Chapter 9

Different time scales

In this chapter we treat some variations on the minimizing-movement scheme motivated by some time-scaling argument.

9.1 Long-time behaviour

We will consider a new parameter $\lambda > 0$ and follow the iterative minimizing scheme from an initial datum x_0 by considering x_k defined recursively as a minimizer of

$$\min\Big\{\frac{1}{\lambda}F_\varepsilon(x) + \frac{1}{2\tau}\|x - x_{k-1}\|^2\Big\}, \tag{9.1}$$

and setting $u^\tau(t) = u^{\tau,\lambda}(t) = x_{\lfloor t/\tau \rfloor}$. Equivalently, we may view this as applying the minimizing-movement scheme to

$$\min\Big\{F_\varepsilon(x) + \frac{\lambda}{2\tau}\|x - x_{k-1}\|^2\Big\}. \tag{9.2}$$

Note that we may compare this scheme with the usual one where x_i are defined as minimizers of the minimizing-movement scheme with time scale $\eta = \tau/\lambda$ giving u^η as a discretization with lattice step η. Then we have

$$u^\tau(t) = x_{\lfloor t/\tau \rfloor} = x_{\lfloor t/\lambda\eta \rfloor} = u^\eta\Big(\frac{t}{\lambda}\Big).$$

Hence, the introduction of λ corresponds to a scaling of time.

Note that this process may be meaningful also if $F_\varepsilon = F$ is independent of ε. In this case, as $\tau \to 0$ we obtain the minimizing movement along $F_\lambda = \frac{1}{\lambda}F$ with λ in place of ε in the notation used hitherto (of course, being a matter of notation, up to a change of parameters we can always suppose that $\lambda \to 0$).

We now first give some simple examples which motivate the study of time-scaled problems, also when the unscaled problems already give a non trivial minimizing movement.

Example 9.1.1 Consider in $\mathbb{R}^2$ the energy

$$F_\varepsilon(x,y) = \frac{1}{2}(x^2 + \varepsilon y^2).$$

The corresponding gradient flow is then

$$\begin{cases} x' = -x \\ y' = -\varepsilon y, \end{cases}$$

with solutions of the form

$$(x_\varepsilon(t), y_\varepsilon(t)) = (x_0 e^{-t}, y_0 e^{-\varepsilon t}).$$

These solutions converge to $(x(t), y(t)) = (x_0 e^{-t}, y_0)$, solving

$$\begin{cases} x' = -x \\ y' = 0, \end{cases}$$

which is the gradient flow of the limit $F(x,y) = \frac{1}{2}x^2$. Note that

$$\lim_{t\to+\infty}(x_\varepsilon(t), y_\varepsilon(t)) = (0,0) \neq (0, y_0) = \lim_{t\to+\infty}(x(t), y(t)).$$

The trajectories of the solutions $(x_\varepsilon, y_\varepsilon)$ lie on the curves

$$\frac{y}{y_0} = \Big(\frac{x}{x_0}\Big)^\varepsilon$$

and are pictured in Fig. 9.1.

The solutions can be seen as superposition of $(x(t), y(t))$ and $\varepsilon(x_\infty(t), y_\infty(t))$, where

$$(x_\infty(t), y_\infty(t)) := (0, e^{-t})$$

is the solution of

$$\begin{cases} x' = 0 \\ y' = -y \\ (x(0), y(0)) = (0, y_0). \end{cases}$$

The solution (x_∞, y_∞) can be obtained by scaling $(x_\varepsilon, y_\varepsilon)$; namely,

$$(x_\infty(t), y_\infty(t)) = \lim_{\varepsilon\to 0}(x_\varepsilon(t/\varepsilon), y_\varepsilon(t/\varepsilon)).$$

In this case the scaled time-scale is $\lambda = \varepsilon$. Note that the limit of the scaled solutions does not satisfy the original initial condition, but its "projection" on the set of (local) minimizers of the limit energy F (or, in other words, the domain of the limit of the energies $\frac{1}{\varepsilon}F_\varepsilon$).

