Local minimization, variational evolution and Γ-convergence

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March-May 2012

Preface

These are the lecture notes of a PhD course given at the University of Rome "Sapienza" from March to May 2012, addressed to an audience of students, some of which with an advanced background (meaning that they were already exposed to the main notions of the Calculus of Variations and of Γ-convergence), and researchers in the field. This was an "advanced" course in that it was meant to address some current (or future) research issues rather than to discuss some subject systematically.

Scope of the course has been the asymptotic analysis of energies depending on a small parameter from the standpoint of local minimization and energy-driven motion. While the study of the limit of global minimizers is by now well understood in terms of Γ-convergence, the description of the behaviour of local minimizers is a more intricate subject. Indeed, at times the fact that Γ-convergence does not capture their limit is mentioned as the proof that Γ-convergence is "wrong". It may well be so. Our standpoint is that it might nevertheless be a good starting point that may be "corrected".

The families of functionals $F_\varepsilon : X \to [-\infty, +\infty]$ that we have in mind are energies that may derive from various types of problems (physical, geometrical, computational) and the parameter ε may be of geometrical nature, or come from modeling argument, or constitutive assumptions, etc. Typical examples are:

1. Elliptic homogenization: $F_\varepsilon(u) = \displaystyle\int_\Omega a(x/\varepsilon)|\nabla u|^2\,dx$ with a 1-periodic. In this case X is (a subset of) $H^1(\Omega)$. The inhomogeneity a represents the fine properties of a composite medium;

2. Oscillating metrics: $F_\varepsilon(u) = \displaystyle\int_0^1 a(u/\varepsilon)|u'|\,dt$, with a as above and X a subspace of $W^{1,\infty}([0,1];\mathbb{R}^n)$. Here we are interested in the overall metric properties of a composite medium;

3. Van-der-Waals theory of phase transitions: $F_\varepsilon(u) = \displaystyle\int_\Omega \Big(\frac{W(u)}{\varepsilon} + \varepsilon|\nabla u|^2\Big)\,dx$ with $X = H^1(\Omega)$. Here W is a double-well potential with minima in ± 1;

4. Atomistic theories: for a $1D$ chain of atoms $F_\varepsilon(u) = \displaystyle\sum_i J(u_i - u_{i-1})$, where J is an interatomic potential (e.g., the Lennard-Jones potential), and u_i represent the position of

the i-th atom of a chain of N atoms, ordered with $u_i > u_{i-1}$. Here $\varepsilon = 1/N$.

Such energies have been analyzed by Γ-convergence methods, which provides a description of the behaviour of global minimizers. Further classical questions regard

• local minimization. Study u_ε such that $F_\varepsilon(u_\varepsilon) = \min\{F_\varepsilon(u) : d(u, u_\varepsilon) \leq \delta\}$ for $\delta > 0$ sufficiently small (here we assume that d is a distance on X);

• stationarity. Study u_ε such that $\nabla F_\varepsilon(u_\varepsilon) = 0$ (we assume in this case that F_ε is differentiable);

• gradient flow. Study $u_\varepsilon = u_\varepsilon(t, x)$ such that $\partial_t u_\varepsilon = -\nabla F(u_\varepsilon)$.

Easy example show that Γ-convergence is not stable for these notions. However, classical results prove that it may be stable if restricted to classes of energies (e.g., it can be proved that convex energies are stable for the gradient flow), or if stronger hypotheses are added (e.g., isolated local minimizers of the Γ-limit provide local minimizers for F_ε). In the course of the lectures we have addressed some questions as:

• find criteria that ensure the convergence of local minimizers and critical points. In case this does not occur then modify the Γ-limit into an equivalent Γ-expansion in order to match this requirement. We note that in this way we "correct" some limit theories, finding other ones present in the literature;

• modify the concept of local minimizer, so that it may be more 'compatible' with the process of Γ-limit. One such concept is the δ-stability of C. Larsen;

• treat evolution problems for energies with many local minima obtained by a time-discrete scheme (minimizing movements). In this case the minimizing movement of the Γ-limit can be always obtained by a choice of the space and time-scale, but more interesting behaviours can be obtained at a critical ratio between them. Furthermore the issues of long-time behaviour and backwards motion can be addressed by suitably choosing Γ-converging sequences.

Rome, June 1, 2012.

Chapter 1

Global minimization

The case of global minimization is by now well understood, and mainly relies on the concept of Γ-limit. In this chapter we review this notion, which will be the starting point of our analysis. In this section we will only be interested in the problem of global minimization. Further properties of Γ-limits will be recalled when necessary.

1.1 Upper and lower bounds

Here and afterwards F_ε will be functionals defined on a separable metric (or metrizable) space X, if not further specified.

Definition 1.1.1 (lower bound) *We say that F is a* lower bound *for the family (F_ε) if for all $u \in X$ we have*

$$F(u) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \text{ for all } u_\varepsilon \to u, \tag{LB}$$

or, equivalently, $F(u) \leq F_\varepsilon(u_\varepsilon) + o(1)$ for all $u_\varepsilon \to u$.

The inequality (LB) is usually referred to as the *liminf inequality.*

If F is a lower bound we obtain a lower bound also for minimum problems on compact sets.

