Chapter 3

Local minimization as a selection criterion

The Γ-limit F of a sequence F_ε is often taken as a simplified description of the energies F_ε, where unimportant details have been averaged out still keeping the relevant information about minimum problems. As far as global minimization problems are concerned this is ensured by the fundamental theorem of Γ-convergence, but this is in general false for local minimization problems. Nevertheless, if some information on the local minima is known, we may use the fidelity of the description of local minimizers as a way to 'correct' Γ-limits. In order to so that, we first introduce some notions of *equivalence by* Γ*-convergence*, and then show how to construct simpler equivalent theories as perturbations of the Γ-limit F in some relent examples.

3.1 Equivalence by Γ-convergence

Definition 3.1.1 *Let* (F_ε) *and* (G_ε) *be sequences of functionals on a separable metric space* X*. We say that they are* equivalent by Γ-convergence *(or* Γ*-equivalent) if there exists a sequence* (m_ε) *of real numbers such that if* $(F_{\varepsilon_j} - m_{\varepsilon_j})$ *and* $(G_{\varepsilon_j} - m_{\varepsilon_j})$ *are* Γ*-converging sequences, their* Γ*-limits coincide and are proper (i.e., not identically* $+\infty$ *and not taking the value* $-\infty$*).*

Remark 3.1.2 (i) since Γ-convergence is sequentially compact (i.e., every sequence has a Γ-converging subsequence), the condition in the definition is never empty. On the set of proper lower-semicontinuous functionals the definition above is indeed an equivalence relation (in particular any sequence (F_ε) is equivalent to itself, regardless to its convergence);

(ii) note that if F_ε Γ-converge to F and G_ε Γ-converge to G then equivalence amounts to $F = G$ and F proper, and (F_ε) is equivalent to the constant sequence F;

(iii) the addition of the constants m_ε allows to consider and discriminate among diverging sequences (whose limit is not proper). For example the sequences of constants $F_\varepsilon = 1/\varepsilon$

and $G_\varepsilon = 1/\varepsilon^2$ are not equivalent, even though they diverge to $+\infty$. Note instead that $F_\varepsilon(x) = x^2/\varepsilon$ and $G_\varepsilon(x) = x^2/\varepsilon^2$ are equivalent.

Definition 3.1.3 (parameterized and uniform equivalence) *For all $\lambda \in \Lambda$ let (F^λ_ε) and (G^λ_ε) be sequences of functionals on a separable metric space X. We say that they are equivalent by Γ-convergence if for all λ they are equivalent according to the definition above. If Λ is a metric space we say that they are* uniformly Γ-equivalent *if there exist (m^λ_ε) such that*

$$\Gamma\text{-}\lim_j (F^{\lambda_j}_{\varepsilon_j} - m^{\lambda_j}_{\varepsilon_j}) = \Gamma\text{-}\lim_j (G^{\lambda_j}_{\varepsilon_j} - m^{\lambda_j}_{\varepsilon_j})$$

and are proper for all $\lambda_j \to \lambda$ and $\varepsilon_j \to 0$.

Remark 3.1.4 Suppose that F^λ_ε Γ-converges to F^λ and (F^λ_ε) and (F^λ) are uniformly Γ-equivalent as above, and that all functionals are equi-coercive and Λ is compact. Then we have

$$\sup_{\lambda\in\Lambda} |\inf F^\lambda_\varepsilon - \min F^\lambda| = o(1)$$

or, equivalently, that $f_\varepsilon(\lambda) = \inf F^\lambda_\varepsilon$ converges uniformly to $f(\lambda) = \min F^\lambda$ on Λ. This follows immediately from the fundamental theorem of Γ-convergence and the compactness of Λ.

Example 3.1.5 Take $\Lambda = [-1, 1]$

$$F^\lambda_\varepsilon(u) = \int_0^1 \Big(\frac{W(u)}{\varepsilon} + \varepsilon|u'|^2\Big)\,dt, \qquad \int_0^1 u\,dt = \lambda$$

with W as in Example 1.5.4. Then we have for fixed λ the Γ-limit

$$F^\lambda(u) = \begin{cases} 0 & \text{if } u(x) = \lambda \\ +\infty & \text{otherwise} \end{cases}$$

if $\lambda = \pm 1$ and

$$F^\lambda(u) = \begin{cases} c_W \#(S(u)) & \text{if } u \in BV((0,1);\{\pm 1\}) \text{ and } \int_0^1 u\,dt = \lambda \\ +\infty & \text{otherwise.} \end{cases}$$

Note that $f_\varepsilon(\lambda) = \inf F^\lambda_\varepsilon$ is a continuous function, while

$$f(\lambda) = \min F^\lambda = \begin{cases} 0 & \text{if } |\lambda| = 1 \\ 1 & \text{otherwise} \end{cases}$$

is not continuous; hence, the convergence $f_\varepsilon \to f$ is not uniform, which implies that (F^λ_ε) and (F^λ) are not uniformly Γ-equivalent.