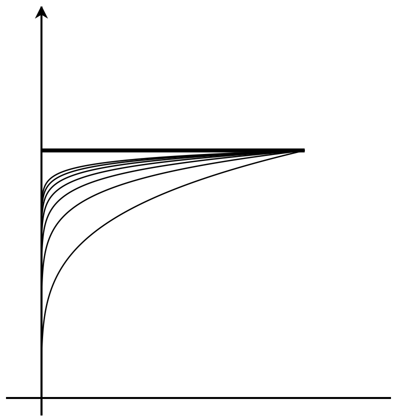

Figure 9.1: trajectories of the solutions, and their pointwise limit

Example 9.1.2 A similar example can be constructed in one dimension, taking, e.g.,

$$F_\varepsilon(x) = \frac{\varepsilon}{2}x^2 + \frac{1}{2}((|x|-1)\vee 0)^2 .$$

If $x_0 < -1$ then the corresponding solutions x_ε satisfy:

- the limit $x(t) = \lim_{\varepsilon\to 0} x_\varepsilon(t)$ solves

$$\begin{cases} x' = -x + 1 \\ x(0) = x_0, \end{cases}$$

which corresponds to the gradient flow of the energy

$$F(x) = \frac{1}{2}((|x|-1)\vee 0)^2 .$$

- the scaled limit $x_\infty(t) = \lim_{\varepsilon\to 0} x_\varepsilon(t/\varepsilon)$ solves

$$\begin{cases} x' = -x \\ x(0) = -1, \end{cases}$$

which corresponds to the gradient flow of the energy

$$F_\infty(x) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} F_\varepsilon(x) .$$

In this case the initial datum is the projection of x_0 on the domain of F_∞.

Remark 9.1.3 In the previous examples we faced the problem of defining a minimizing movement for a sequence of functionals F_ε (Γ-)converging to a limit F when the initial data

x_0^ε converge to a point $x_0 \notin \mathrm{dom} F$. Note that in this case the approximating trajectories u^ε are always defined if one can define x_1^ε; i.e., a solution of

$$\min\Big\{F_\varepsilon(x) + \frac{1}{2\tau}\|x - x_0\|^2\Big\},$$

or equivalently of

$$\min\{2\tau F_\varepsilon(x) + \|x - x_0\|^2 : x \in \mathrm{dom}\, F\},$$

after which $x_1^\tau \in \mathrm{dom} F$ and we apply the theory already studied. Note that if $\mathrm{dom} F$ is a closed set in X then x_1^τ converge to the projection x_1 of x_0 on dom F, so it may be meaningful to directly study the minimizing movements from that point. Note however that, as always, the choice of initial data $x_1^\tau \to x_1$ may provide a choice among the minimizing movements from x_1.

We now give more examples with families of energies F_ε Γ-converging to a limit F. Since we are mainly interested in highlighting the existence of a time scale $\lambda = \lambda_\varepsilon$ at which the scaled motion is not trivial, we will make some simplifying assumptions, one of which is that the initial datum be a local minimizer for F, so that the (unscaled) minimizing movement for the limit from that point is trivial.

Example 9.1.4 We take as F the 1D Mumford-Shah functional on $(0,1)$ defined by

$$F(u) = \int_0^1 |u'|^2 dt + \#(S(u)),$$

with domain the set of piecewise-H^1 functions. We take

$$F_\varepsilon(u) = \int_0^1 |u'|^2 dt + \sum_{S(u)} g\Big(\frac{|u^+ - u^-|}{\varepsilon}\Big),$$

where g is a positive concave function with

$$\lim_{z\to+\infty} g(z) = 1.$$

We also consider the boundary conditions

$$u(0-) = 0, \qquad u(1+) = 1.$$

We suppose that

- u_0 is a local minimizer for F; i.e., it is piecewise constant;
- $\#(S(u_0)) = \{x_0, x_1\}$ (the simplest non-trivial local minimizer) with $0 \le x_0 < x_1 \le 1$;
- competing functions are also piecewise constant.

With these conditions, all minimizers u_k obtained by iterative minimization satisfy:

• $S(u_k) \subset \{x_0, x_1\}$.

