Chapter 5

Stability

The notion of local minimizer is ‘scale-independent’; i.e., it does not depend on the rate at which energies converge, so that it does not discriminate, e.g., between energies

$$F_\varepsilon(x) = x^2 + \sin^2\Big(\frac{x}{\varepsilon}\Big) \qquad \text{or} \qquad F_\varepsilon(x) = x^2 + \sqrt{\varepsilon}\sin^2\Big(\frac{x}{\varepsilon}\Big).$$

We now examine a notion of *stability* such that, loosely speaking, a point is *stable* if it is not possible to reach a lower energy state from that point without crossing an energy barrier of a specified height. In this case the local minimizers in the first of the two sequence of energies are stable as $\varepsilon \to 0$, while those in the second sequence are not.

5.1 Stable points

We first introduce a notion of stability that often can be related to notions of local minimality.

Definition 5.1.1 (slide) *Let* $F : X \to [0, +\infty]$ *and* $\delta > 0$. *A continuous function* $\phi : [0, 1] \to X$ *is a* δ-slide *for* F *at* $u_0 \in X$ *if*

- $\phi(0) = u_0$ *and* $F(\phi(1)) < F(\phi(0)) = F(u_0)$*;*
- *there exits* $\delta' < \delta$ *such that* $F(\phi(t)) \le F(\phi(s)) + \delta'$ *if* $0 \le s \le t \le 1$.

Definition 5.1.2 (stability) *Let* $F : X \to [0, +\infty]$ *and* $\delta > 0$. *A point* $u_0 \in X$ *is* δ-stable *for* F *if no* δ*-slide exists for* F *at* u_0.

A point $u_0 \in X$ *is* stable *for* F *if it is* δ*-stable for some* $\delta > 0$ *(and hence for all* δ *small enough).*

Let $F_\varepsilon : X \to [0, +\infty]$. *A sequence of points* (u_ε) *in* X *is* uniformly stable *for* (F_ε) *if there exists* $\delta > 0$ *such that all* u_ε *are* δ*-stable for* ε *small.*

Example 5.1.3 (1) $F(x) = \begin{cases} 0 & x = 0 \\ \sin\Big(\dfrac{1}{x}\Big) & \text{otherwise.} \end{cases}$ The point 0 is not a local minimizer but it is δ-stable for $\delta < 1$;

(2) Similarly for $F(x) = \begin{cases} 0 & x = 0 \\ -x^2 + \sin^2\Big(\dfrac{1}{x}\Big) & \text{otherwise;} \end{cases}$

(3) Let $X = \mathbb{C}$ and $F(z) = F(\rho e^{i\theta}) = \begin{cases} \theta\rho & |z| \le 1 \\ -1 & \text{otherwise,} \end{cases}$
where we have chosen the representation $z = \rho e^{i\theta}$ with $0 < \theta \le 2\pi$. Then 0 is an isolated local minimum, but it is not stable; e.g., taking $\phi(t) = 2te^{i\delta/2}$. Note in fact that $\phi(0) = 0$, $F(\phi(1)) = -1 < 0$, and $\sup F(\phi(t)) = F(\phi(1/2)) = \delta/2$;

(4) We can generalize example (3) to an infinite-dimensional example. Take $X = L^2(-\pi, \pi)$ and

$$F(u) = \begin{cases} \displaystyle\sum_k \frac{1}{k^2}|c_k|^2 & \text{if } u = \sum_k c_k e^{ikx} \text{ and } \|u\|_{L^2} < 1 \\ -1 & \text{otherwise.} \end{cases}$$

The constant 0 is an isolated minimum point. F is lower semicontinuous, and continuous in $\{\|u\|_{L^2} < 1\}$. Note that $F(e^{ikx}) = \frac{1}{k^2}$ so that $\phi_k(t) = 2te^{ikx}$ is a δ-slide for $k^2 > 1/\delta$;

(5) $F_\varepsilon(x) = x^2 + \sin^2\Big(\dfrac{x}{\varepsilon}\Big)$. Each bounded sequence of local minimizers is uniformly stable;

(6) $F_\varepsilon(x) = x^2 + \varepsilon^\alpha \sin^2\Big(\dfrac{x}{\varepsilon}\Big)$ with $0 < \alpha < 1$. No bounded sequence of local minimizers is uniformly stable (except the constant sequence of global minimizers $x_\varepsilon = 0$).