3.2 A selection criterion

We use the concept of equivalence as above to formalize a problem of the form: given F_ε find "simpler" G_ε equivalent to F_ε, which capture the "relevant" features of F_ε.

We will proceed as follows:

• compute the Γ-limit F of F_ε. This suggests a limit domain and a class of energies (e.g., energies with sharp interfaces in place of diffuse ones; convex homogeneous functionals in place of oscillating integrals, etc.);

• if the description given by F is not "satisfactory," then "perturb" F so as to obtain a family (G_ε) Γ-equivalent to (F_ε).

The same procedure may apply to parameterized families (F^λ_ε).

Of course, the criteria for the construction of G_ε as above may be of different types. In the following example we consider the parameterized family of Example 3.1.5, and the criterion of uniform equivalence.

Example 3.2.1 We consider the functionals F^λ in Example 3.1.5, which have been shown to be not uniformly equivalent to the sequence F^λ_ε. We wish to construct energies of the same form of F^λ; i.e., with domain $u \in BV((0,1);\{\pm 1\})$ with $\int_0^1 u\,dt = \lambda$, and uniformly Γ-equivalent to the sequence F^λ_ε. These energies must then depend on ε. Suppose that $W \in C^2$. If we look for energies of the form

$$G^\lambda_\varepsilon = \begin{cases} c^\lambda_\varepsilon \#(S(u)) & \text{if } u \in BV((0,1);\{\pm 1\}) \text{ and } \int_0^1 u\,dx = \lambda \\ +\infty & \text{otherwise,} \end{cases}$$

then it is possible to show that the choice

$$c^\lambda_\varepsilon = \min\Big\{\frac{W(\lambda)}{\varepsilon}, c_W\Big\}.$$

gives G^λ_ε uniformly Γ-equivalent to F^λ_ε. This choice is not unique, even within energies of the form prescribed; in fact we may also take the Taylor expansions of W at ± 1 in place of W

$$c^\lambda_\varepsilon = \min\Big\{\frac{W''(-1)}{2\varepsilon}(\lambda+1)^2, \frac{W''(1)}{2\varepsilon}(\lambda-1)^2, c_W\Big\},$$

or any other function with the same Taylor expansion. The form of c^λ_ε highlights that minimizers for F^λ_ε can either be close to a sharp interface (in which case their value is c_W), or close to the constant λ (which gives the energy value $W(\lambda)/\varepsilon$). When $\lambda = \pm 1 + O(\sqrt{\varepsilon})$ the second type of minimizers may have lower energy. Nevertheless they are never detected by F^λ.

We may also take G^λ_ε of a slightly more complex form, defined on piecewise-constant functions, setting

$$G^\lambda_\varepsilon = \begin{cases} \displaystyle\int_0^1 \frac{W(u)}{\varepsilon}\,dx + c_W\#(S(u)) & \text{if } u \text{ piecewise constant and } \int_0^1 u\,dx = \lambda \\ +\infty & \text{otherwise.} \end{cases}$$

This choice gives a better description of the minimizers of F^λ_ε.

In the rest of the chapter a "unsatisfactory description" will mean a partial description of local minimizers. We will then try to perturb the Γ-limits so as to satisfy this requirement.

3.3 A 'quantitative' example: phase transitions

We consider the same type of energies as in Examples 1.5.4 and 3.1.5

$$F_\varepsilon(u) = \int_0^1 \Big(\frac{W(u)}{\varepsilon} + \varepsilon|u'|^2\Big)\,dt$$

with W a double-well potential with wells in ± 1. For the sake of simplicity, in the present example the domain of F_ε is restricted to 1-periodic functions (i.e., u such that $u(1) = u(0)$). This constraint is compatible with the Γ-limit, which is then given by

$$F(u) = c_W\#(S(u)\cap[0,1)) \qquad u \in BV((0,1);\{\pm 1\})$$

(again, u is extended to a periodic function, so that it may have a jump at 0, which then is taken into account in the limit energy).