We may use the constant value z_k of u_k on (x_0, x_1) as a one-dimensional parameter. The minimum problem defining z_k is then (supposing that $z_0 > 0$ so that all $z_k > 0$)

$$\min\Big\{\frac{1}{\lambda}\Big(g\Big(\frac{z}{\varepsilon}\Big) + g\Big(\frac{1-z}{\varepsilon}\Big)\Big) + \frac{1}{2\tau}(x_1 - x_0)(z - z_{k-1})^2\Big\},$$

which gives

$$(x_1 - x_0)\frac{z_k - z_{k-1}}{\tau} = -\frac{1}{\varepsilon\lambda}\Big(g'\Big(\frac{z_k}{\varepsilon}\Big) - g'\Big(\frac{1-z_k}{\varepsilon}\Big)\Big).$$

As an example, we may take

$$g(z) = \frac{z}{1+z},$$

so that the equation for z_k becomes

$$(x_1 - x_0)\frac{z_k - z_{k-1}}{\tau} = -\frac{\varepsilon}{\lambda}\Big(\frac{1}{\varepsilon^2 + z_k^2} - \frac{1}{\varepsilon^2 + (z_k - 1)^2}\Big)\Big).$$

This suggests the scale

$$\lambda = \varepsilon,$$

and with this choice gives the limit equation for $z(t)$

$$z' = -\frac{1 - 2z}{(x_1 - x_0)z^2(z-1)^2}.$$

In this time scale, unless we are in the equilibrium $z = \frac{1}{2}$ the middle value moves towards the closest value between 0 and 1.

As a side remark, note that a simple qualitative study of this equation shows that if the initial datum is not 1/2 then $z = 0$ or $z = 1$ after a finite time, after which the motion is trivial. Note that the limit state is a local minimum with only one jump.

Example 9.1.5 We consider the same functionals F and F_ε as in Example 9.1.4 with an initial datum with three jumps satisfying the same Dirichlet boundary conditions $u(0^-) = 0$, $u(1^+) = 1$ and the same assumptions as before.

With the notation used above, the minimum problem is

$$\begin{aligned}\min\Big\{&\frac{1}{\lambda}\left(g\left(\frac{z_0 - 0}{\varepsilon}\right) + g\left(\frac{z_1 - z_0}{\varepsilon}\right) + g\left(\frac{1 - z_1}{\varepsilon}\right)\right)\\ &+\frac{1}{2\tau}\left((x_1 - x_0)|z_0 - z_0^{k-1}|^2 + (x_2 - x_1)|z_1 - z_1^{k-1}|^2\right)\Big\}. \qquad (9.3)\end{aligned}$$

Differently from the previous case, now we have to compute a gradient as a function of z_0 and z_1, the constant values of u respectively on (x_0, x_1) and (x_1, x_2). Hence, the Euler equations for (9.3) give the following system for z_0^k and z_1^k:

$$(x_1 - x_0)\frac{z_0^k - z_0^{k-1}}{\tau} = -\frac{1}{\lambda\varepsilon}\left(g'\left(\frac{z_0^k}{\varepsilon}\right) - g'\left(\frac{z_1^k - z_0^k}{\varepsilon}\right)\right), \tag{9.4}$$

$$(x_2 - x_1)\frac{z_1^k - z_1^{k-1}}{\tau} = -\frac{1}{\lambda\varepsilon}\left(g'\left(\frac{z_1^k - z_0^k}{\varepsilon}\right) - g'\left(\frac{1 - z_1^k}{\varepsilon}\right)\right). \tag{9.5}$$

For the sake of illustration, we may take the same g as in the previous example, so that equations (9.4) and (9.5) become

$$(x_1 - x_0)\frac{z_0^k - z_0^{k-1}}{\tau} = -\frac{\varepsilon}{\lambda}\left(\frac{1}{(\varepsilon + z_0^k)^2} - \frac{1}{(\varepsilon + z_1^k - z_0^k)^2}\right), \tag{9.6}$$

$$(x_2 - x_1)\frac{z_1^k - z_1^{k-1}}{\tau} = -\frac{\varepsilon}{\lambda}\left(\frac{1}{(\varepsilon + z_1^k - z_0^k)^2} - \frac{1}{(\varepsilon + 1 - z_1^k)^2}\right). \tag{9.7}$$

This suggests the scale

$$\lambda = \varepsilon, \tag{9.8}$$

and with this choice the limit equations for $z_0(t)$ and $z_1(t)$ are

$$z_0' = -\frac{z_1(z_1 - 2z_0)}{(x_1 - x_0)z_0^2(z_1 - z_0)^2}, \tag{9.9}$$

$$z_1' = -\frac{1 - z_0^2 - 2z_1(1 - z_0)}{(x_2 - x_1)(z_1 - z_0)^2(1 - z_1)^2}. \tag{9.10}$$

In this time scale, it is easy to see that the gradient is zero when $(z_0, z_1) = (\frac{z_1}{2}, \frac{1+z_0}{2})$, so we can have the following different behaviors:

- Equilibrium point. For the initial datum $(\overline{z}_0, \overline{z}_1) = (\frac{1}{3}, \frac{2}{3})$ the motion is trivial;
- If z_0 is larger than the equilibrium point, then $z_0' > 0$ and the constant value z_0 will increase towards z_1, otherwise it will decrease towards zero. The same holds for z_1 between z_0 and 1.