Proposition 1.1.2 *Let F be a lower bound for F_ε and K be a compact subset of X. Then*

$$\inf_K F \leq \liminf_{\varepsilon\to 0} \inf_K F_\varepsilon. \tag{1.1}$$

Proof. Let $u_{\varepsilon_k} \in K$ be such that $u_{\varepsilon_k} \to \overline{u}$ and

$$\lim_k F_{\varepsilon_k}(u_{\varepsilon_k}) = \liminf_{\varepsilon\to 0} \inf_K F_\varepsilon.$$

We set

$$\widetilde{u}_\varepsilon = \begin{cases} u_{\varepsilon_k} & \text{if } \varepsilon = \varepsilon_k \\ \overline{u} & \text{otherwise.} \end{cases}$$

Then by (LB) we have

$$\inf_K F \le F(\overline{u}) \le \liminf_{\varepsilon\to 0} F_\varepsilon(\widetilde{u}_\varepsilon) \le \lim_k F_{\varepsilon_k}(u_{\varepsilon_k}) = \liminf_{\varepsilon\to 0} \inf_K F_\varepsilon, \tag{1.2}$$

as desired. □

Remark 1.1.3 Note that the hypothesis that K be compact cannot altogether be removed. A trivial example on the real line is:

$$F_\varepsilon(x) = \begin{cases} -1 & \text{if } x = 1/\varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

Then $F = 0$ is a lower bound according to Definition 1.1.1, but (1.1) fails taking $\mathbb{R}$ in place of K.

Remark 1.1.4 The hypothesis that K be compact can be substituted by the hypothesis that K be closed and the sequence (F_ε) be *equi-coercive*; i.e., that

$$\text{if } \sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty \text{ then } (u_\varepsilon) \text{ is precompact,} \tag{1.3}$$

the proof being the same.

Definition 1.1.5 (upper bound) *We say that F is a* upper bound *for the family (F_ε) if for all $u \in X$ we have*

$$\textit{there exists } u_\varepsilon \to u \textit{ such that} \qquad F(u) \ge \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon). \tag{UB}$$

or, equivalently, $F(u) \ge F_\varepsilon(u_\varepsilon) + o(1)$.

The inequality (UB) is usually referred to as the *limsup inequality.*

If F is an upper bound we obtain an upper bound also for minimum problems on open sets.

Proposition 1.1.6 *Let F be an uper bound for F_ε and A be an open subset of X. Then*

$$\inf_A F \ge \limsup_{\varepsilon\to 0} \inf_A F_\varepsilon. \tag{1.4}$$

Proof. The proof is immediately derived from the definition after remarking that if $u \in A$ then we may suppose also that $u_\varepsilon \in A$ so that

$$F(u) \ge \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \ge \limsup_{\varepsilon\to 0} \inf_A F_\varepsilon$$

and (1.4) follows by the arbitrariness of u. □

Remark 1.1.7 Again, note that the hypothesis that A be open cannot be removed. A trivial example on the real line is:

$$F_\varepsilon(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

(independent of ε). Then $F = 0$ is an upper bound according to Definition 1.1.5 (and also a lower bound!), but (1.4) fails taking $A = \{0\}$.

Note that the remark above shows that an upper bound at a point can be actually lower that any element of the family F_ε at that point.

1.2 Γ-convergence

In this section we introduce the concept of Γ-limit.

Definition 1.2.1 (Γ-limit) *We say that F is the* Γ-limit *of the sequence (F_ε) if it is both a lower and an upper bound according to Definitions* 1.1.1 *and* 1.1.5*.*

If (LB) *and* (UB) *hold at a point u then we say that F is the* Γ-limit at u*, and we write*

$$F(u) = \Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u).$$

Note that this notation does does not imply that u is in any of the domains of F_ε, even if $F(u)$ is finite.

Remark 1.2.2 (alternate upper bound inequalities) If F is a lower bound then requiring that (UB) holds is equivalent to any of the following

there exists $u_\varepsilon \to u$ such that $\quad F(u) = \lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon);$ (RS)

for all $\eta > 0$ there exists $u_\varepsilon \to u$ such that $\quad F(u) + \eta \geq \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon).$ (AUB)

The latter is called the *approximate limsup inequality*, and is more handy in computations. A sequence satisfying (RS) is called a *recovery sequence*.

Example 1.2.3 We analyze some simple examples on the real line.

1. As remarked above the constant sequence

$$F_\varepsilon(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

Γ-converges to the constant 0.

2. The sequence

$$F_\varepsilon(x) = \begin{cases} 1 & \text{if } x = \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

again Γ-converges to the constant 0. This is clearly a lower and an upper bound at all $x \neq 0$. At $x = 0$ any sequence $x_\varepsilon \neq \varepsilon$ is a recovery sequence.

3. The sequence

$$F_\varepsilon(x) = \begin{cases} -1 & \text{if } x = \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

Γ-converges to

$$F(x) = \begin{cases} -1 & \text{if } x = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Again, F is clearly a lower and an upper bound at all $x \neq 0$. At $x = 0$ the sequence $x_\varepsilon = \varepsilon$ is a recovery sequence.