• Note that all functions in $BV((0,1);\{\pm 1\})$ are L^1-local minimizers (even though not isolated). This is a general fact when we have a lower-semicontinuous function taking discrete values.

• We now show that F_ε has no non-trivial L^1-local minimizer. We consider the simplified case

$$W(u) = (|u| - 1)^2\,.$$

In this case $c_W = 2$. Suppose otherwise that u is a local minimizer. If $u \geq 0$ (equivalently, $u \leq 0$) then

$$F_\varepsilon(u) = \int_0^1 \Big(\frac{(u-1)^2}{\varepsilon} + \varepsilon|u'|^2\Big)\,dt.$$

Since this functional is convex, its only local minimizer is the global minimizer $u = 1$. Otherwise, we can suppose, up to a translation, that there exists $L \in (0,1)$ such that $u(\pm L/2) = 0$ and $u(x) > 0$ for $|x| < L/2$. Again, using the convexity of

$$F^L_\varepsilon(u) = \int_{-L/2}^{L/2} \Big(\frac{(u-1)^2}{\varepsilon} + \varepsilon|u'|^2\Big)\,dt$$

we conclude that u must be the global minimizer of F_L with zero boundary data; i.e., the solution of

$$\begin{cases} u'' = \dfrac{1}{\varepsilon^2}(u-1) \\ u(\pm\frac{L}{2}) = 0. \end{cases}$$

This gives

$$u(x) = 1 - \Big(\cosh\Big(\frac{L}{2\varepsilon}\Big)\Big)^{-1}\cosh\Big(\frac{x}{\varepsilon}\Big)$$

and

$$F^L_\varepsilon(u) = 2\frac{\sinh\Big(\frac{L}{\varepsilon}\Big)}{\Big(\cosh\Big(\frac{L}{2\varepsilon}\Big)\Big)^2}\,.$$

Note that

$$\frac{d^2}{dL^2}F^L_\varepsilon(u) = -\frac{2}{\varepsilon^2}\frac{\sinh\Big(\frac{L}{2\varepsilon}\Big)}{\Big(\cosh\Big(\frac{L}{2\varepsilon}\Big)\Big)^3};$$

i.e., this minimum value is a concave function of L. This immediately implies that no local minimizer may exist with changing sign; in fact, such a minimizer would be a local minimizer of the function

$$f(L_1,\dots,L_K) = 2\sum_{k=1}^{K}\frac{\sinh\Big(\frac{L_k}{\varepsilon}\Big)}{\Big(\cosh\Big(\frac{L_k}{2\varepsilon}\Big)\Big)^2}, \tag{3.1}$$

for some $K > 0$ under the constraint $L_k > 0$ and $\sum_k L_k = 1$, which is forbidden by the negative definiteness of its Hessian matrix. Note moreover that

$$F^L_\varepsilon(u) = 2 - 4e^{-\frac{L}{\varepsilon}} + O(e^{-\frac{2L}{\varepsilon}})$$

and that $-4e^{-\frac{L}{\varepsilon}}$ is still a concave function of L.

- We can now propose a 'correction' to F by considering in its place

$$G_\varepsilon(u) = c_W\#(S(u)) - \sum_{x\in S(u)\cap[0,1)} 4e^{-\frac{1}{\varepsilon}|x-\max(S(u)\cap(-\infty,x)|}$$

defined on periodic functions with $u \in BV((0,1);\{\pm1\})$. It is easily seen that G_ε Γ-converges to F, and is hence equivalent to F_ε; thanks to the concavity of the second term the same argument as above shows that we have no non-trivial local minimizers. As a side remark note that this approximation also maintains the stationary points of F_ε, which are functions with K jumps at distance $1/K$. This is easily seen after remarking that the distances between consecutive points must be a stationary point for (3.1). Moreover, the additional terms can also be computed as a development by Γ-convergence, which extends this equivalence to 'higher order'.

