Chapter 6

Minimizing movements

6.1 An energy-driven implicit-time discretization

We now introduce a notion of energy-based motion which generalizes an implicit-time scheme for the approximation of solutions of gradient flows to general (also non differentiable) energies. We will use the terminology of *minimizing movements*, introduced by De Giorgi, even though we will not use the precise notation used in the literature.

Definition 6.1.1 (minimizing movements) *Let X be a separable Hilbert space and let $F : X \to [0, +\infty]$ be coercive and lower semicontinuous. Given x_0 and $\tau > 0$ we define recursively x_k as a minimizer for the problem*

$$\min\Big\{F(x) + \frac{1}{2\tau}\|x - x_{k-1}\|^2\Big\}, \tag{6.1}$$

and the piecewise-constant trajectory $u^\tau : [0, +\infty) \to X$ given by

$$u^\tau(t) = x_{\lfloor t/\tau \rfloor}\,. \tag{6.2}$$

A minimizing movement *for F from x_0 is any limit of a subsequence u^{τ_j} uniform on compact sets of $[0, +\infty)$.*

In this definiton we have taken $F \geq 0$ and X Hilbert for the sale of simplicity. In particular we can take X a metric space and the (power of the) distance in place of the squared norm.

Remark 6.1.2 A heuristic explanation of the definition above is given when F is smooth. In this case, with the due notation, a minimizer for (6.1) solves the equation

$$\frac{x_k - x_{k-1}}{\tau} = -\nabla F(u_k); \tag{6.3}$$

i.e., u^τ solves the equation

$$\frac{u^\tau(t) - u^\tau(t-\tau)}{\tau} = -\nabla F(u^\tau(t)). \tag{6.4}$$

If we may pass to the limit in this equation as $u^\tau \to u$ then

$$\frac{\partial u}{\partial t} = -\nabla F(u). \tag{6.5}$$

This is easily shown if $X = \mathbb{R}^n$ and $F \in C^2(\mathbb{R}^n)$. In this case by taking any $\varphi \in C_0^\infty((0,T);\mathbb{R}^n)$ we have

$$-\int_0^T \langle \nabla F(u^\tau), \varphi\rangle dt = \int_0^T \Big\langle \frac{u^\tau(t) - u^\tau(t-\tau)}{\tau}, \varphi\Big\rangle dt = -\int_0^T \Big\langle u^\tau(t), \frac{\varphi(t) - \varphi(t+\tau)}{\tau}\Big\rangle dt,$$

from which, passing to the limit

$$\int_0^T \langle \nabla F(u), \varphi\rangle dt = \int_0^T \langle u, \varphi'\rangle dt;$$

i.e., (6.5) is satisfied in the sense of distributions, and hence in the classical sense.

Remark 6.1.3 (stationary solutions) Let x_0 be a local minimizer for F, then the only minimizing movement for F from x_0 is the constant function $u(t) = x_0$.

Indeed, if x_0 is a minimizer for F when $\|x - x_0\| \leq \delta$ by the positiveness of F it is the only minimizer of $F(x) + \frac{1}{2\tau}\|x - x_0\|^2$ for $\tau \leq \delta^2/F(x_0)$ if $F(x_0) > 0$ (any τ if $F(x_0) = 0$). So that $x_k = x_0$ for all k for these τ.

Proposition 6.1.4 (existence of minimizing movements) *For all F and x_0 as above there exists a minimizing movement $u \in C^{1/2}([0,+\infty);X)$.*

