Local minimization, variational evolution and Γ-convergence

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Preface

These are the lecture notes of a PhD course given at the University of Pavia, which are themselves an elaboration of a previous course held at Rome "Sapienza" University from March to May 2012, addressed to an audience of students, some of which with an advanced background (meaning that they were already exposed to the main notions of the Calculus of Variations and of Γ-convergence), and researchers in the field. This was an "advanced" course in that it was meant to address some current (or future) research issues rather than to discuss some subject systematically. Part of the notes has been also reworked during a ten-hour course at the University of Narvik on October 25-30, 2012.

Scope of the course has been the asymptotic analysis of energies depending on a small parameter from the standpoint of local minimization and related energy-driven motions. While the study of the limit of global minimizers is by now well understood in terms of Γ-convergence, the description of the behaviour of local minimizers is a more intricate subject. Similarly, a rather complete theory of quasistatic motion "compatible" with Γ-convergence has been developed and analyzed, while in general stability of gradient flows with respect to Γ-convergence can be obtained only under special hypotheses. Indeed, at times the fact that Γ-convergence does not capture the behavior of local minimizers or of gradient flows is mentioned as the proof that Γ-convergence is "wrong". It may well be so. Our standpoint is that it might nevertheless be a good starting point that may be systematically "corrected".

For sequences F_ε for which Γ-convergence methods provide a description of the behaviour of global minimizers, further classical questions regard

• local minimization. Study u_ε such that $F_\varepsilon(u_\varepsilon) = \min\{F_\varepsilon(u) : d(u, u_\varepsilon) \leq \delta\}$ for $\delta > 0$ sufficiently small (here we assume that d is a distance on X);

• stationarity. Study u_ε such that $\nabla F_\varepsilon(u_\varepsilon) = 0$ (we assume in this case that F_ε is differentiable);

• gradient flow. Study $u_\varepsilon = u_\varepsilon(t, x)$ such that $\partial_t u_\varepsilon = -\nabla F_\varepsilon(u_\varepsilon)$.

Easy example show that Γ-convergence is not stable for these notions. However, classical results prove that it may be stable if restricted to classes of energies (e.g., it can be proved that convex energies are stable for the gradient flow), or if stronger hypotheses are added (e.g., isolated local minimizers of the Γ-limit provide local minimizers for F_ε). In the course of the lectures we have addressed some questions as:

• find criteria that ensure the convergence of local minimizers and critical points. In case this does not occur then modify the Γ-limit into an equivalent Γ-expansion in order to match this requirement. We note that in this way we "correct" some limit theories, finding (or "validating") other ones present in the literature;

• modify the concept of local minimizer, so that it may be more 'compatible' with the process of Γ-limit. One such concept is the δ-stability of C. Larsen;

• treat evolution problems for energies with many local minima obtained by a time-discrete scheme (minimizing movements along a sequence of functionals). In this case the minimizing movement of the Γ-limit can be always obtained by a choice of the space and time-scale, but more interesting behaviours can be obtained at a critical ratio between them. Furthermore the issues of long-time behaviour and backwards motion can be addressed by suitably choosing Γ-converging sequences.

Since the scope of the course is to highlight the phenomena and issues linked to local minimization and variational evolution, we have focused our attention on those aspects, rather than on the details of the Γ-convergence process, or the optimal hypotheses for the definition of gradient flows, for which we refer to the existing literature.

Pavia, November 2012.

Introduction

The theory of Γ-convergence was conceived by Ennio De Giorgi at the beginning of the 1970s. It originated from previous notions of convergence related mainly to elliptic operators as G-convergence or H-convergence or to convex functionals as Mosco convergence. The main issue in the definition of Γ-convergence is tracking the behaviour of global minimum problems (minimum values and minimizers) of a sequence F_ε by the computation of an "effective" minimum problem related to the (suitably defined) Γ-limit of this sequence. Even though the definition of such a limit is local (in that in defining its value at a point x we only take into account sequences converging to x), its computation in general does not describe the behaviour of local minimizers of F_ε (i.e., points x_ε which are absolute minimizers of the restriction of F_ε to a small neighbourhood of x_ε itself). The situation, in a very simplified picture, is that in Figure 1, where the original F_ε possess many local minimizers,