It must be noted that if the initial datum is not an equilibrium point then after a finite time one of the jump sizes vanishes, after which we are back to the previous example. In Figures 9.2–9.5 we picture four stages of the evolution computed numerically.

A further simplified example is obtained by taking symmetric initial data $x_2 - x_1 = x_1 - x_0 =: L$ and $z_0(0) = \frac{1}{2} - w_0$ and $z_1(0) = \frac{1}{2} + w_0$ with $0 < w_0 < 1/2$, for which the motion is described by a single parameter $w(t)$ satisfying

$$w' = \frac{3(\frac{1}{2} + w)(w - \frac{1}{6})}{4L(\frac{1}{2} - w)w^2},$$

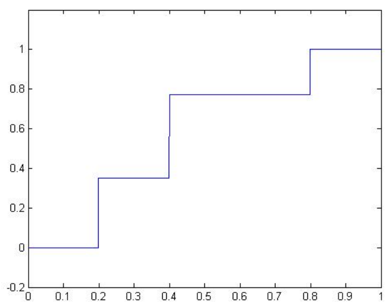

Figure 9.2: Initial conditions

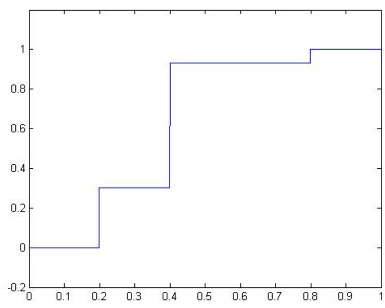

Figure 9.3: Iteration n. 30

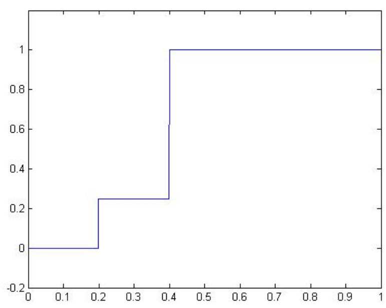

Figure 9.4: Iteration n. 60

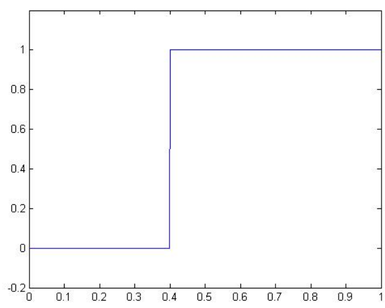

Figure 9.5: Iteration n. 100

in which case the equilibrium point corresponds to $w_0 = 1/6$, and otherwise after a finite either we have $w = 0$ (which gives $z_0 = z_1 = 1/2$; i.e., the equilibrium point with two jumps) or $w = \frac{1}{2}$ (which gives $z_0 = 0$ and $z_1 = 1$; i.e., a final state with only one jump point

Example 9.1.6 In the framework of the energies considered in the previous example, we not consider the case when we do not impose boundary conditions. To make it easier we consider the case in which there are only two jumps in x_0 and x_1, with $0 < x_0 < x_1 < 1$, and a piecewise-constant initial value:

$$u_0(x) = \begin{cases} z_0 & \text{if } 0 \le x < x_0 \\ z_1 & \text{if } x_0 < x < x_1 \\ z_2 & \text{if } x_1 < x \le 1 \end{cases} \tag{9.11}$$

where we consider $0 < z_0 < z_1 < z_2 < 1$ for simplicity. In this case, differently from the previous one, since the value of u at the boundary points is not prescribed, then all these values can change in time. In order to study the minimizing movement, we now need to consider the derivatives with respect to $z_0,\ z_1,\ z_2$. What we get is:

$$-\frac{1}{\varepsilon\lambda}\cdot\frac{1}{(1+\frac{z_1-z_0}{\varepsilon})^2}+\frac{1}{\tau}x_0(z_0^k-z_0^{k-1}) \quad = \quad 0 \tag{9.12}$$