Remark 5.1.4 (local minimality and stability) (i) If $F : X \to \mathbb{R}$ is continuous and u stable; then u is a local minimizer;

(ii) Let F be lower semicontinuous and coercive. Then every isolated local minimizer of F is stable.

(iii) if u is just a local minimizer then u may not be stable.

To check (i) suppose that u is not a local minimum for F. Then let ρ be such that $|F(u) - F(w)| < \delta/2$ if $w \in B_\rho(u)$, and let $u_\rho \in B_\rho(u)$ be such that $F(u_\rho) < F(u)$. Then it suffices to take $\phi(t) = u + t(u_\rho - u)$.

To check (ii), let $\eta > 0$ be such that u_0 is an isolated minimum point in $B_\eta(u_0)$. If u_0 is not stable then there exist $1/k$ slides ϕ_k with final point outside $B_\eta(u_0)$. This implies that there exist $u_k = \phi_k(t_k)$ for some t_k with $u_k \in \partial B_\eta(u_0)$, so that $F(u_k) \le F(u_0) + 1/k$. By coerciveness, upon extraction of a subsequence $u_\varepsilon \to \overline{u} \in \partial B_\eta(u_0)$, and by lower semicontinuity $F(\overline{u}) \le \liminf_k F(u_k) \le F(u_0)$, which is a contradiction.

For (iii) take for example $u = 0$ for $F(u) = (1 - |u|) \wedge 0$ on $\mathbb{R}$.

5.2 Stable sequences of functionals

We now give a notion of stability of parameterized functionals.

Definition 5.2.1 (relative (sub)stability) *We say that a sequence (F_ε) is* (sub)stable relative to F *if the following holds*

• *if u_0 has a δ-slide for F and $u_\varepsilon \to u_0$, then each u_ε has a δ-slide for F_ε (for ε small enough).*

Remark 5.2.2 (relative (super)stability) The condition of sub-stability above can be compared to the lower bound for Γ-convergence. With this parallel in mind we can introduce a notion of *(super)stability relative to F* by requiring that

• if u_0 is an isolated local minimum for F then there exists $u_\varepsilon \to u$ such that (u_ε) is uniformly stable for F_ε.

Remark 5.2.3 (i) Note that if F is a constant then all (F_ε) are stable relative to F;

(ii) In general if $F_\varepsilon = F$ for all ε then (F_ε) may not be stable relative to F. Take for example

$$F_\varepsilon(x) = F(x) = \begin{cases} \sin\Big(\dfrac{1}{x}\Big) & \text{if } x > 0 \\ x & \text{if } x \le 0; \end{cases}$$

then $x_0 = 0$ has δ-slides for all $\delta > 0$, while taking $x_\varepsilon = (2\pi\lfloor\frac{1}{\varepsilon}\rfloor - \frac{\pi}{2})^{-1}$ we have $x_\varepsilon \to x_0$ and x_ε has no δ-slide for $\delta < 2$.

The following proposition is in a sense the converse of Theorem 4.1.1 with Γ-convergence substituted with stability.

Proposition 5.2.4 *Let (F_ε) be (sub)stable relative to F and u_ε be a sequence of uniformly stable points for F_ε with $u_\varepsilon \to u$. Then u is stable for F.*

Proof. If $u_\varepsilon \to u$ and u_ε is uniformly stable then it is stable for some $\delta > 0$. By the (sub)stability of (F_ε) then u is δ' stable for all $0 < \delta' < \delta$; i.e., it is stable. □

Remark 5.2.5 The main drawback of the notion of stability of energies is that it is not in general compatible with the addition of (continuous) perturbations. Take for example $F_\varepsilon(x) = \sin^2\Big(\dfrac{x}{\varepsilon}\Big)$ and $F = 0$. Then F_ε is stable relative to F, but $G_\varepsilon(x) = F_\varepsilon(x) + x$ is not stable with respect to $G(x) = x$: each x has a δ-slide for all $\delta > 0$, but if $x_\varepsilon \to x$ is a sequence of local minimizers of G_ε then they are δ-stable for $\delta < 1$.