Proof. By the coerciveness and lower semicontinuity of F we obtain that u_k are well defined for all k. Moreover, since

$$F(x_k) + \frac{1}{2\tau}\|x_k - x_{k-1}\|^2 \leq F(x_{k-1}),$$

we have $F(x_k) \leq F(x_{k-1})$ and

$$\|x_k - x_{k-1}\|^2 \leq 2\tau(F(x_{k-1}) - F(x_k)), \tag{6.6}$$

so that for $t > s$

$$\begin{aligned}
\|u^\tau(t) - u^\tau(s)\| &\leq \sum_{k=\lfloor s/\tau\rfloor+1}^{\lfloor t/\tau\rfloor} \|x_k - x_{k-1}\| \\
&\leq \sqrt{\lfloor t/\tau\rfloor - \lfloor s/\tau\rfloor}\sqrt{\sum_{k=\lfloor s/\tau\rfloor+1}^{\lfloor t/\tau\rfloor} \|x_k - x_{k-1}\|^2} \\
&\leq \sqrt{\lfloor t/\tau\rfloor - \lfloor s/\tau\rfloor}\sqrt{2\tau \sum_{k=\lfloor s/\tau\rfloor+1}^{\lfloor t/\tau\rfloor} (F(x_{k-1}) - F(x_k))} \\
&= \sqrt{\lfloor t/\tau\rfloor - \lfloor s/\tau\rfloor}\sqrt{2\tau(F(x_{\lfloor s/\tau\rfloor}) - F(x_{\lfloor t/\tau\rfloor}))} \\
&\leq \sqrt{2F(x_0)}\sqrt{\tau(\lfloor t/\tau\rfloor - \lfloor s/\tau\rfloor)} \\
&\leq \sqrt{2F(x_0)}\sqrt{t - s + \tau}
\end{aligned}$$

This shows that the functions u^τ are (almost) equicontinuous and equibounded in $C([0, +\infty); X)$. Hence, they converge uniformly. Moreover, passing to the limit we obtain

$$\|u(t) - u(s)\| \leq \sqrt{2F(x_0)}\sqrt{|t - s|}$$

so that $u \in C^{1/2}([0, +\infty); X)$. □

Remark 6.1.5 (growth conditions) The positiveness of F can be substituted by the requirement that for all $\overline{x}$ the functionals

$$x \mapsto F(x) + \frac{1}{2\tau}\|x - \overline{x}\|^2$$

be bounded from below; i.e., that there exists $C > 0$ such that

$$x \mapsto F(x) + C\|x - \overline{x}\|^2$$

be bounded from below.

Example 6.1.6 (non-uniqueness of minimizing movements) If F is not C^2 we may have more than one minimizing movement.

(i) *Bifurcation at times with multiple minimizers.* A simple example is $F(x) = -\frac{1}{\alpha}|x|^\alpha$ with $0 < \alpha < 2$, which is not C^2 at $x = 0$. In this case, for $x_0 = 0$ we have a double choice for minimum problem (6.1); i.e.,

$$x_1 = \pm\tau^{1/(2-\alpha)}.$$

Once x_1 is chosen all other valued are determined, and it can be seen that either $x_k > 0$ for all k or $x_k < 0$ for all k (for $\alpha = 1$, e.g., we have $x_k = \pm k\tau$), and that in the limit we have the two solutions of

$$\begin{cases} u' = |u|^{(\alpha-2)}u \\ u(0) = 0 \end{cases}$$

with $u(t) \neq 0$ for $t > 0$. Note in particular that we do not have the trivial solution $u = 0$. In this example we do not have to pass to a subsequence of τ.

(i) *Different movements depending on subsequences of* τ. Discrete trajectories can be different depending on the time step τ. We give an explicit example, close in spirit to the previous one. In this example the function F is asymmetric, so that x_1 is unique but may take positive or negative values depending on τ.

We define F as the Lipschitz function taking value 0 at $x = 0$, for $x > 0$

$$F'(x) = \begin{cases} -1 & \text{if } 2^{-2k-1} < x < 2^{-2k},\ k \in \mathbb{N} \\ -2 & \text{otherwise for } x > 0 \end{cases}$$

and $F'(x) = 3 + F'(-x)$ for $x < 0$. It is easily seen that for $x_0 = 0$ we may have a unique minimizer x_1 with $x_1 > 0$ or $x_1 < 0$ depending on τ. In particular we have $x_1 = -2^{-2k} < 0$ for $\tau = 2^{-2k-1}$ and $x_1 = 2^{-2k+1} > 0$ for $\tau = 2^{-2k}$. In the two cases we then have again the solutions to

$$\begin{cases} u' = -F(u) \\ u(0) = 0 \end{cases}$$

with $u(t) < 0$ for all $t > 0$ or $u(t) > 0$ for all $t > 0$, respectively.