$$\frac{1}{\varepsilon\lambda}\left(\frac{1}{(1+\frac{z_1-z_0}{\varepsilon})^2}-\frac{1}{(1+\frac{z_2-z_1}{\varepsilon})^2}\right)+\frac{1}{\tau}(x_1-x_0)(z_1^k-z_1^{k-1}) \quad = \quad 0 \tag{9.13}$$

$$\frac{1}{\varepsilon\lambda}\cdot\frac{1}{(1+\frac{z_2-z_1}{\varepsilon})^2}+\frac{1}{\tau}(1-x_1)(z_2^k-z_2^{k-1}) \quad = \quad 0 \tag{9.14}$$

This suggests again the scale $\lambda=\varepsilon$. With this choice, we find:

$$z_0' \quad = \quad \frac{1}{x_0}\frac{1}{(z_1-z_0)^2} \tag{9.15}$$

$$\begin{aligned} z_1' &= -\frac{1}{(x_1-x_0)}\cdot\frac{(z_2+z_0-2z_1)(z_2-z_0)}{(z_1-z_0)^2(z_2-z_1)^2} \\ &= -\frac{z_2-z_0}{(x_1-x_0)(z_1-z_0)(z_2-z_1)}\left(\frac{1}{z_1-z_0}-\frac{1}{z_2-z_1}\right) \end{aligned} \tag{9.16}$$

$$z_2' \quad = \quad -\frac{1}{(1-x_1)}\frac{1}{(z_2-z_1)^2} \tag{9.17}$$

We observe that z_0' is always positive, while z_2' is always negative, which means that the first constant will increase in time and the second one will decrease, trying to reduce again the number of the jumps. z_1' is zero at $\frac{z_0+z_2}{2}$; above this value it becomes positive, and under this value it becomes negative.

Example 9.1.7 We consider another approximation of the Mumford-Shah functional: the (scaled) Perona-Malik functional. In the notation for discrete functionals (see Section 3.4), we may define

$$F_\varepsilon(u)=\sum_{i=1}^{N}\frac{1}{|\log\varepsilon|}\log\Big(1+\varepsilon|\log\varepsilon|\Big|\frac{u_i-u_{i-1}}{\varepsilon}\Big|^2\Big).$$

Note that also the pointwise limit on piecewise-H^1 functions gives the Mumford-Shah functional since

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon|\log\varepsilon|}\log\Big(1+\varepsilon|\log\varepsilon|z^2\Big)=z^2$$

and

$$\lim_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}\log\Big(1+|\log\varepsilon|\frac{w^2}{\varepsilon}\Big)=1$$

for all $w\neq 0$.

As in the Example 9.1.4, we consider the case when competing functions are non-negative piecewise constants with $S(u) \subset S(u_0) = \{x_0, x_1\}$ and with the boundary conditions $u(0-) = 0, u(1+) = 1$. The computation is then reduced to a one-dimensional problem with unknown the constant value z_k defined by the minimization

$$\min\Big\{\frac{1}{\lambda|\log\varepsilon|}\Big(\log\Big(1+|\log\varepsilon|\frac{z^2}{\varepsilon}\Big)+\log\Big(1+|\log\varepsilon|\frac{(z-1)^2}{\varepsilon}\Big)\Big)+\frac{1}{2\tau}(x_1-x_0)(z-z_{k-1})^2\Big\},$$

which gives the equation

$$(x_1-x_0)\frac{z_k-z_{k-1}}{\tau}=-\frac{2}{\lambda}\Big(\frac{z}{\varepsilon+|\log\varepsilon|z^2}+\frac{z-1}{\varepsilon+|\log\varepsilon|(z-1)^2}\Big).$$

This suggests the time scale

$$\lambda=\frac{1}{|\log\varepsilon|},$$

and gives the equation for $z(t)$

$$z'=-\frac{2}{(x_1-x_0)}\cdot\frac{1-2z}{z(1-z)},$$

which provides a qualitative behaviour of z similar to the previous example.

Example 9.1.8 We now consider the sharp-interface model with

$$F(u)=\#(S(u)\cap[0,1))$$

defined on all piecewise-constant 1-periodic functions with values in ±1. For F all functions are local minimizers.