4. Take the sum of the energies in Examples 2 and 3 above. This is identically 0, so is its limit, while the sum of the Γ-limits is the function F in Example 3. The same Γ-limit is obtained by taking the function $G_\varepsilon(x) = F_\varepsilon(x) + F_\varepsilon(-x)$ (F_ε in Example 3).

5. Let $F_\varepsilon(x) = \sin(x/\varepsilon)$. Then the Γ-limit is the constant -1. This is clearly a lower bound. A recovery sequence for a fixed x is $x_\varepsilon = 2\pi\varepsilon\lfloor x/(2\pi\varepsilon)\rfloor - \varepsilon\pi/2$ ($\lfloor t\rfloor$ is the integer part of t).

It may be useful to define the lower and upper Γ-limits, so that the existence of a Γ-limit can be viewed as their equality

Definition 1.2.4 (lower and upper Γ-limits) *We define*

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon(u) = \inf\{\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) : u_\varepsilon \to u\} \tag{1.5}$$

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(u) = \inf\{\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) : u_\varepsilon \to u\} \tag{1.6}$$

Remark 1.2.5 1. We immediately obtain that the Γ-limit exists at a point u if and only if

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon(u) = \Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(u).$$

2. Comparing with the trivial sequence $u_\varepsilon = u$ we obtain

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon(u) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(u)$$

(and analogously for the Γ-lim sup). More in general, note that the Γ-limit depends on the topology on X. If we change topology, converging sequences change and the value of the Γ-limit changes. A weaker topology will have more converging sequences and the value will decrease, a stronger topology will have less converging sequences and the value will increase. The pointwise limit above corresponding to the Γ-limit with respect to the discrete topology.

3. From the formulas above it is immediate to check that a constant sequence $F_\varepsilon = F$ Γ-converges to itself if and only if F is lower semicontinuous; i.e., (LB) holds with $F_\varepsilon = F$. Indeed this is equivalent to say that (1.5) holds, while F is always an upper bound.

The following fundamental property of Γ-convergence derives directly from its definition

Proposition 1.2.6 (stability under continuous perturbations) *Let F_ε Γ-converge to F and G_ε converge continuously to G (i.e., $G_\varepsilon(u_\varepsilon) \to G(u)$ if $u_\varepsilon \to u$); then $F_\varepsilon + G_\varepsilon \to F + G$.*

Note that this proposition applies to $G_\varepsilon = G$ if G is continuous, but is in general false for $G_\varepsilon = G$ even if G is lower semicontinuous.

Example 1.2.7 The functions $\sin(x/\varepsilon)+x^2+1$ Γ-converge to x^2. In this case we may apply the proposition above with $F_\varepsilon(x) = \sin(x/\varepsilon)$ (see Example 1.2.3(5)) and $G_\varepsilon(x) = x^2 + 1$. Note for future reference that F_ε has countably many local minimizers, which tend to be dense in the real line, while F has only one global minimizer.

1.3 Convergence of minimum problems

As we have already remarked, the Γ-convergence of F_ε will not imply convergence of minimizers if minimizers (or 'almost minimizers') do not converge. It is necessary then to assume a *compactness* (or 'mild coerciveness') property as follows:

$$\text{there exists a precompact sequence } (u_\varepsilon) \text{ with } F_\varepsilon(u_\varepsilon) = \inf F_\varepsilon + o(1), \tag{1.7}$$

which is implied by the following stronger condition

$$\text{there exists a compact set } K \text{ such that } \inf F_\varepsilon = \inf\nolimits_K F_\varepsilon \text{ for all } \varepsilon > 0. \tag{1.8}$$

This condition is implied by the *equi-coerciveness* hypothesis (1.3); i.e., if for all c there exists a compact set K such that the sublevel sets $\{F_\varepsilon \leq c\}$ are all contained in K.

By arguing as for Propositions 1.1.2 and 1.1.6 we will deduce the convergence of minima. This result is further made precise in the following theorem.

Theorem 1.3.1 (Fundamental Theorem of Γ-convergence) *Let (F_ε) satisfy the compactness property* (1.7) *and Γ-converge to F. Then*

(i) *F admits minimum, and* $\min F = \lim_{\varepsilon\to 0} \inf F_\varepsilon$

(ii) *if (u_{ε_k}) is a* minimizing sequence *for some subsequence (F_{ε_k}) (i.e., is such that $F_{\varepsilon_k}(u_{\varepsilon_k}) = \inf F_\varepsilon + o(1)$) which converges to some $\overline{u}$ then its limit point is a minimizer for F.*

Proof. By condition (1.7) we can argue as in the proof of Proposition 1.1.2 with $K = X$ and also apply Proposition 1.1.6 with $A = X$ to deduce that

$$\inf F \geq \limsup_{\varepsilon\to 0} \inf F_\varepsilon \geq \liminf_{\varepsilon\to 0} \inf F_\varepsilon \geq \inf F\,. \tag{1.9}$$