3.4 A 'qualitative' example: Lennard-Jones atomistic systems

As in Example 1.5.5, we consider a scaled systems of one-dimensional nearest-neighbour atomistic interactions through a Lennard-Jones type interaction. Let J be a C^2 potential as in Figure 3.1, with domain $(-1,+\infty)$ (we set $J(w)=+\infty$ for $w\le -1$), minimum in 0 with $J''(0)>0$, convex in $(-1,w_0)$, concave in $(w_0,+\infty)$ and tending to $J(\infty)<+\infty$ at $+\infty$. We consider the energy

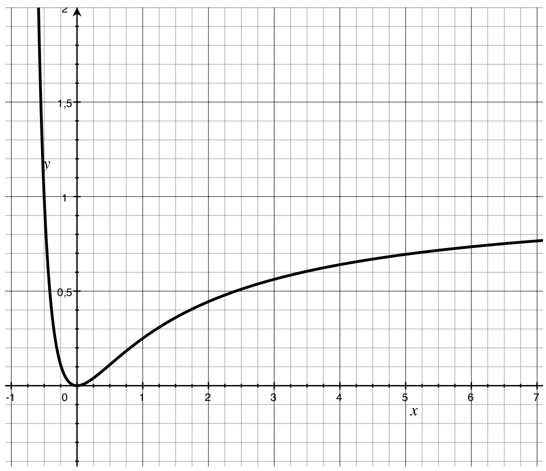

Figure 3.1: a (translation of a) Lennard-Jones potential

$$F^\lambda_\varepsilon(u)=\sum_{i=1}^N J\Big(\frac{u_i-u_{i-1}}{\sqrt{\varepsilon}}\Big)$$

with the boundary conditions $u_0=0$ and $u_N=\lambda\ge 0$. Here $\varepsilon=1/N$ with $N\in\mathbb{N}$. The vector $(u_0,\dots,u_N)$ is identified with a discrete function defined on $\varepsilon\mathbb{Z}\cap[0,1]$ or with its piecewise-affine interpolation. With this last identification, F^λ_ε can be viewed as functionals in $L^1(0,1)$, and their Γ-limit computed with respect to that topology.

Taking into account the boundary conditions, we can extend all functions to $u(x)=0$ for $x\le 0$ and $u(x)=\lambda$ for $x\ge\lambda$, and denote by $S(u)$ (set of discontinuity points of u) the minimal set such that $u\in H^1((-s,1+s)\setminus S(u))$ for $s>0$. With this notation, the same arguments as in Example 1.5.5 give that the Γ-limit is defined on piecewise-$H^1(0,1)$ functions by

$$F^\lambda(u)=\frac{1}{2}J''(0)\int_0^1|u'|^2\,dt+J(\infty)\#(S(u)\cap[0,1])$$

with the constraint that $u^+ > u^-$ on $S(u)$ and the boundary conditions $u^-(0) = 0$, $u^+(1) = \lambda$ (so that $S(u)$ is understood to contain also 0 or 1 if $u^+(0) > 0$ or $u - (1) < \lambda$). For simplicity of notation we suppose

$$\frac{1}{2}J''(0) = J(\infty) = 1.$$

• *Local minimizers of* F^λ. By the strict convexity of $\int_0^1 |u'|^2\,dt$ this part of the energy is minimized, given the average $z = \int_0^1 u'\,dt$, by the piecewise-constant gradient $u' = z$. From now on we tacitly assume that u' is constant. We then have two cases depending on the number of jumps:

(i) if $S(u) = \emptyset$ then $z = \lambda$, and this is a strict local minimizer since any L^1 perturbation with a jump of size w and (average) gradient z has energy $z^2 + 1$ independent of w, which is strictly larger than λ^2 if the perturbation is small;

(ii) if $\#S(u) \geq 1$ then L^1 local minimizers are all functions with $u' = 0$ (since otherwise we can strictly decrease the energy by taking a small perturbation v with the same set of discontinuity points and $v' = su'$ with $s < 1$).

The energy of the local minima in dependence of λ is pictured in Figure 3.2.