We take

$$F_\varepsilon(u)=\#(S(u)\cap[0,1))-\sum_{x_i\in[0,1)\cap S(u)}e^{-\frac{x_{i+1}-x_i}{\varepsilon}},$$

where $\{x_i\} = S(u)$ is a numbering of $S(u)$ with $x_i < x_{i+1}$.

We take as initial datum u_0 with $\#(S(u_0)) = 2$; hence, $S(u_0) = \{x_0, y_0\}$, and, after identifying u_0 with $A_0 = [x_0, y_0]$, apply the Almgren-Taylor-Wang variant of the iterative minimization process, where the distance term $\frac{1}{2\tau}\|u - u_{k-1}\|^2$ is substituted by

$$\frac{1}{\tau}\int_{A\triangle A_{k-1}}\mathrm{dist}(x,\partial A_{k-1})\,dx.$$

The computation of $A_1 = [x_1, y_1]$ is obtained by the minimization problem

$$\min\Big\{-\frac{1}{\lambda}\Big(e^{-\frac{(y-x)}{\varepsilon}}+e^{-\frac{(1+x-y)}{\varepsilon}}\Big)+\frac{1}{2\tau}((x-x_0)^2+(y-y_0)^2)\Big\},$$

which gives

$$\begin{aligned}\frac{x_1-x_0}{\tau} &= \frac{1}{\varepsilon\lambda}\Big(e^{-\frac{(y_1-x_1)}{\varepsilon}}-e^{-\frac{(1+x_1-y_1)}{\varepsilon}}\Big)\\ \frac{y_1-y_0}{\tau} &= -\frac{1}{\varepsilon\lambda}\Big(e^{-\frac{(y_1-x_1)}{\varepsilon}}-e^{-\frac{(1+x_1-y_1)}{\varepsilon}}\Big).\end{aligned}$$

Let $y_0-x_0<1/2$; we argue that the scaled time scale is

$$\lambda=\frac{1}{\varepsilon}e^{-\frac{y_0-x_0}{\varepsilon}},$$

for which we have

$$\begin{aligned}\frac{x_1-x_0}{\tau} &= \Big(e^{-\frac{(y_1-y_0-x_1+x_0)}{\varepsilon}}-e^{-\frac{(1+x_1-x_0-y_1+y_0)}{\varepsilon}}\Big)\\ \frac{y_1-y_0}{\tau} &= -\Big(e^{-\frac{(y_1-y_0-x_1+x_0)}{\varepsilon}}-e^{-\frac{(1+x_1-x_0-y_1+y_0)}{\varepsilon}}\Big).\end{aligned}$$

In terms of $L_k=y_k-x_k$ this can be written as

$$\frac{L_1-L_0}{\tau} = -2\Big(e^{-\frac{(L_1-L_0)}{\varepsilon}}-e^{-\frac{(1+L_0-L_1)}{\varepsilon}}\Big).$$

Under the assumption $\tau<<\varepsilon$ we have in the limit

$$L'=-2\Big(e^{o(1)}-e^{-\frac{1}{\varepsilon}+o(1)}\Big)=-2\,,$$

which shows that the two closer interfaces move towards each other shortening linearly their distance.

9.2 Reversed time

In a finite-dimensional setting a condition to be able to define a minimizing movement for F is that

$$u\mapsto F(u)+\frac{1}{2\tau}|u-\overline{u}|^2 \tag{9.18}$$

be lower semicontinuous and coercive for all $\overline{u}$ and for τ sufficiently small. This is not in contrast with requiring that also

$$u\mapsto -F(u)+\frac{1}{2\tau}|u-\overline{u}|^2 \tag{9.19}$$

satisfy the same conditions; for example if F is continuous and of quadratic growth. Note that this can be seen as a further extension of the time-scaling argument in the previous sections with $\lambda=-1$. If the iterative scheme gives a solution for the gradient flow, a minimizing movement u for the second scheme produces a solution $v(t) = u(-t)$ to the backward problem

$$\begin{cases} v'(t) = -F(v(t)) & \text{for } t \leq 0 \\ v(0) = u_0 \end{cases}$$

In an infinite-dimensional setting the two requirements of being able to define both the minimizing movement (9.18) and (9.19) greatly limits the choice of F, and rules out all interesting cases. A possible approach to the definition of a backward minimizing movement is then to introduce a (finite-dimensional) approximation F_ε to F, for which we can define a minimizing motion along $-F_\varepsilon$.

We now give an example in the context of crystalline motion, where we consider a negative scaling of time.