We then have that there exists the limit

$$\lim_{\varepsilon\to 0} \inf F_\varepsilon = \inf F.$$

Since from (1.7) there exists a minimizing sequence (u_ε) from which we can extract a converging subsequence, it suffices to prove (ii). We can then follow the proof of Proposition 1.1.2 to deduce as in (1.2) that

$$\inf F \le F(\overline{u}) \le \lim_k F_{\varepsilon_k}(u_{\varepsilon_k}) = \lim_{\varepsilon\to 0} \inf F_\varepsilon = \inf F;$$

i.e., $F(\overline{u}) = \inf F$ as desired. □

Corollary 1.3.2 *In the hypotheses of Theorem* 1.3.1 *the minimizers of F are all the limits of converging minimizing sequences.*

Proof. If $\overline{u}$ is a limit of a converging minimizing sequence then it is a minimizer of F by (ii) in Theorem 1.3.1. Conversely, if $\overline{u}$ is a minimizer of F, then every its recovery sequence (u_ε) is a minimizing sequence. □

Remark 1.3.3 Trivially, it is not true that all minimizers of F are limits of minimizers of F_ε, since this is not true even for (locally) uniformly converging sequences on the line. Take for example:

1) $F_\varepsilon(x) = \varepsilon x^2$ or $F_\varepsilon(x) = \varepsilon e^x$ and $F(x) = 0$. All points minimize the limit but only $x = 0$ minimizes F_ε in the first case, and we have no minimizer for the second case. Note also that the functionals in the second case still satisfy the compactness condition (1.7);

2) $F(x) = (x^2 - 1)^2$ and $F_\varepsilon(x) = F(x) + \varepsilon(x-1)^2$. F is minimized by 1 and -1, but the only minimum of F_ε is 1. Note however that -1 is the limit of strong local minimizers for F_ε.

1.4 An example: homogenization

The theory of homogenization of integral functional is a very wide subject in itself. We will refer to monographs on the subject for details if needed. In this context, we want only to highlight some facts and give a hint of the behaviour in the case of elliptic energies.

We consider $a : \mathbb{R}^n \to [\alpha, \beta]$, with $0 < \alpha < \beta < +\infty$ 1-periodic in the coordinate directions, and the integrals

$$F_\varepsilon(u) = \int_\Omega a\Big(\frac{x}{\varepsilon}\Big)|\nabla u|^2\,dx$$

defined in $H^1(\Omega)$, where Ω is a bounded open subset of $\mathbb{R}^n$. The computation of the Γ-limit of F_ε is referred to as their *homogenization*, implying that a simpler 'homogeneous' functional can be used to capture the relevant features of F_ε. The limit can be computed both with respect to the L^1- topology, but it can also be improved; e.g., in 1D it coincides with the limit in the L^∞ topology. This means that the liminf inequality holds for u_ε converging in the L^1 topology (actually, by the Poincaré inequality sequences converging in $L^1(\Omega)$ with bounded Dirichlet integral converge in the L^2-topology), while there exist a recovery sequence with u_ε tending to u in the L^∞ sense.

An upper bound is given by the pointwise limit of F_ε, whose computation in this case can be obtained by the following non-trivial but well-known result.

Proposition 1.4.1 (Riemann-Lebesgue lemma) *The functions $a_\varepsilon(x) = a\Big(\frac{x}{\varepsilon}\Big)$ converge weakly* in L^∞ to their average*

$$\overline{a} = \int_{(0,1)^n} a(y)\,dy \tag{1.10}$$

For fixed u the pointwise limit of $F_\varepsilon(u)$ is then simply $\overline{a}\int_\Omega |\nabla u|^2\,dx$, which then gives an upper bound for the Γ-limit.

In a one-dimensional setting, the Γ-limit is completely described by a, and is given by

$$F_{\rm hom}(u) = \underline{a}\int_\Omega |u'|^2\,dx, \qquad \text{where} \qquad \underline{a} = \Big(\int_0^1 \frac{1}{a(y)}\,dy\Big)^{-1}$$

is the *harmonic mean* of a. Recovery sequences oscillate around the target function. By optimizing such oscillations we obtain the value of $\underline{a}$.

In the higher-dimensional case the limit can still be described by an elliptic integral, of the form

$$F_{\rm hom}(u) = \int_\Omega \langle A\nabla u, \nabla u\rangle\,dx,$$

where A is a constant symmetric matrix with $\underline{a}I \le A \le \overline{a}I$ (I the identity matrix) with strict inequalities unless a is constant.

In order to make minimum problems meaningful, we may consider the affine space $X = \varphi + H^1_0(\Omega)$ (i.e., we consider only functions with $u = \varphi$ on $\partial\Omega$). It can be proved that this boundary condition is 'compatible' with the Γ-limit; i.e., that the Γ-limit is the restriction to X of the previous one. As a consequence of Thorem 1.3.1 we then conclude that oscillating minimum problems for F_ε with fixed boundary data are approximated by a simpler minimum problem with the same boundary data. Note however that all energies, both F_ε and $F_{\rm hom}$, are strictly convex, which implies that they have no local non global minimizer.