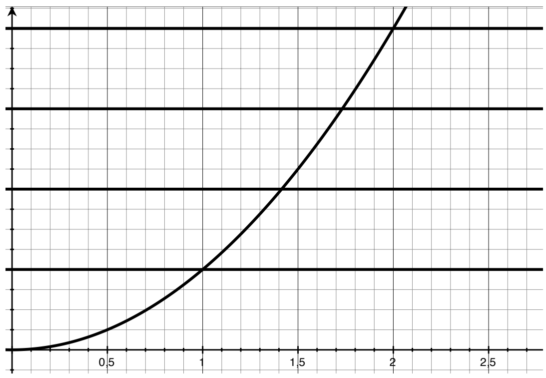

Figure 3.2: local minima for F^λ

• *Local minimizers of* F^λ_ε. This is a finite-dimensional problem, whose stationarity condition is

$$J'\Big(\frac{u_i - u_{i-1}}{\sqrt{\varepsilon}}\Big) = \sigma \qquad \text{for all } i,$$

for some $\sigma > 0$. The shape of J' is pictured in Figure 3.3; its maximum is achieved for $w = w_0$. Note that for all $0 < \sigma < J'(w_0)$ we have two solutions of $J'(w) = \sigma$, while we have no solution for $\sigma > J'(w_0)$.

We have three cases:

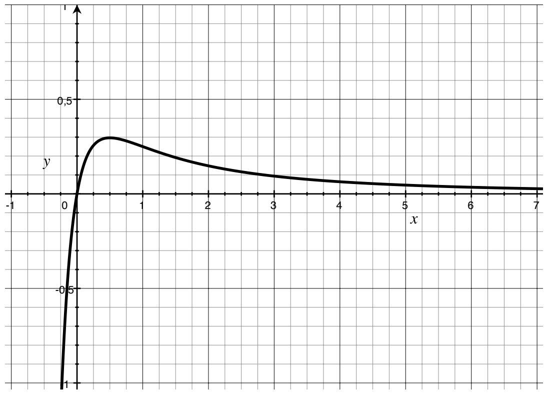

Figure 3.3: derivative of J

(i) we have

$$\frac{u_i - u_{i-1}}{\sqrt{\varepsilon}} \leq w_0 \tag{3.2}$$

for all i. In this case the boundary condition gives $\frac{u_i - u_{i-1}}{\varepsilon} = \lambda$ for all i, so that we have the constraint.

$$\lambda \leq \frac{w_0}{\sqrt{\varepsilon}}. \tag{3.3}$$

This solution is a local minimum. This is easily checked when $\lambda < \frac{w_0}{\sqrt{\varepsilon}}$ since small perturbations maintain the condition (3.2). In the limit case $\lambda = \frac{w_0}{\sqrt{\varepsilon}}$ we may consider only perturbations where (3.2) is violated at exactly one index (see (ii) below), to which there corresponds an energy

$$J(w_0 + t) + (N-1)J\Big(w_0 - \frac{t}{N-1}\Big),$$

for $t \geq 0$, which has a local minimum at 0.

(ii) condition (3.2) is violated by two (or more) indices j and k. Let $\overline{w}$ be such that

$$\frac{u_j - u_{j-1}}{\sqrt{\varepsilon}} = \frac{u_k - u_{k-1}}{\sqrt{\varepsilon}} = \overline{w} > w_0.$$

We may perturb $u_i - u_{i-1}$ only for $i = j, k$, so that the energy varies by

$$f(s) := J(\overline{w} + s) + J(\overline{w} - s) - 2J(\overline{w})\,. \tag{3.4}$$

We have $f'(0) = 0$ and $f''(0) = 2J''(\overline{w}) < 0$, which contradicts the minimality of u.

(iii) condition (3.2) is violated exactly by one index. The value of $w = \frac{u_i - u_{i-1}}{\sqrt{\varepsilon}}$ for the $N-1$ indices satisfying (3.2) is obtained by computing local minimizers of the energy on such functions, which is

$$f_\varepsilon^\lambda(w) := (N-1)J(w) + J\Big(\frac{\lambda}{\sqrt{\varepsilon}} - (N-1)w\Big)$$

defined for $0 \le w \le \min\Big\{w_0, \frac{1}{N-1}\Big(\frac{\lambda}{\sqrt{\varepsilon}} - w_0\Big)\Big\}$. We compute

$$(f_\varepsilon^\lambda)'(w) := (N-1)\Big(J'(w) - J'\Big(\frac{\lambda}{\sqrt{\varepsilon}} - (N-1)w\Big)\Big).$$

Note that

$$f_\varepsilon^\lambda(0) = J\Big(\frac{\lambda}{\sqrt{\varepsilon}}\Big) = 1 - o(1)$$

and $(f_\varepsilon^\lambda)'(0) < 0$. If $\lambda > w_0/\sqrt{\varepsilon}$ then $(f_\varepsilon^\lambda)'(w) = 0$ has a unique solution, which is a local minimizer, while if $\lambda \le w_0/\sqrt{\varepsilon}$ we have two solutions $w_1 < w_2$, of which the first one is a local minimizer. We then have a unique curve of local minimizers with one jump.