5.3 Stability and Γ-convergence

In this section we will couple stability with Γ-convergence, and try to derive some criteria in order to guarantee the compatibility with respect to the addition of continuous perturbations. The main issue is to construct δ-slides for the approximating functionals starting from δ-slides for the Γ-limit.

Example 5.3.1 We consider the one-dimensional energies

$$F_\varepsilon(u)=\int_0^1 a\Big(\frac{x}{\varepsilon}\Big)|u'|^2\,dx,$$

where a is a 1-periodic function with $0<\inf a<\sup a<+\infty$, so that F_ε Γ-converge to the Dirichlet integral

$$F(u)=\underline{a}\int_0^1 |u'|^2\,dx\,.$$

We will also consider a perturbation of F_ε with

$$G(u)=\int_0^1 g(x,u)\,dx,$$

where g is a Carathéodory function with $|g(x,u)|\le C(1+|u|^2)$ (this guarantees that G is L^2-continuous).

We want to check that $F_\varepsilon+G$ is stable relative to $F+G$. To this end consider a point u_0 such that a δ-slide ϕ for $F+G$ exists at u_0, and points $u_\varepsilon\to u_0$. We wish to construct a δ-slide for $F_\varepsilon+G$ at u_ε.

With fixed $K\in\mathbb{N}$ we consider points $x_i^K=i/K$ for $i=0,\dots K$ and denote for every t with $\phi^K(t)$ the piecewise affine interpolation of $\phi(t)$ on the points x_i^K. Note that we have

- for all K we have $F(\phi^K(t))\le F(\phi(t))$ by Jensen's inequality;
- $F(\phi^K(t))\to F(\phi(t))$ as $K\to+\infty$;
- for fixed K the map $t\mapsto\phi^K(t)$ is continuous with respect to the *strong* H^1-convergence.

Indeed its gradient is piecewise constant and is weakly continuous in t, hence it is strongly continuous.

We fix $\delta'<\delta$ such that

$$F(\phi(t))+G(\phi(t))\le F(\phi(s))+G(\phi(s))+\delta' \text{ if } 0\le s\le t\le 1,$$

choose $\delta''>0$ such that $\delta'+2\delta''<\delta$ and

$$F(\phi(1))+G(\phi(1))<F(u_0)+G(u_0)-2\delta''.$$

Let K be large enough so that (if $u_0^K=\phi^K(0)$ denotes the interpolation of u_0)

$$F(u_0^K)+G(u_0^K)\ge F(u_0)+G(u_0)-\delta''$$

and

$$|G(\phi^K(t)) - G(\phi(t))| < \delta''$$

for all t. We then have

$$F(\phi^K(t)) + G(\phi^K(t)) \leq F(\phi(t)) + G(\phi(t)) + \delta''\,.$$

We then claim that, up to a reparameterization, ϕ^K is a δ-slide for $F+G$ from u_0^K.

Indeed, let $M = \inf\{t : F(\phi^K(t)) + G(\phi^K(t)) < F(u_0^K) + G(u_0^K)\}$. This set is not empty since it contains the point 1. If $0 \leq s \leq t \leq M$ then we have

$$\begin{aligned}
& F(\phi^K(t)) + G(\phi^K(t)) - F(\phi^K(s)) + G(\phi^K(s)) \\
\leq\ & \sup\{F(\phi^K(r)) + G(\phi^K(r)) : 0 \leq r \leq M\} - F(u_0^K) + G(u_0^K) \\
\leq\ & \sup\{F(\phi(r)) + G(\phi(r)) : 0 \leq r \leq M\} + 2\delta'' - F(u_0) + G(u_0) \\
\leq\ & \delta' + 2\delta'' < \delta
\end{aligned}$$

By the continuity of $t \mapsto F(\phi^K(t)) + G(\phi^K(t))$ we can then find $\bar t > M$ such that $F(\phi^K(\bar t)) + G(\phi^K(\bar t)) < F(u_0^K) + G(u_0^K)$ and $s \mapsto \Phi^K(s\bar t)$ is a δ-slide. For the following, we suppose that $\bar t = 1$, so that we do not need any reparameterization.