Example 1.4.2 We can add some continuously converging perturbation to obtain some more convergence result. For example, we can add perturbations of the form

$$G_\varepsilon(u) = \int_\Omega g\Big(\frac{x}{\varepsilon}, u\Big)\,dx.$$

On g we make the following hypothesis:

g is a Borel function 1-periodic in the first variable and uniformly Lipschitz in the second one; i.e.,

$$|g(y,z) - g(y,z')| \le L|z - z'|.$$

We then have a perturbed homogenization result as follows.

Proposition 1.4.3 *The functionals $F_\varepsilon + G_\varepsilon$ Γ-converge both in the L^1 topology to the functional $F_{\rm hom} + G$, where*

$$G(u) = \int_\Omega \overline{g}(u)\,dx, \qquad \text{and} \qquad \overline{g}(z) = \int_{(0,1)^n} g(y,z)\,dy$$

is simply the average of $g(\cdot, z)$.

Proof. By Proposition 1.2.6 it suffices to show that G_ε converges continuously with respect to the L^1-convergence. If $u_\varepsilon \to u$ in L^1 then

$$\begin{aligned} |G_\varepsilon(u_\varepsilon) - G(u)| &\le \int_\Omega \Big| g\Big(\frac{x}{\varepsilon}, u_\varepsilon\Big) - g\Big(\frac{x}{\varepsilon}, u\Big)\Big|\,dx + |G_\varepsilon(u) - G(u)| \\ &\le L\int_\Omega |u_\varepsilon - u|\,dx + |G_\varepsilon(u) - G(u)|. \end{aligned}$$

It suffices then to show that G_ε converges pointwise to G. If u is piecewise constant then this follows immediately from the Riemann-Lebesgue Lemma. Noting that also $|\overline{g}(z) - \overline{g}(z')| \le L|z - z'|$ we easily obtain the convergence for $u \in L^1(\Omega)$ by the density of piecewise-constant functions. □

Note that with a slightly more technical proof we can improve the Lipschitz continuity condition to a local Lipschitz continuity of the form

$$|g(y,z) - g(y,z')| \le L(1 + |z| + |z'|)|z - z'|.$$

In particular in 1D we can apply the result for $g(y,z) = a(y)|z|^2$ and we have that

$$\int_\Omega a\Big(\frac{t}{\varepsilon}\Big)(|u'|^2 + |u|^2)\,dt$$

Γ-converges to

$$\int_\Omega (\underline{a}|u'|^2 + \overline{a}|u|^2)\,dt.$$

As a consequence of Theorem 1.3.1, under the condition of coerciveness

$$\lim_{z \to \pm\infty} \inf g(\cdot, z) = +\infty,$$

we obtain a convergence result as follows.

Proposition 1.4.4 *The solutions to the minimum problems*

$$\min\Big\{F_\varepsilon(u)+G_\varepsilon(u) : u\in H^1(\Omega)\Big\}$$

converge (up to subsequences) to a constant function $\overline{u}$, whose constant value minimizes $\overline{g}$.

Proof. The proof of the proposition follows immediately from Theorem 1.3.1, once we observe that by the coerciveness and continuity of $\overline{g}$ a minimizer for that function exists, and the constant function $\overline{u}$ defined above minimizes both $F_{\rm hom}$ and G. □

If g is differentiable then by computing the Euler-Lagrange equations of $F_\varepsilon+G_\varepsilon$ we conclude that we obtain solutions of

$$-\sum_{ij}\frac{\partial}{\partial x_i}\Big(a\Big(\frac{x}{\varepsilon}\Big)\frac{\partial u_\varepsilon}{\partial x_i}\Big)+\frac{\partial}{\partial u}g\Big(\frac{x}{\varepsilon},u_\varepsilon\Big)=0 \tag{1.11}$$

with Neumann boundary conditions, converging to the constant $\overline{u}$.

1.5 Higher-order Γ-limits and a choice criterion

If the hypotheses of Theorem 1.3.1 are satisfied then we have noticed that every minimum point of the limit F corresponds to a minimizing sequence for F_ε. However, not all points may be limits of minimizers for F_ε, and it may be interesting to discriminate between limits of minimizing sequences with different speeds of convergence. To this end, we may look at scaled Γ-limits. If we suppose that, say, u is a limit of a sequence (u_ε) with

$$F_\varepsilon(u_\varepsilon)=\min F+O(\varepsilon^\alpha) \tag{1.12}$$

for some $\alpha>0$ (but, of course, the rate of convergence may also no be polynomial) then we may look at the Γ-limit of the scaled functionals

$$F^\alpha_\varepsilon(u)=\frac{F_\varepsilon(u_\varepsilon)-\min F}{\varepsilon^\alpha}. \tag{1.13}$$

Suppose that F^α_ε Γ-converges to some F^α not taking the value $-\infty$. Then:

(i) the domain of F^α is contained in the set of minimizers of F (but may as well be empty);

(ii) $F^\alpha(u)\neq+\infty$ if and only if there exists a recovery sequence for u satisfying (1.12).

Moreover, we can apply Theorem 1.3.1 to F^α_ε and obtain the following result, which gives a choice criterion among minimizers of F.