The energy of the local minima in dependence of λ is schematically pictured in Fig. 3.4.

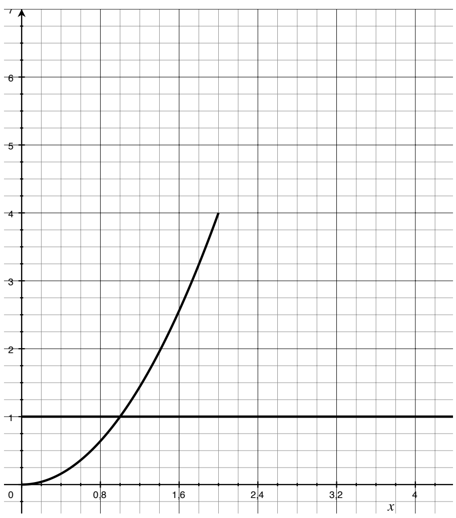

Figure 3.4: local minima for F_ε^λ

• *A qualitative comparison of local minimization.* First, the local minimizer for F_ε^λ which never exceeds the convexity threshold (corresponding to the minimizer with $S(u) = \emptyset$ for F^λ) exists only for $\lambda \le w_0/\sqrt{\varepsilon}$; second, we only have one curve of local minimizers for F_ε^λ

which exceed the convexity threshold for only one index (corresponding to the minimizers with $\#S(u) = 1$ for F^λ).

• Γ-*equivalent energies.* We choose to look for energies defined on piecewise-H^1 functions of the form

$$G_\varepsilon^\lambda(u) = \int_0^1 |u'|^2\,dt + \sum_{t\in S(u)} g\Big(\frac{u^+ - u^-}{\sqrt{\varepsilon}}\Big),$$

again with the constraint that $u^+ > u^-$ on $S(u)$ and the boundary conditions $u^-(0) = 0$, $u^+(1) = \lambda$. In order that local minimizers satisfy $\#(S(u)) \leq 1$ we require that $g : (0, +\infty) \to (0, +\infty)$ be strictly concave. In fact, with this condition the existence of two points in $S(u)$ is ruled out by noticing that given $w_1, w_2 > 0$ the function $t \mapsto g(w_1 + t) + g(w_2 - t)$ is concave. Moreover, we also require that g satisfy

$$\lim_{w\to+\infty} g(w) = 1.$$

With this condition is is easily seen that we have the Γ-convergence of G_ε^λ to F^λ.

In order to make a comparison with the local minimizers of F_ε^λ we first consider local minimizers with $S(u) = \emptyset$; i.e., $u(t) = \lambda t$. Such a function is a local minimizer if it is not energetically favourable to introduce a small jump of size w; i.e., if 0 is a local minimizer for

$$g_\varepsilon^\lambda(w) := (\lambda - w)^2 + g\Big(\frac{w}{\sqrt{\varepsilon}}\Big),$$

where we have extended the definition of g by setting $g(0) = 0$. Note that if g is not continuous in 0 then 0 is a strict local minimizer for g_ε^λ for all λ. Otherwise, we can compute the derivative, and obtain that

$$\frac{d}{dw} g_\varepsilon^\lambda(0) = -2\lambda + \frac{1}{\sqrt{\varepsilon}} g'(0).$$

For ε small enough, 0 is a (isolated) local minimizer if and only if $\frac{d}{dw} g_\varepsilon^\lambda(0) > 0$; i.e.,

$$\lambda < \frac{1}{2\sqrt{\varepsilon}} g'(0)\,.$$

If we choose

$$g'(0) = 2w_0$$

we obtain the desired constraint on this type of local minimizers. A possible simple choice of g is

$$g(w) = \frac{2w_0 w}{1 + 2w_0 w}.$$

We finally consider local minimizers with $\#(S(u)) = 1$. If w denotes the size of the jump then again computing the derivative of the energy, we conclude the existence of a single local minimizer w with

$$2(\lambda - w) = \frac{1}{\sqrt{\varepsilon}} g'\Big(\frac{w}{\sqrt{\varepsilon}}\Big),$$

and energy approaching 1 as $\varepsilon \to 0$.