Next, we construct a δ-slide for $F_\varepsilon + G$. To this end, for the sake of simplicity, we assume that $N = \frac{1}{\varepsilon K} \in \mathbb{N}$. Let v be a function in $H_0^1(0,1)$ such that

$$\int_0^1 a(y)|v'+1|^2\,dy = \min\Big\{\int_0^1 a(y)|w'+1|^2\,dy : w \in H_0^1(0,1)\Big\} = \underline{a}.$$

Note that we also have

$$\int_0^N a(y)|v'+1|^2\,dy = \min\Big\{\int_0^N a(y)|w'+1|^2\,dy : w \in H_0^1(0,1)\Big\} = N\underline{a}.$$

We then define the function $\phi_\varepsilon^K(t)$ by setting on $[x_i^K, x_{i+1}^K]$

$$\phi_\varepsilon^K(t)(x_i^K + s) = \phi(t)(x_i^K) + K(\phi(t)(x_{i+1}^K) - \phi(t)(x_i^K))\Big(s + \varepsilon v(\frac{s}{\varepsilon})\Big), \quad 0 \leq s \leq \frac{1}{K}\,,$$

so that

$$F_\varepsilon(\phi_\varepsilon^K(t)) = F(\phi^K(t)).$$

Note again that we may suppose ε small enough so that $|G(\phi_\varepsilon^K(t)) - G(\phi^K(t))| = o(1)$ uniformly in t so that ϕ_ε^K is a δ-slide for $F_\varepsilon + G$ at $\phi_\varepsilon^K(0)$.

It now remains to construct a L^2-continuous function $\psi_\varepsilon : [0,1] \to H^1(0,1)$ with $\psi_\varepsilon(0) = u_\varepsilon$ and $\psi_\varepsilon(1) = \phi_\varepsilon^K(0)$ such that concatenating ψ_ε with ϕ_ε^K we have a δ-slide. This is achieved by taking the affine interpolation (in t) of u_ε and u_ε^K defined by setting on $[x_i^K, x_{i+1}^K]$

$$u_\varepsilon^K(x_i^K + s) = u_\varepsilon(x_i^K) + K(u_\varepsilon(x_{i+1}^K) - u_\varepsilon(x_i^K))\Big(s + \varepsilon v(\frac{s}{\varepsilon})\Big), \quad 0 \leq s \leq \frac{1}{K}\,,$$

on $(0, 1/2)$ and of u_ε^K and $\phi_\varepsilon^K(0)$ on $(1/2, 1)$.

Example 5.3.2 We consider the oscillating perimeter functionals F_ε and F of Example 3.5.1 We now show that if A has a δ-slide for F and $A_\varepsilon \to A$, then each A_ε has a $(\delta + o(1))$-slide for F_ε (and so a δ-slide for ε sufficiently small). If is easily checked that the same argument can be used if we add to F_ε a continuous perturbation

$$G(A) = \int_A f(x)\,dx,$$

where f is a (smooth) bounded function, so that the stability can be used also for $F_\varepsilon + G$.

We first observe that an arbitrary sequence A_ε of Lipschitz sets converging to a set A can be substituted by a sequence in $\mathcal{A}_\varepsilon$ with the same limit. To check this, consider a connected component of ∂A_ε. Note that for ε small enough every portion of ∂A_ε parameterized by a curve $\gamma : [0,1] \to \mathbb{R}^2$ such that $a(\gamma(0)/\varepsilon) = a(\gamma(1)/\varepsilon) = 1$ and $a(\gamma(t)/\varepsilon) = 2$ for $0 < t < 1$ can be deformed continuously to a curve lying on $\varepsilon a^{-1}(1)$ and with the same endpoints. If otherwise a portion of ∂A_ε lies completely inside a cube Q_i^ε it can be shrunk to a point or expanded to the whole cube Q_i^ε. In both cases this process can be obtained by a $O(\varepsilon)$-slide, since either the lengths of the curves are bounded by 2ε, or the deformation can be performed so that the lengths are decreasing.