Example 6.1.7 (heat equation) Taking $X = L^2(\Omega)$ and the Dirichlet integral $F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx$, with fixed $u_0 \in H^1(\Omega)$ and $\tau > 0$ we can solve iteratively

$$\min\Big\{\frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2\tau}\int_\Omega |u - u_{k-1}|^2\,dx\Big\},$$

whose unique minimizer u_k solves the Euler-Lagrange equation

$$\frac{u_k - u_{k-1}}{\tau} = \Delta u_k, \qquad \frac{\partial u_k}{\partial \nu} = 0 \text{ on } \partial\Omega, \tag{6.7}$$

where ν is the inner normal to Ω. We then set $u^\tau(x,t) = u_{\lfloor t/\tau \rfloor}(x)$, which converges, up to subsequences, to $u(x,t)$. We can then pass to the limit in (6.8) in the sense of distributions to obtain the heat equation

$$\frac{\partial u}{\partial t} = \Delta u, \qquad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \tag{6.8}$$

combined with the initial datum $u(x,0) = u_0(x)$. Due to the uniqueness of the solution to the heat equation we also obtain that the whole sequence converges as $\tau \to 0$.

Example 6.1.8 (One-dimensional fracture energies) In dimension one, we still consider $X = L^2(0,1)$ and the Griffith (or Mumford-Shah) energy

$$F(u) = \frac{1}{2}\int_0^1 |u'|^2\,dx + \#(S(u))$$

with domain piecewise-H^1 functions. We fix u_0 piecewise-H^1 and $\tau > 0$. In this case we solve iteratively

$$\min\Big\{\frac{1}{2}\int_0^1 |u'|^2\,dx + \#(S(u)) + \frac{1}{2\tau}\int_0^1 |u - u_{k-1}|^2\,dx\Big\}.$$

This problem is not convex, and may have multiple minimizers. Nevertheless in this simpler case we can prove iteratively that for τ small enough we have $S(u_k) = S(u_0)$ for all k, and hence reduce to the independent iterated minimization problems of the Dirichlet integral on each component of $(0,1)\setminus S(u_0)$, giving the heat equation in $(0,1)\setminus S(u_0)$ with Neumann boundary conditions on 0, 1 and $S(u_0)$. The description holds until the first time T such that $u(x^-,T) = u(x^+,T)$ at some point $x \in S(u(\cdot,T))$.

We check this with some simplifying hypotheses:

(1) that $\int_0^1 |u_0'|^2 dx < 2$. This implies that $\#(S(u_k)) \le \#S(u_0)$ since by the monotonicity of the energy we have $\#(S(u_k)) \le F(u_k) \le F(u_0) < \#S(u_0) + 1$. This hypothesis can be removed with a localization argument;

(2) that there exists $\eta > 0$ such that $|u_0(x) - u_0(x')| \ge \eta$ if $(x,x') \cap S(u_0) \ne \emptyset$. This will imply that $|u_k(x^+) - u_k(x^-)| \ge \eta$ at all $x \in S(u_k)$ so that $T = +\infty$ in the notation above.

Furthermore, we suppose that $S(u_0) = \{x_0\}$ (a single point) and $u_0(x_0^+) > u_0(x_0^-)$, for simplicity of notation.

