x}{\varepsilon}\right) dx\, dy$$

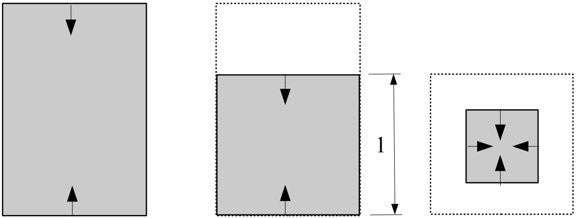
We still have

$$F_\varepsilon(A) \xrightarrow{\Gamma} F(A) = \int_{\partial A} \|\nu\|_1 d\mathcal{H}^1$$

Limit equation

We only consider the case $\tau/\varepsilon \to 0$ (limits of M.M.) and an initial datum a rectangle R_0. The evolution is still a rectangle $R(t)$ with

- horizontal sides moving inwards with velocity $v = \kappa$
- vertical sides moving inwards with velocity $v = \max\left\{\kappa - \frac{1}{\kappa}, 0\right\}$



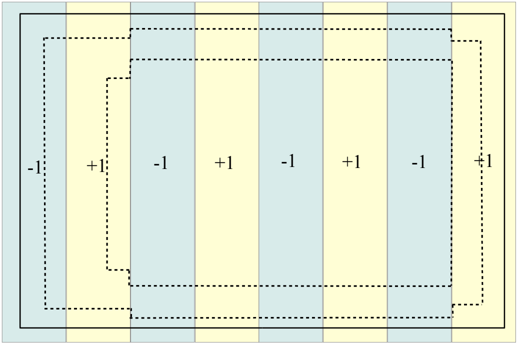
Microscopic mechanism: homogenzation of velocities

For $\kappa < 1$ the microscopic velocity of the vertical sides are

$$v = \kappa - 1 \qquad \text{or } v = \kappa + 1$$

hence they have contrasting directions $\Rightarrow$ pinning

Technical difficulty: even for rectangles initial data the discrete evolutions at fixed ε are not rectangles

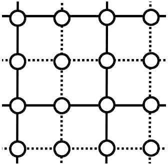


Example 3. Spin systems with weak inclusions / Motion by mushy layers

(B-Solci, 2015)

Setting: square lattice $\mathbb{Z}^2$, spin variable, ε-depending (scaled) energy

$$E_\varepsilon(u) = \sum_{(i,j)} \varepsilon\, c^\varepsilon_{ij}(u_i - u_j)^2 \qquad\qquad c^\varepsilon_{ij} = \begin{cases} \varepsilon \\ 1 \end{cases}$$



Upon normalizing E_ε, we still have

$$E_\varepsilon(u) \xrightarrow{\Gamma} F(u) = \int_{\partial\{u=1\}} \|\nu\|_1 d\mathcal{H}^1$$

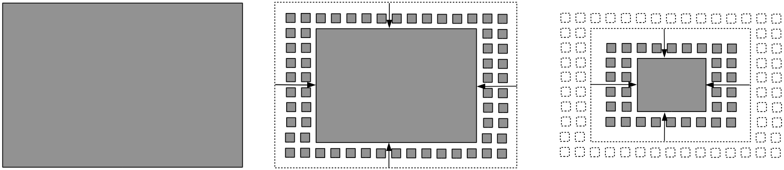
Asymptotic motion

We consider only the case $\tau/\varepsilon \to +\infty$.
The effective motion is

$$v = \max\left\{\kappa, \frac{4}{3}(\kappa - 1)\right\}$$

Microscopic mechanism: short-time pinning

E.g., taking as initial datum a rectangle R_0



Note: In this case the "general result" on the extreme M.M. for perimeters "fails": the limit as $\tau/\varepsilon \to +\infty$ is not the ATW motion of the Γ-limit as $\varepsilon \to 0$.

This is explained by a "lack of equi-coerciveness": as a result the limit of $E_\varepsilon + \frac{1}{\tau} D_\varepsilon$ is not $F + \frac{1}{\tau} D$

$\Longrightarrow$ **necessity to define an "ATW" motion also when we do not have a "reference dissipation D"**

(i.e., D is such that we have a compact convergence for which $E_\varepsilon \xrightarrow{\Gamma} F$ and D_ε converges continuously to D)

(B. *Local Minimization, Variational Motion and Gamma-convergence*, LNM 2013)

Conclusions

We have examined three cases of pinning for geometric motion due to microstructure

- pinning by local minimization
- pinning by barriers
- short-time pinning

A general framework proposed to study such phenomena are minimizing movements along a sequence of functional at given time scale. For pinned geometric motion issues are

- computing the critical time scale for depinning
- describe effective motions
- develop homogenization techniques for the velocity law
- extend the ATW scheme to cases without a reference dissipation
- etc.

Thank you for your attention!