We can therefore assume that $A_\varepsilon \in \mathcal{A}_\varepsilon$ and that there exist a δ-slide for E at A obtained by a continuous family $A(t)$ with $0 \le t \le 1$.

We fix $N \in \mathbb{N}$ and set $t_j^N = j/N$. For all $j \in \{1, \dots, N\}$ let $A_\varepsilon^{N,j}$ be a recovery sequence in $\mathcal{A}_\varepsilon$ for $A(t_j^N)$. Furthermore we set $A_n^{N,0} = A_\varepsilon$. Note that, since $A_\varepsilon^{N,j} \to A(t_j^N)$ and $A(t)$ is continuous, we have $|A_\varepsilon^{N,j} \triangle A_\varepsilon^{N,j+1}| = o(1)$ as $N \to +\infty$. We may suppose that the set $A_\varepsilon^{N,j+1}$ is the union of $A_\varepsilon^{N,j}$ and a family of cubes $Q_i^{N,j}$. We may order the indices i and construct a continuous family of sets $A^{N,j,i}(t)$ such that $A^{N,j,i}(0) = A_\varepsilon^{N,j} \cup \bigcup_{k<i} Q_k^{N,j}$, $A^{N,j,i}(1) = A_\varepsilon^{N,j} \cup \bigcup_{k\le i} Q_k^{N,j}$,

$$\Big(\mathcal{H}^1(A_\varepsilon^{N,j}) \wedge \mathcal{H}^1(A_\varepsilon^{N,j+1})\Big) - C\varepsilon \le \mathcal{H}^1(A^{N,j,i}(t)) \le \Big(\mathcal{H}^1(A_\varepsilon^{N,j}) \vee \mathcal{H}^1(A_\varepsilon^{N,j+1})\Big) + C\varepsilon.$$

Since also $|A^{N,j,i}(t)|$ differs from $|A_\varepsilon^{N,j}|$ and $|A_\varepsilon^{N,j+1}|$ by at most $o(1)$ as $N \to +\infty$, by concatenating all these families, upon reparametrization we obtain a family $A_n^N(t)$ such that $A_n^N(0) = A_\varepsilon$, $A_n^N(1) = A_\varepsilon(1)$, and, if $s < t$ then we have, for some $j < k$

$$F_\varepsilon(A_n^N(s)) \ge F(A(t_j^N)) - C\varepsilon - o(1),$$

$$F_\varepsilon(A_n^N(t)) \le F(A(t_k^N)) + C\varepsilon + o(1).$$

Since $A(t)$ is a δ-slide for E we have

$$F(A(t_k^N)) \le F(A(t_j^N)) + \varepsilon,$$

so that

$$F_\varepsilon(A_n^N(t)) \le F_\varepsilon(A_n^N(s)) + \delta + C\varepsilon + o(1).$$

By choosing N large enough and ε small enough we obtain the desired $(\delta + o(1))$-slide.

The previous example suggests a criterion for 'strong' stability (i.e., compatible with continuous perturbations), which is sometimes satisfied by Γ-converging sequences. We have constructed δ-slides for the approximating functionals in two steps: one in which we have transformed a limit δ-slide $\phi(t)$ considering recovery sequences (essentially, setting $\phi_\varepsilon(t) = u^t_\varepsilon$, where (u^t_ε) is a recovery sequence for $\phi(t)$), another where we have constructed an "almost-decreasing" path from u_ε to $\phi_\varepsilon(0)$. Note that this step, conversely, is possible thanks to the liminf inequality.