We reason by induction. We first examine the properties of u_1; checking that it has a jump point close to x_0. Suppose otherwise that there exists $\delta > 0$ such that $u \in H^1(x_0-\delta, x_0+\delta)$. We take δ small enough so that

$$u_0(x) \le u(x_0^-) + \frac{1}{4}(u(x_0^+) - u(x_0^-)) \text{ for } x_0 - \delta < x < x_0$$
$$u_0(x) \ge u(x_0^+) - \frac{1}{4}(u(x_0^+) - u(x_0^-)) \text{ for } x_0 < x < x_0 + \delta.$$

In this case

$$\begin{aligned} &\frac{1}{2}\int_{x_0-\delta}^{x_0+\delta} |u_1'|^2\,dx + \frac{1}{2\tau}\int_{x_0-\delta}^{x_0+\delta} |u_1 - u_0|^2 dx \\ \ge\ & \frac{1}{2}\min\Big\{\int_0^\delta |v'|^2\,dx + \frac{1}{\tau}\int_0^\delta |v|^2 dx : v(\delta) = \frac{1}{4}(u(x_0^+) - u(x_0^-))\Big\} \\ =\ & \frac{(u(x_0^+) - u(x_0^-))^2}{8\sqrt{\tau}}\tanh\Big(\frac{\delta}{\sqrt{\tau}}\Big), \end{aligned}$$

the last equality easily obtained by computing the solution of the Euler-Lagrange equation. This shows that for each such $\delta > 0$ fixed we have $(x_0-\delta, x_0+\delta)\cap S(u_1) \neq \emptyset$ for τ sufficiently small. Note that the smallness of τ depends only on the size of $u(x_0^+) - u(x_0^-)$ (which is larger than η). Since $\#S(u_1) \leq \#S(u_0)$ we then have $(x_0-\delta, x_0+\delta)\cap S(u_1) = \{x_1\}$; we may suppose that $x_1 \geq x_0$.

We now check that $x_1 = x_0$. Suppose otherwise; then note that by the Hölder continuity of u_1 we have that for δ small enough (depending only on the size of $u(x_0^+) - u(x_0^-)$) we have

$$u_1(x) \leq u_0(x_0^-) + \frac{1}{8}(u(x_0^+) - u(x_0^-)) \text{ for } x_0 < x < x_1$$

and

$$u_1(x_1^+) \geq u_0(x_1^+) - \frac{1}{8}(u(x_0^+) - u(x_0^-)).$$

We may then consider the function $\widetilde{u}$ coinciding with u_1 on $(0, x_0)$, with $\widetilde{u}' - U_1'$ and $S(\widetilde{u}) = \{x_0\}$. Then $F(\widetilde{u}) = F(u_1)$ and

$$\int_0^1 |\widetilde{u} - u_0|^2\,dx < \int_0^1 |u_1 - u_0|^2\,dx,$$

contradicting the minimality of u_1. Hence, we have $S(u_1) = S(u_0)$.

Note that u_1 is obtained by separately minimizing the problems with the Dirichlet integral on $(0, x_0)$ and $(x_0, 1)$, and in particular that on each such interval $\sup u_1 \leq \sup u_0$ and $\inf u_1 \geq \inf u_0$, so that the condition that $|u_1(x) - u_1(x')| \geq \eta$ if $(x, x') \cap S(u_1) \neq \emptyset$ still hold. This shows that we can iterate the scheme obtaining u_k which satisfy $\inf u_0 \leq u_k \leq \sup u_0$ on each component of $(0,1) \setminus S(u_0)$ and $u_k' = 0$ on 0, 1 and $S(u_0)$. In particular $|u_k^+ - u_k^-| \geq \eta$ on $S(u_0)$, which shows that the limit satisfies the heat equation with Neumann conditions on $S(u_0)$ for all times.

6.2 Time-dependent minimizing movements

We can generalize the definition of minimizing movement to include forcing terms or varying boundary conditions, by considering time-parameterized energies $F(x,t)$ and, given τ and an initial datum x_0, define x_k recursively by choosing x_k as a minimizer of

$$\min\Big\{F(x, k\tau) + \frac{1}{2\tau}\|x - x_{k-1}\|^2\Big\}, \tag{6.9}$$

and eventually define $u^\tau(t) = x_{\lfloor t/\tau \rfloor}$. We may define, up to subseqences, a limit u of u^τ as $\tau \to 0$ if some technical hypothesis is added to F. One such is that in the scheme above

$$F(u_k, k\tau) + \frac{1}{2\tau}\|x_k - x_{k-1}\|^2 \leq (1 + C\tau)F(u_{k-1}, (k-1)\tau) + C\tau, \tag{6.10}$$

for some C (at least if $k\tau$ remains bounded). With such a condition we can repeat the convergence argument as for the time-independent case and obtain a limit minimizing movement u.