Theorem 1.5.1 *Let the hypotheses of Theorem* 1.3.1 *be satisfied and the functionals in* (1.13) Γ-*converge to some F^α not taking the value $-\infty$ and not identically $+\infty$. Then*

(i) $\inf F_\varepsilon=\min F+\varepsilon^\alpha\min F^\alpha+o(\varepsilon^\alpha)$*;*

(ii) *if $F_\varepsilon(u_\varepsilon)=\min F_\varepsilon+o(\varepsilon^\alpha)$ and $u_\varepsilon\to u$ then u minimizes both F and F^α.*

Proof. We can apply Theorem 1.3.1 to a (subsequence of a) converging minimizing sequence for F^α_ε; i.e., a sequence satisfying hypothesis (ii). Its limit point u satisfies

$$F^\alpha(u) = \min F^\alpha = \lim_{\varepsilon\to 0} \min F^\alpha_\varepsilon = \lim_{\varepsilon\to 0} \frac{\min F_\varepsilon - \min F}{\varepsilon^\alpha},$$

which proves (i). Since, as already remarked u is also a minimizer of F, we also have (ii). □

Example 1.5.2 Simple examples in the real line:

(1) if $F_\varepsilon(x) = \varepsilon x^2$ then $F(x) = 0$. We have $F^\alpha(x) = 0$ if $0 < \alpha < 1$, $F^1(x) = x^2$ (if $\alpha = 1$), and

$$F^\alpha(x) = \begin{cases} 0 & x = 0 \\ +\infty & x \neq 0 \end{cases}$$

if $\alpha > 1$;

(2) if $F_\varepsilon(x) = (x^2-1)^2 + \varepsilon(x-1)^2$ then $F(x) = (x^2-1)^2$. We have

$$F^\alpha(x) = \begin{cases} 0 & |x| = 1 \\ +\infty & |x| \neq 1 \end{cases}$$

if $0 < \alpha < 1$,

$$F^1(x) = \begin{cases} 0 & x = 1 \\ 4 & x = -1 \\ +\infty & |x| \neq 1 \end{cases}$$

if $\alpha = 1$,

$$F^\alpha(x) = \begin{cases} 0 & x = 1 \\ +\infty & x \neq 1 \end{cases}$$

if $\alpha > 1$.

Remark 1.5.3 It must be observed that the functionals F^α_ε in Theorem 1.5.1 are often equicoercive with respect to a stronger topology than the original F_ε, so that we can improve the convergence in (ii).

Example 1.5.4 (Gradient theory of phase transitions) Let

$$F_\varepsilon(u) = \int_\Omega (W(u) + \varepsilon^2|\nabla u|^2)\,dx \tag{1.14}$$

be defined in $L^1(\Omega)$ with domain in $H^1(\Omega)$. Here $W(u) = (u^2-1)^2$ (or a more general double-well potential). Then (F_ε) is equicoercive with respect to the weak L^1-convergence. Since this convergence is metrizable on bounded sets, we can consider $L^1(\Omega)$ equipped with this convergence. The Γ-limit is then simply

$$F^0(u) = \int_\Omega W^{**}(u)\,dx,$$

where W^{**} is the convex envelope of W; i.e. $W^{**}(u) = ((u^2-1)\vee 0)^2$. All functions with $\|u\|_\infty \le 1$ are minimizers of F^0.

We take $\alpha = 1$ and consider

$$F^1_\varepsilon(u) = \int_\Omega \Big(\frac{W(u)}{\varepsilon} + \varepsilon|\nabla u|^2\Big)\, dx. \tag{1.15}$$

Then (F^1_ε) is equicoercive with respect to the strong L^1-convergence, and its Γ-limit is

$$F^1(u) = c_W \mathcal{H}^{n-1}(\partial\{u=1\}\cap\Omega) \text{ for } u \in BV(\Omega;\{\pm1\}), \tag{1.16}$$

and $+\infty$ otherwise, where $c_W = 8/3$ (in general $c_W = 2\int_{-1}^{1}\sqrt{W(s)}\,ds$). This results states that recovery sequences (u_ε) tend to sit in the bottom of the wells (i.e., $u \in \pm 1$) in order to make $\frac{W(u_\varepsilon)}{\varepsilon}$ finite, and to minimize the interface between the phases $\{u=1\}$ and $\{u=-1\}$. By balancing the effects of the two terms in the integral one obtains the optimal 'surface tension' c_W.

Note that

(i) we have an improved convergence of recovery sequences from weak to strong L^1-convergence;

(ii) the domain of F^1 is almost disjoint from that of the F^1_ε, the only two functions in common being the constants ±1;

(iii) in order to make the Γ-limit properly defined we have to use the space of *functions of bounded variation* or, equivalently, the family of *sets of finite perimeter* if we take as parameter the set $A = \{u=1\}$. In this context the set $\partial\{u=1\}$ is properly defined in a measure-theoretical way, as well as its $(n-1)$-dimensional Hausdorff measure.