Theorem 5.3.3 (a criterion of strong stability) *Suppose that F_ε and F satisfy the following requirements:*

if ϕ is a path from u (i.e., $\phi : [0,1] \to X$, $\phi(0) = u$, and ϕ is continuous) and $u_\varepsilon \to u$, then there exist paths ψ_ε from u_ε and ϕ_ε from $\psi_\varepsilon(1)$ such that

(i) *$\tau \mapsto F_\varepsilon(\psi_\varepsilon(\tau))$ is decreasing up to $o(1)$ as $n \to +\infty$; i.e.,*

$$\sup_{0\le\tau_1<\tau_2\le 1}\Big(F_\varepsilon(\psi_\varepsilon(\tau_2)) - F_\varepsilon(\psi_\varepsilon(\tau_1))\Big) \to 0 \text{ as } n \to \infty$$

(ii) $\sup_{\tau\in[0,1]}\operatorname{dist}(\phi_\varepsilon(\tau), \phi(\tau)) = o(1)$

(iii) *there exist $0 = \tau^\varepsilon_1 < \tau^\varepsilon_2 < \dots < \tau^\varepsilon_\varepsilon = 1$ with $\max_i[\tau^\varepsilon_i - \tau^\varepsilon_{i-1}] = o(1)$ such that $\max_i |F_\varepsilon(\phi_\varepsilon(\tau^\varepsilon_i)) - F(\phi(\tau^\varepsilon_i))| = o(1)$ and $F_\varepsilon(\phi_\varepsilon(\tau))$ is between $F_\varepsilon(\phi_\varepsilon(\tau^\varepsilon_i))$ and $F_\varepsilon(\phi_\varepsilon(\tau^\varepsilon_{i+1}))$ for $\tau \in (\tau^\varepsilon_i, \tau^\varepsilon_{i+1})$, up to $o(1)$; i.e., there exist infinitesimal $\beta_n > 0$ such that*

$$\min\Big\{F_\varepsilon(\phi_\varepsilon(\tau^\varepsilon_i)), F_\varepsilon(\phi_\varepsilon(\tau^\varepsilon_{i+1}))\Big\} - \beta_n \le F_\varepsilon(\phi_\varepsilon(\tau)) \le \max\Big\{F_\varepsilon(\phi_\varepsilon(\tau^\varepsilon_i)), F_\varepsilon(\phi_\varepsilon(\tau^\varepsilon_{i+1}))\Big\} + \beta_n$$

Then $(F_\varepsilon + G)$ is stable relative to $(E + G)$ for every continuous G such that $(F_\varepsilon + G)$ is coercive.

Proof. Suppose that u has a δ-slide ϕ for $F + G$ (and therefore a $(\delta - \delta')$-slide for some $\delta' > 0$) and $u_\varepsilon \to u$. Then we choose $\psi_\varepsilon, \phi_\varepsilon$ as in (i)–(iii) above and set $\phi'_\varepsilon(\tau) := \psi_\varepsilon(\tau)$ for $\tau \in [0,1]$, and $\phi'_\varepsilon(\tau) := \phi_\varepsilon(\tau - 1)$ for $\tau > 1$. We then consider $\tau_1 < \tau_2 \in [0,T]$. If $\tau_1, \tau_2 \in [0,1]$, then

$$F_\varepsilon(\phi'_\varepsilon(\tau_2)) - F_\varepsilon(\phi'_\varepsilon(\tau_1)) = F_\varepsilon(\psi_\varepsilon(\tau_2)) - F_\varepsilon(\psi_\varepsilon(\tau_1)) \le o(1).$$

If $\tau_1, \tau_2 > 1$, then

$$F_\varepsilon(\phi'_\varepsilon(\tau_2)) - F_\varepsilon(\phi'_\varepsilon(\tau_1)) = F_\varepsilon(\phi_\varepsilon(\tau_2)) - F_\varepsilon(\phi_\varepsilon(\tau_1)) \le E(\phi(\tau^\varepsilon_j)) - E(\phi(\tau^\varepsilon_i)) + o(1)$$

for some $\tau^\varepsilon_i \le \tau^\varepsilon_j$. If $\tau_1 < 1 < \tau_2$, then