Indeed, with such a condition we have

$$\|x_k - x_{k-1}\|^2 \le 2\tau\Big((1+C\tau)F(u_{k-1},(k-1)\tau) - F(u_k,k\tau) + C\tau\Big), \tag{6.11}$$

and the inequality (for τ small enough)

$$F(u_k,k\tau) \le (1+C\tau)F(u_{k-1},(k-1)\tau) + C\tau \le (1+C\tau)(F(u_{k-1},(k-1)\tau)+1) \tag{6.12}$$

that implies that $F(u_k,k\tau)$ is equibounded for $k\tau$ bounded. We fix $T>0$; from (6.11) we obtain (for $0 \le s \le t \le T$)

$$\begin{aligned}
\|u^\tau(t) - u^\tau(s)\| &\le \sum_{k=\lfloor s/\tau\rfloor+1}^{\lfloor t/\tau\rfloor} \|x_k - x_{k-1}\| \\
&\le \sqrt{\lfloor t/\tau\rfloor - \lfloor s/\tau\rfloor}\sqrt{\sum_{k=\lfloor s/\tau\rfloor+1}^{\lfloor t/\tau\rfloor} \|x_k - x_{k-1}\|^2} \\
&\le \sqrt{\lfloor t/\tau\rfloor - \lfloor s/\tau\rfloor}\sqrt{2\tau \sum_{k=\lfloor s/\tau\rfloor+1}^{\lfloor t/\tau\rfloor} \Big(F(x_{k-1},(k-1)\tau) - F(x_k,k\tau) + C_T\tau\Big)} \\
&= \sqrt{\lfloor t/\tau\rfloor - \lfloor s/\tau\rfloor}\sqrt{2\tau F(x_0,0) + C_T\tau(\lfloor t/\tau\rfloor - \lfloor s/\tau\rfloor))} \\
&\le \sqrt{2F(x_0,0)(t-s+\tau) + C_T(t-s+\tau)^2},
\end{aligned}$$

which gives an equicontinuity condition sufficient to pass to the limit as $\tau \to 0$.

Note that from (6.12) for $s<t$ we obtain the estimate

$$F(u^\tau(t),\lfloor t/\tau\rfloor\tau) \le e^{C(t-s+\tau)}(F(u^\tau(s),\lfloor s/\tau\rfloor\tau) + C\tau). \tag{6.13}$$

Example 6.2.1 (heat equation with varying boundary conditions) We can take

$$F(u,t) = \frac{1}{2}\int_0^1 |u'|^2\,dx,$$

with domain all H^1-functions with $u(0)=0$ and $u(1)=t$.

Then if we can test the problem defining u_k with the function $\tilde u = u_{k-1} + \tau x$. We then have

$$F(u_k,k\tau) + \frac{1}{2\tau}\|u_k - u_{k-1}\|^2 \le F(\tilde u,k\tau) + \frac{1}{2\tau}\|\tilde u - u_{k-1}\|^2$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^1 |u'_{k-1}+\tau|^2\,dx + \frac{1}{6}\tau^2 \\
&\le (1+\tau)\frac{1}{2}\int_0^1 |u'_{k-1}|^2\,dx + \tau + \frac{1}{6}\tau^2 \\
&\le (1+\tau)\frac{1}{2}F(u_{k-1},(k-1)\tau) + \Big(1+\frac{1}{6}\tau\Big)\tau.
\end{aligned}
$$

which gives (6.10).

We then have the convergence of u^τ to the solution u of the equation

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} \\
u(0,t) = 0, u(1,t) = t \\
u(x,0) = u_0(x).
\end{cases}
\tag{6.14}
$$

Clearly in this example we may take any Lipschitz function $g(t)$ in place of t as boundary condition.