Example 9.2.1 We consider in $\mathbb{R}^2$

$$F(A) = \int_{\partial A} \|\nu\|_1 d\mathcal{H}^1,$$

and F_ε the restriction of F to the sets of the form

$$\bigcup \Big\{ \varepsilon i + \Big(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\Big)^2 : i \in B \Big\},$$

where B is a subset of $\mathbb{Z}^2$. Hence, we may identify these union of ε-cubes with the corresponding B. Even though this is not a finite-dimensional space, we will be able to apply the Almgren-Taylor-Wang scheme.

We choose (with the identifications with subsets of $\mathbb{Z}^2$) as initial datum

$$A_0^\varepsilon = \{(0,0)\} = \Big(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\Big)^2,$$

and solve iteratively

$$\min \Big\{ -\frac{1}{\lambda} F_\varepsilon(A) + \frac{1}{\tau} \int_{A \setminus A_{k-1}^\varepsilon} \mathrm{dist}_\infty(x, \partial A_{k-1}^\varepsilon)\, dx \Big\}.$$

with $\lambda = \lambda_\varepsilon > 0$ to be determined. In the interpretation as a reversed-time scheme, this means that we are solving a problem imposing the extinction at time 0.

Note that taking F in place of F_ε would immediately give the value $-\infty$ in the minimum problem above; e.g., by considering sets of the form (in polar coordinates)

$$A_j = \{(\rho, \theta) : \rho \leq 3\varepsilon + \varepsilon \sin(j\theta)\},$$

which contain A_0^ε, are contained in $B_{4\varepsilon}(0)$ and have a perimeter larger than $4j\varepsilon$.

Under the assumption that $\varepsilon << \tau$ all minimizing sets are the checkerboard structure corresponding to indices $i \in \mathbb{Z}^2$ with $i_1 + i_2$ even contained in a square Q_k centered in 0

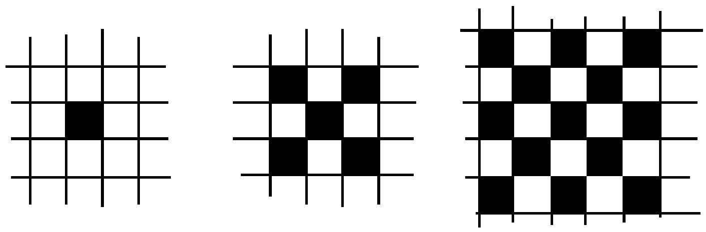

Figure 9.6: enucleating sets

(see Fig. 9.6). We may take the sides L_k of those squares as unknown. The incremental problems can be rewritten as

$$\min\Big\{-\frac{2}{\varepsilon\lambda}((L_{k-1}+\Delta L)^2 - L_{k-1}^2) + \frac{1}{\tau}(L_{k-1}(\Delta L)^2 + r_k(\Delta L)^2)\Big\},$$

with r_k negligible as $\tau \to 0$. For the interfacial part, we have taken into account that for ε small the number of squares contained in a rectangle is equal to its area divided by $2\varepsilon^2$ and each of the squares gives an energy contribution of 4ε; for the distance part, we note that the integral can be equivalently taken on half of $Q_k \setminus Q_{k-1}$. Minimization in ΔL gives

$$\frac{\Delta L}{\tau} = \frac{2}{\varepsilon\lambda}\Big(1 + \frac{\Delta L}{L_{k-1}}\Big).$$

Choosing $\lambda = \frac{1}{\varepsilon}$, we obtain a linear growth

$$L(s) = 2s\,.$$

What we have obtained is the description of the structure of ε-squares (the checkerboard one) along which the increase of the perimeter is maximal (and, in a sense, the decrease of the perimeter is maximal for the reverse-time problem).

9.3 Reference to Chapter 9

The literature on long-time behaviour and backward equations, even though not by the approach by minimizing movements, is huge. The long-time motion of interfaces in one space dimension by energy methods has been studied in

L. Bronsard and R.V. Kohn. On the slowness of phase boundary motion in one space dimension. Comm. Pure Appl. Math. 43 (1990), 983–997.

Examples 9.1.5 and 9.1.6 have been part of the course exam of C. Sorgentone and S. Tozza at Sapienza University in Rome.

I acknowledge the suggestion of J.W. Cahn to use finite-dimensional approximations to define backward motion of sets.