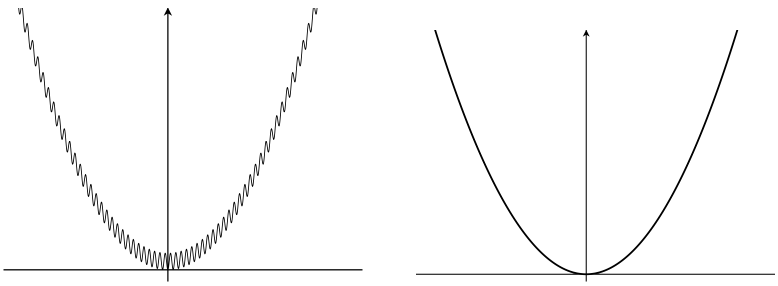

Figure 1: many local minima may disappear in the limit

some (or all) of which are "integrated out" in the Γ-convergence process (note that this happens even when the oscillations depth does not vanish). A notable exception is when we have an isolated local minimizer x of the Γ-limit: in that case we may track the behaviour of local minimizers as absolute minimizers of F_ε restricted to a fixed neighbourhood of x and conclude the existence of local minimizers for F_ε close to x. The possibility of the actual application of such a general principle has been envisaged by Kohn and Sternberg, who first used it to deduce the existence of local minimizers of the Allen-Cahn equation by exhibiting local area minimizing sets.

A recent different but related direction of research concerns the study of gradient flows. A general variational theory based on the solution of Euler schemes has been developed by Ambrosio, Gigli and Savaré. The stability of such schemes by Γ-perturbations is linked to the absence of local minimizers, which could generate "pinned" flows (i.e., stationary solutions or solutions "attracted" by a local minimum) that are not detected by the limit. Conditions that guarantee such a stability are of convexity type on the energies. These conditons can be removed under other special assumptions on the gradient flows and for "well-prepared" initial data following the scheme proposed for Ginzburg-Landau energies by Sandier and Serfaty. Unfortunately, as remarked by those authors, this scheme is often hard to verify.

Taking the above-mentioned results as a starting point we have explored some different directions. The standpoint of the analysis is that even though the Γ-limit may not give the correct description of the effect of local minimizers, it may nevertheless be "corrected" in some systematic way.

Before dealing with local minima, we have first examined a question related to global minimization, concerning the behavior of quasistatic motions, where the functions $u_\varepsilon = u_\varepsilon(t,x)$ minimize at each value of the parameter t (which is understood as a "slow time") a total energy of the form $F_\varepsilon(u,v)+D_\varepsilon(v)$ subject to a t-depending forcing condition. Here, v is an additional parameter satisfying some monotonicity constraint and D_ε is a dissipation. We examine conditions that imply that the separate Γ-convergences of F_ε to F and of D_ε to D guarantee the convergence of u_ε to a quasi static motion related to F and D, as envisaged by Mielke, Roubiček and Stefanelli. In general such conditions do not hold, and a relaxed formulation taking into account the interaction of F_ε and D_ε must be used. In this case, quasi static motion results to be compatible with Γ-convergence provided that is understood as that of the functionals $G_\varepsilon(u,v) = F_\varepsilon(u,v) + D_\varepsilon(v)$ of the variable (u,v).

A first issue beyond global minimization is the problem of distinguishing 'meaningful local minimizers' from those that may 'rightfully' considered to disappear in the limit. Taking Figure 1 as a pictorial example, local minimizers deriving from vanishing oscillations should be considered as different from those 'trapped' by energy barriers. To that end, we study the notion of δ-stable state as recently introduced by C. Larsen, and the related notion of stable sequences of energies. We show that Γ-convergence allows to exhibit classes of stable sequences.

Another issue takes into account the notion of "equivalence by Γ-convergence" as introduced and studied by Braides and Truskinovsky: in the case that a Γ-limit or a Γ-development may be insufficient to capture some desired feature of the minimum problems F_ε we may introduce equivalent energies $\widetilde{F}_\varepsilon$. These energies still integrate out the unimportant details of F_ε but maintain the desired feature and are equivalent to the original F_ε in that they have the same Γ-limit or Γ-development. One of the requirements that may be asked to $\widetilde{F}_\varepsilon$ is that they have the same landscape of local minimizers as F_ε. As an example, we highlight that a Γ-development taking into account interactions between neighbouring transitions recovers the local minimizers of Allen-Cahn energies that are integrated out by the usual sharp models of phase transitions.