$$F_\varepsilon(\phi'_\varepsilon(\tau_2)) - F_\varepsilon(\phi'_\varepsilon(\tau_1)) = F_\varepsilon(\phi_\varepsilon(\tau_2)) - F_\varepsilon(\psi_\varepsilon(\tau_1)) \le E(\phi(\tau^\varepsilon_i)) - E(\phi(0)) + o(1)$$

for some τ_i^ε, so that in any case

$$\begin{aligned}
&(F_\varepsilon(\phi'_\varepsilon(\tau_2)) + G(\phi'_\varepsilon(\tau_2))) - (F_\varepsilon(\phi'_\varepsilon(\tau_1)) + G(\phi'_\varepsilon(\tau_1))) \\
&\qquad\le (E(\phi(\tau_j)) + G(\phi(\tau_j))) - (E(\phi(\tau_i)) + G(\phi(\tau_i))) + o(1) \qquad (5.1)\\
&\qquad< \delta - \delta' + o(1)
\end{aligned}$$

for some $\tau_i \le \tau_j$, where we used the continuity of G together with (ii) and (iii), as well as the fact that ϕ is a δ-slide for u. The same argument gives

$$(F_\varepsilon + G)(\phi'_\varepsilon(1)) - (F_\varepsilon + G)(\phi'_\varepsilon(0)) \le (E + G)(\phi(1)) - (E + G)(\phi(0)) + o(1),$$

so that ϕ'_ε is a δ-slide for $F_\varepsilon + G$, for ε sufficiently small. □

5.4 Delta-stable evolution

The notion of δ-slide (or some of its modification) can be used to define evolutions in a similar way as in the case of quasi static motion, in cases when the presence of energy barriers may be relevant in the model under consideration. To that end, one can proceed by discrete approximation as in Remark 2.2.6:

• with fixed $\mathcal{F}(t, U)$ and $\mathcal{D}(U)$ energy and dissipation as in Section 2.2, time step $\tau > 0$ and *maximal barrier height* $\delta > 0$, define U_k^τ recursively by setting $U_0^\tau = U_0$, and choosing U_k^τ as a solution of the minimum problem

$$\min_U \Big\{ \mathcal{F}(\tau k, U) + \mathcal{D}(U - U_{k-1}^\tau) \Big\}$$

on the class of U such that there exists a path ϕ from U_{k-1}^τ to U such that

$$\mathcal{F}(\tau k, \phi(t)) + \mathcal{D}(\phi(s) - U_{k-1}^\tau) \le \mathcal{F}(\tau k, \phi(s)) + \mathcal{D}(\phi(s) - U_{k-1}^\tau) + \delta \qquad (5.2)$$

if $0 \le s < t \le 1$.

• define the continuous trajectory $U^{\delta,\tau}(t) = U^\tau_{\lfloor t/\tau \rfloor}$;

• define the *δ-stable evolutions* as the limits U^δ of (subsequences of) $U^{\delta,\tau}$ (which exists under suitable assumptions).

In order to ensure the existence of the minimizer U_k^τ some additional properties of the functionals

$$F_k(U) = \mathcal{F}(\tau k, U) + \mathcal{D}(U - U_{k-1}^\tau)$$

must be assumed; namely, that if U_j is a sequence converging to $\overline{U}$ and $F_k(U_j) < F_k(U_{k-1}^\tau) - C$ for some positive constant C such that there exists paths ϕ_j from U_{k-1}^τ to U_j satisfying (5.2) then there exists a path ϕ satisfying (5.2) from U_{k-1}^τ to $\overline{U}$.

Remark 5.4.1 It must be noted that stable evolution gives a different notion from the global minimization approach even when $\mathcal{D} = 0$, in which case the quasistatic approach just gives a parameterized choice of minimizers of $F(t, \cdot)$.