Chapter 2

Local minimization

We will now consider some local minimization issues. By a *local minimizer* of F we mean a point u such that there exists $\delta > 0$ such that

$$F(u_0) \leq F(u) \quad \text{if} \quad d(u, u_0) \leq \delta. \tag{2.1}$$

2.1 Convergence to isolated local minimizers

The following theorem shows that we may extend (part of) the fundamental theorem of Γ-convergence to *isolated local minimizers* of the Γ-limit F; i.e., to points u_0 such that there exists $\delta > 0$ such that

$$F(u_0) < F(u) \quad \text{if} \quad 0 < d(u, u_0) \leq \delta. \tag{2.2}$$

The proof of this theorem essentially consists in remarking that we may at the same time apply Proposition 1.1.2 (more precisely, Remark 1.1.4) to the closed ball of center u_0 and radius δ, and Proposition 1.1.6 to the open ball of center u_0 and radius δ.

Theorem 2.1.1 *Suppose that each F_ε is coercive and lower semicontinuous and the sequence (F_ε) Γ-converge to F and is equicoercive. If u_0 is an isolated local minimizer of F then there exist a sequence (u_ε) converging to u_0 with u_ε a local minimizer of F_ε for ε small enough.*

Proof. Let $\delta > 0$ satisfy (2.2). Note that by the coerciveness and lower semicontinuity of F_ε there exists a minimizer u_ε of F_ε on $\overline{B_\delta(u_0)}$, the closure of $B_\delta(u_0) = \{u : d(u, u_0) \leq \delta\}$. By the equicoerciveness of (F_ε), upon extracting a subsequence, we can suppose that $u_\varepsilon \to \overline{u}$. Since $\overline{u} \in \overline{B_\delta(u_0)}$ we then have

$$\begin{aligned} F(u_0) &\leq F(\overline{u}) \leq \liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) = \liminf_{\varepsilon \to 0} \min_{\overline{B_\delta(u_0)}} F_\varepsilon \\ &\leq \limsup_{\varepsilon \to 0} \inf_{B_\delta(u_0)} F_\varepsilon \leq \inf_{B_\delta(u_0)} F = F(u_0), \end{aligned} \tag{2.3}$$

where we have used Proposition 1.1.6 in the last inequality. By (2.2) we have that $\overline{u} = u_0$ and $u_\varepsilon \in B_\delta(u_0)$ for ε small enough, which proves the thesis. □

Remark 2.1.2 Clearly, the existence of an isolated (local) minimizer in the limit does not imply that the converging (local) minimizers are isolated. It suffices to consider $F_\varepsilon(x) = ((x-\varepsilon)\vee 0)^2$ converging to $F(x) = x^2$.

2.2 Two examples

We use Theorem 2.1.1 to prove the existence of sequences of converging local minima.

Example 2.2.1 (local minimizers for elliptic homogenization) Consider the functionals in Example 1.4.2. Suppose furthermore that $\overline{g}$ has an isolated local minimum at z_0. We will show that the constant function $u_0(x) = z_0$ is a L^1-local minimizer of $F_{\rm hom} + G$. Thanks to Theorem 2.1.1 we then deduce that there exists a sequence of local minimizers of $F_\varepsilon + G_\varepsilon$ (in particular, if g is differentiable with respect to u, of solutions of the Euler-Lagrange equation (1.11)) converging to u_0.

We only prove the statement in the one-dimensional case, for which $\Omega = (0, L)$. We now consider $\delta > 0$ and u such that

$$\|u - u_0\|_{L^1(0,L)} \leq \delta.$$

Since z_0 is an isolated local minimum of $\overline{g}$ there exists $h > 0$ such that $g(z_0) < g(z)$ if $0 < |z - z_0| \leq h$. If $\|u - u_0\|_\infty \leq h$ then $G(u) \geq G(u_0)$ with equality only if $u = u_0$ a.e., so that the thesis is verified. Suppose otherwise that there exists a set of positive measure A such that $|u - u_0| > h$ on A. We then have

$$h|A| \leq \int_A |u - u_0|\, dt \leq \delta,$$

so that $|A| \leq \delta/h$. We can then estimate

$$G(u) \geq \min \overline{g}|A| + (L - |A|)\overline{g}(z_0) \geq G(u_0) - \frac{\overline{g}(z_0) - \min \overline{g}}{h}\delta\,.$$

On the other hand, there exists a set of positive measure B such that

$$|u(x) - u_0| \leq \frac{\delta}{L}$$

(otherwise the L^1 estimate doe not hold). Let $x_1 \in B$ and $x_2 \in A$, we can estimate (we can assume $x_1 < x_2$)

$$F_{\rm hom}(u) \geq \alpha \int_{[x_1,x_2]} |u'|^2\, dt \geq \alpha \frac{(u(x_2) - u(x_1))^2}{x_2 - x_1} \geq \alpha \frac{\left(h - \frac{\delta}{L}\right)^2}{L}$$

(using Jensen's inequality). Summing up we have

$$\begin{aligned} F_{\text{hom}}(u) + G(u) &\geq F_{\text{hom}}(u_0) + G(u_0) + \alpha \frac{\left(h - \frac{\delta}{L}\right)^2}{L} - \frac{\overline{g}(z_0) - \min \overline{g}}{h}\delta \\ &= F_{\text{hom}}(u_0) + G(u_0) + \alpha \frac{h^2}{L} + O(\delta) \\ &> F_{\text{hom}}(u_0) + G(u_0) \end{aligned}$$

for δ small as desired.