• With the choice above the pictures of the local minimizers for G^λ_ε and for F^λ_ε are of the same type, but may vary in quantitative details. We have not addressed the problem of the uniformity of this description, for which a refinement of the choice of g could be necessary.

• As a conclusion, we remark that this example has some modeling implications. The functional F^λ can be seen as a one-dimensional version of the energy of a brittle elastic medium according to Griffith's theory of Fracture ($S(u)$ represents the fracture site in the reference configuration), which is then interpreted as a continuum approximation of an atomistic model with Lennard Jones interactions. The requirement that also local minima may be reproduced by the limit theory has made us modify our functional F^λ obtaining another sequence of energies, which maintain an internal parameter ε. Energies of the form G^λ_ε are present in the literature, and are related to Barenblatt's theory of ductile Fracture. Note that in all these considerations the parameter λ appears in the functionals only as a boundary condition, and does not influence the form of the energy.

3.5 A negative example: oscillating perimeters

The procedure described above cannot be always performed in a simple fashion. This may happen if the structure of the Γ-limit F cannot be easily modified to follow the pattern of the local minimizers of F_ε. We include an example where local minimizers if F_ε tend to be a dense set, while functionals with the structure of F have no local minimizers.

Example 3.5.1 We consider the function $a : \mathbb{Z}^2 \to \{1, 2\}$

$$a(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 \in \mathbb{Z} \text{ or } x_2 \in \mathbb{Z} \\ 2 & \text{otherwise,} \end{cases}$$

and the related scaled-perimeter functionals

$$F_\varepsilon(A) = \int_{\partial A} a\Big(\frac{x}{\varepsilon}\Big)\, d\mathcal{H}^1$$

defined on Lipschitz sets A. The energies F_ε Γ-converge, with respect to the convergence $A_\varepsilon \to A$, understood as the L^1 convergence of the corresponding characteristic functions, to an energy of the form

$$F(A) = \int_{\partial^* A} g(\nu) d\mathcal{H}^1 \tag{3.5}$$

defined on all sets of finite perimeter (ν denotes the normal to $\partial^* A$). A direct computation shows that actually

$$g(\nu) = \|\nu\|_1 = |\nu_1| + |\nu_2|.$$

Furthermore, it is easily seen that the same F is equivalently the Γ-limit of

$$\widetilde{F}_\varepsilon(A) = \mathcal{H}^1(\partial A),$$

defined on A which are the union of cubes $Q_i^\varepsilon := \varepsilon(i + (0,1)^2)$ with $i \in \mathbb{Z}^2$. We denote by $\mathcal{A}_\varepsilon$ the family of such A. Note that $\widetilde{F}_\varepsilon$ is the restriction of F_ε to $\mathcal{A}_\varepsilon$.

If $A \in \mathcal{A}_\varepsilon$ then A is trivially a L^1-local minimizer for $\widetilde{F}_\varepsilon$ with $\delta < \varepsilon^2$, since any two distinct elements of $\mathcal{A}_\varepsilon$ are at least at L^1-distance ε^2 (the area of a single ε-square). It can be proved also that all $A \in \mathcal{A}_\varepsilon$ are L^1-local minimizer for F_ε with $\delta = C\varepsilon^2$ for $C > 0$ sufficiently small.

3.6 References to Chapter 3

The notion of equivalence by Γ-convergence is introduced and analyzed in

A. Braides and L. Truskinovsky. Asymptotic expansions by Gamma-convergence. Cont. Mech. Therm. 20 (2008), 21–62

Local minimizers for Lennard-Jones type potentials (also with external forces) are studied in

A. Braides, G. Dal Maso and A. Garroni. Variational formulation of softening phenomena in fracture mechanics: the one-dimensional case. Arch. Rational Mech. Anal. 146 (1999), 23–58.

More details on the derivation of fracture energies from interatomic potentials and the explanation of the $\sqrt{\varepsilon}$-scaling can be found in

A. Braides, A.J. Lew and M. Ortiz. Effective cohesive behavior of layers of interatomic planes. Arch. Ration. Mech. Anal. 180 (2006), 151–182

(see also the quoted paper by Braides and Truskinovsky for an explanation in terms of uniform Γ-equivalence)