Example 6.2.2 (minimizing movements vs quasi static evolution for fracture) We can take

$$
F(u,t) = \frac{1}{2}\int_0^1 |u'|^2\,dx + \#S(u),
$$

with domain all piecewise-H^1-loc functions with $u(x) = 0$ if $x \le 0$ and $u(x) = t$ for $x \ge 1$, so that $S(u) \subset [0,1]$, and the fracture may also appear at the boundary points 0 and 1. As in the previous example we can test the problem defining u_k with the function $\tilde u = u_{k-1} + \tau x$ since $\#S(\tilde u) = \#S(u_{k-1})$, to obtain (6.10).

We consider the initial datum $u_0 = 0$. Note that the minimum problems for $F(\cdot,t)$ correspond to the definition of quasi static evolution in Remark 2.3.2. We now show that for problems (6.9) the solution does not develop fracture.

Indeed, consider

$$
k_\tau = \min\{k : u_k \notin H^1_{\rm loc}(\mathbb{R})\},
$$

and suppose that $\tau k_\tau \to \overline{t} \in [0,+\infty)$. Then we have that u^τ converges on $[0,\overline{t}]$ to u described by (6.14) in the previous example. Moreover we may suppose that

$$
\lim_{\tau\to 0}\int_0^1 |u'_{k_\tau-1}|^2\,dx = \int_0^1 |u'(x,\overline{t})|^2\,dx,
$$

and since also $u_{k_\tau} \to u(\cdot,\overline{t})$ as $\tau \to 0$, we have

$$
\int_0^1 |u'(x,\overline{t})|^2\,dx \le \liminf_{\tau\to 0}\int_0^1 |u'_{k_\tau}|^2\,dx.
$$

We deduce that

$$F(u_{k_\tau}, k_\tau\tau) \geq F(u_{k_\tau-1}, (k_\tau - 1)\tau) + 1 + o(1)$$

as $\tau \to 0$, which contradicts (6.12).

From the analysis above we can compare various ways to define the evolutive response of a brittle elastic material to applied increasing boundary displacements (at least in a one-dimensional setting):

• **(quasistatic motion)** the response is purely elastic until a threshold (depending on the size of the specimen) is reached, after which we have brutal fracture;

• **(stable evolution)** the response is purely elastic, and corresponds to minimizing the elastic energy at fixed boundary displacement;

• **(minimizing movement)** in this case the solution does not develop fracture, but follows the heat equation with given boundary conditions.

6.3 References to Chapter 6

The terminology "(generalized) minimizing movement" has been introduced by De Giorgi in a series of papers devoted to mathematical conjectures. We also refer to the original treatment by L. Ambrosio.

E. De Giorgi. New problems on minimizing movements. In "E. De Giorgi. Selected Papers. Springer, 2006".

L. Ambrosio. Minimizing movements. Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. 19 (1995), 191–246.

A theory of gradient flows in metric spaces using minimizing movements is described in the book

L. Ambrosio, N. Gigli and G. Savarè, Gradient flows in metric spaces and in the space of probability measures, Lectures in Mathematics ETH Zürich. Birkhäuser, Basel, 2005.

Minimizing movements for the Mumford-Shah functional in more that one space dimension (and hence also for the Griffith fracture energy) with the condition of increasing fracture have been defined in

L. Ambrosio and A. Braides. Energies in SBV and variational models in fracture mechanics. In *Homogenization and Applications to Material Sciences*, (D. Cioranescu, A. Damlamian, P. Donato eds.), GAKUTO, Gakkōtosho, Tokio, Japan, 1997, p. 1–22,

and partly analyzed in a two-dimensional setting in

A. Chambolle and F. Doveri. Minimizing movements of the Mumford and Shah energy. Discr. Cont. Dynamical Syst. 3 (1997), 153–174.

In this case the heat equation with Neumann boundary conditions still holds outside $S(u)$, but the characterization of the (possible) motion of the crack is still an open problem. As compared with the corresponding Francfort-Marigo theory, here an analog of the Francfort-Larsen transfer lemma is missing.