As a simple example take the one-dimensional energy

$$F(t, x) = \min\{x^2, (x-1)^2\} - 2tx,$$

and $x_0 = 0$. Then the trajectory of parameterized minimizers of $F(t, x)$ from x_0 is

$$u(t) = \begin{cases} 0 & \text{if } t = 0 \\ 1 + t & \text{if } t > 0. \end{cases}$$

On the contrary, the limit $\overline{u}$ as $\delta \to 0$ of the corresponding δ-stable evolutions $u_\delta(t)$ is

$$\overline{u}(t) = \begin{cases} t & \text{if } t < \frac{1}{2} \\ \\ 1 + t & \text{if } t \geq \frac{1}{2}. \end{cases}$$

Example 5.4.2 (the long-bar paradox in Fracture Mechanics) As shown in Remark 2.3.2, for one-dimensional fracture problems with applied boundary displacement; i.e., for the energies and dissipations

$$\mathcal{F}(t, u) = \int_0^1 |u'|^2\, dt, \qquad \mathcal{D}(S(u)) = \#(S(u))$$

defined on piecewise-H^1 functions u with $u(0^-) = 0$ and $u(1^+) = t$ (and $S(u)$ denotes the set of discontinuity points of u), fracture is brutal and appears at a critical value of the displacement t. If instead of a bar of unit length we take a bar of length L and we consider the normalized boundary conditions $u(0-) = 0$ and $u(L^+) = tL$, then the critical value for fracture is $t = \frac{1}{\sqrt{L}}$ for which the energy of the (unfractured) linear solution $u(t) = tx$ equals the energy of a piecewise-constant solution with one discontinuity. In other words a long bar fractures at lower values of the strain (the gradient of the linear solution). In order to overcome this drawback of the theory one may consider δ-stable evolutions, or, rather, a small variation from it necessary due to the fact that the domains of the functionals $\mathcal{F}(t, \cdot)$ are disjoint for different t. In the iterated minimization scheme above we consider minimization among functions u such that there exists a L^2-continuous path ϕ from the elastic solution $u_{k\tau}(x) = k\tau x$ (we again consider only the case $L = 1$) to u such that $\phi(t)(0) = 0$, $\phi(t)(1) = k\tau$ for all t and

$$\mathcal{F}(\tau k, \phi(t)) + \#(S(\phi(t)) \leq \mathcal{F}(\tau k, \phi(s)) + \#(S(\phi(s)) + \delta \tag{5.3}$$

if $0 \leq s < t \leq 1$. This set of u is contained in H^1. Indeed, otherwise there would be a $\bar{t} \in (0,1]$ such that $\#(S(\phi(t_j)) \geq 1$ for a non-increasing sequence of t_j converging to t, and $\phi(t) \in H^1$ for $t \leq \bar{t}$. By the lower semicontinuity of $\mathcal{F}$ and the minimality of $u_{k\tau}$ we have

$$\mathcal{F}(k\tau, u_{k\tau}) \leq \mathcal{F}(k\tau, \phi(\bar{t})) \leq \mathcal{F}(k\tau, \phi(t_j)) + o(1),$$

which gives

$$\mathcal{F}(k\tau, \phi(0)) + 1 \leq \mathcal{F}(k\tau, \phi(t_j)) + \#(S(\phi(t_j)) + o(1)$$

contradicting (5.3) for $s = 0$ and $t = t_j$ if $\delta < 1$.

We conclude that for all k the minimizer is exactly $u_{k\tau}$, and we may pass to the limit obtaining the elastic solution $u(t,x) = tx$. As a conclusion we have that no fracture appears, and this conclusion is clearly independent of the length of the bar.

5.5 References to Chapter 5

The notion of stable points has been introduced in

C. J. Larsen, Epsilon-stable quasi-static brittle fracture evolution, Comm. Pure Appl. Math. 63 (2010), 630–654,

where also stable fracture evolution has been studied; in particular there it is shown that the scheme in Section 5.4 can be applied to Griffith fracture energies.

The notions of stability for sequences of functionals have been analyzed in

A. Braides and C.J. Larsen. Γ-convergence for stable states and local minimizers. Ann. SNS Pisa 10 (2011), 193–206

These notions are further investigated in

M. Focardi. Γ-convergence: a tool to investigate physical phenomena across scales, Math. Mod. Meth. Appl. Sci. **35** (2012), 1613–1658