Linked to the study of local minimizers is the variational motion defined by the limit of Euler schemes at vanishing time step. This motion has been usually defined for a single functional F (and is sometimes referred to as a *minimizing movement*), by introducing a time step τ and define discrete trajectories (u_k^τ) iteratively as solutions of

$$\min\Big\{F(u)+\frac{1}{2\tau}\|u-u_{k-1}^\tau\|^2\Big\}$$

(for simplicity assume that the energies be defined on a Hilbert space, and $\|u\|$ the related norm). A minimizing motion is a suitably defined continuum limit of such discrete trajectories. We examine a variation of this scheme with two parameters: one is the time step τ, and the second one is the parameter ε (that we may ofter regard as a space scale) appearing in the Γ-converging sequence F_ε. The Euler scheme is then applied at fixed τ with $\varepsilon=\varepsilon(\tau)$ and F_ε in place of F, so that the result may depend on the interaction between the two scales. A general result, directly derived from the properties of Γ-convergence allows to deduce the existence of a 'fast' space scale such that the limit of the ε-τ Euler scheme is just a minimizing movement for the Γ-limit. For 'slow' space scales often the motion is 'pinned' at local minimizers. This observation highlights the existence of one or more critical ε-τ regimes which capture the most interesting features of the motion connected to these energies. Another property of these Euler schemes for Γ-converging energies is the existence of more superposed time scales, whose motions can themselves be interpreted as derived from Euler schemes for scaled functionals. Moreover, an appropriate choice of Γ-approximating sequences to a given F may be used to define a 'backward' motion.

The course is organized around some fundamental examples. We have analyzed a number of prototypical Γ-converging sequences $F_\varepsilon : X\to[-\infty,+\infty]$, highlighting different phenomena.

1. Elliptic homogenization:

$$F_\varepsilon(u)=\int_\Omega a\Big(\frac{x}{\varepsilon}\Big)|\nabla u|^2\,dx$$

with a 1-periodic. In this case X is (a subset of) $H^1(\Omega)$. The inhomogeneity a represents the fine properties of a composite medium. These energies are convex so that they do not possess local (non-global) minimizers. We may nevertheless introduce non-trivial perturbations for which we exhibit an isolated local minimizer for the Γ-limit, deducing existence of nontrivial local minimizers. Their convexity ensures that for the functionals F_ε Γ-convergence commutes with the minimizing movement schemes. As an application we can deduce a parabolic homogenization theorem.

2. Oscillating metrics:

$$F_\varepsilon(u)=\int_0^1 a\Big(\frac{u}{\varepsilon}\Big)|u'|\,dt,$$

with a as above and X a subspace of $W^{1,\infty}([0,1];\mathbb{R}^n)$, and the analog oscillating perimeter energies

$$F_\varepsilon(A) = \int_{\partial A} a\Big(\frac{x}{\varepsilon}\Big) d\mathcal{H}^{d-1}(x).$$

Here we are interested in the overall metric properties, or, in the case of perimeters, of the averaged interfacial energies in a composite medium. These energies have many local minimizers (which tend to be dense); however they are not stable as $\varepsilon \to 0$. As a consequence we have a variety of limit minimizing movements that range from trivial (i.e., constant) motions, to 'crystalline' motion, passing through interesting regimes where the scale ε interacts with the timescale τ. In the case of perimeter energies the limit motion is a kind of motion by curvature with a discontinuous dependence of the velocity on the curvature. Conversely, they may be "incompatible" with simple types of dissipation, and lead to relaxed quasi static evolutions.

3. Van-der-Waals theory of phase transitions:

$$F_\varepsilon(u) = \int_\Omega \Big(\frac{W(u)}{\varepsilon} + \varepsilon|\nabla u|^2\Big)\, dx$$

with $X = H^1(\Omega)$. Here W is a double-well potential with minima in ± 1. In one dimension this is an example where the limit sharp-interface energy has many local minimizers (actually, all functions in the domain of the limit are local minimizers), so that a correction of the Γ-limit is needed, which takes into account the interaction of interfaces, through an exponentially decaying term. It must be noted that this corrected energy also provides an example of a scaled time-scale at which we have motion of interfaces (which otherwise stay pinned).

4. Atomistic theories: for a $1D$ chain of atoms

$$F_\varepsilon(u) = \sum_i J(u_i - u_{i-1}),$$

where J is an interatomic potential (e.g., the Lennard-Jones potential), and u_i represent the position of the i-th atom of a chain of N atoms, ordered with $u_i > u_{i-1}$. Here $\varepsilon = 1/N$. Starting from Lennard-Jones atomic interactions we show that the resulting Γ-limit in one dimension (the Mumford-Shah functional or Griffith brittle fracture energy) must be modified to a Barenblatt cohesive fracture energy to maintain the features of the local minimizers. Even though these energies are not convex, the Euler scheme commutes with Γ-convergence. Another feature of the variational motion is the appearance of a rescaled time scale, in which 'multiple fractures' (that correspond to 'extra' local minimizers introduced by the limit process) tend to interact. A similar behavior is shown for Perona-Malik scaled energies, when $J(z)$ is of the form $\log(1+z^2)$.

We believe that the examples and results highlighted provide a nice overview of the type of issues these notes want to be a stimulus to think about.