Example 2.2.2 (Kohn-Sternberg) In order to prove the existence of L^1 local minimizers for the energies F_ε in (1.14) by Theorem 1.4.2 it suffices to prove the existence of isolated local minimizers for the minimal interface problem related to the energy (1.16). In order for this to hold we need some hypothesis on the set Ω (for example, it can be proved that no non-trivial local minimizer exists when Ω is convex).

We treat the two-dimensional case only. We suppose that Ω is bounded, regular, and has an "isolated neck"; i.e., it contains a straight segment whose endpoints meet $\partial\Omega$ perpendicularly, and $\partial\Omega$ is strictly concave at those endpoints (see Fig. 2.1). We will show

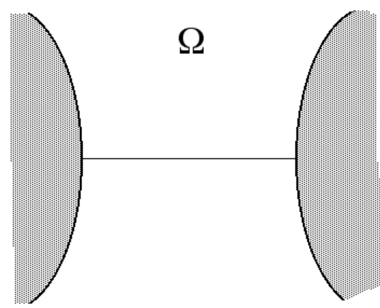

Figure 2.1: a neck in the open set Ω

that the set with boundary that segment is an isolated local minimizer for the perimeter functional.

We can think that the segment is $(0, L) \times \{0\}$. By the strict concavity of $\partial\Omega$ there exist $h > 0$ such that in a rectangular neighbourhood of the form $(a, b) \times (-2h, 2h)$ the lines $x = 0$ and $x = L$ meet $\partial\Omega$ only at $(0, 0)$ and $(0, L)$ respectively. The candidate strict local minimizer is $A_0 = \{(x, y) \in \Omega; x > 0\}$, which we identify with the function $u_0 = -1 + 2\chi_{A_0}$, taking the value $+1$ in A_0 and -1 in $\Omega \setminus A_0$.

Take another test set A. The L^1 closeness condition for functions translates into

$$|A \triangle A_0| \leq \delta.$$

We may suppose that A is sufficiently regular (some minor extra care must be taken when A is a set of finite perimeter, but the proof may be repeated essentially unchanged).

Consider first the case that A contains a horizontal segment $y = M$ with $M \in [h, 2h]$ and its complement contains a horizontal segment $y = m$ with $m \in [-2h, h]$. Then a portion of the boundary ∂A is contained in the part of Ω in the strip $|y| \leq 2h$, and its length is strictly greater than L, unless it is exactly the minimal segment (see Fig. 2.2).

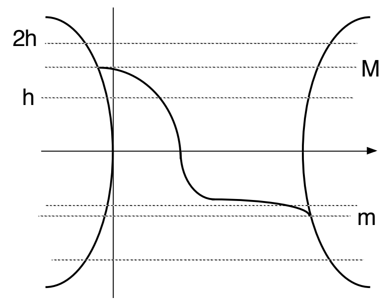

Figure 2.2: comparison with a uniformly close test set

If the condition above is not satisfied then A must not contain, e.g., any horizontal segment $y = t$ with $t \in [h, 2h]$ (see Fig. 2.3). In particular, the length of the portion of ∂A

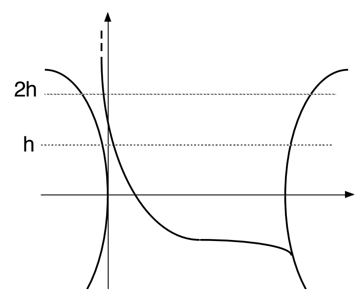

Figure 2.3: comparison with a L^1-close test set

contained with $h \leq y \leq 2h$ is not less than h. Consider now the one-dimensional set

$$B = \{t \in (0, L) : \partial A \cap (\{t\} \times (-h, h)) = \emptyset\}.$$

We have

$$\delta \geq |A \triangle A_0| \geq h|B|,$$

so that $|B| \leq \delta/h$, and the portion of ∂A with $h \leq y \leq 2h$ is not less than $L - \delta/h$. Summing up we have

$$\mathcal{H}^1(\partial A) \geq h + L - \frac{\delta}{h} = \mathcal{H}^1(\partial A_0) + h - \frac{\delta}{h},$$

and the desired strict inequality for δ small enough.

2.3 Generalizations

We can give some generalizations of Theorem 2.1.1 in terms of scaled energies.

Proposition 2.3.1 *Let F_ε satisfy the coerciveness and lower-semicontinuity assumptions of Theorem* 2.1.1. *Suppose furthermore that a bounded positive function $f : (0,+\infty) \to (0,+\infty)$ exists and constants m_ε such that the scaled functionals*

$$\widetilde{F}_\varepsilon(u) = \frac{F_\varepsilon(u) - m_\varepsilon}{f(\varepsilon)} \tag{2.4}$$

are equicoercive and Γ-converge on $\overline{B_\delta(u_0)}$ to $\widetilde{F}_0$ given by

$$\widetilde{F}_0(u) = \begin{cases} 0 & \textit{if } u = u_0 \\ +\infty & \textit{otherwise} \end{cases} \tag{2.5}$$

in $\overline{B_\delta(u_0)}$. Then there exists a sequence (u_ε) converging to u_0 of local minimizers of F_ε.

Remark 2.3.2 (i) First note that the functionals F_ε in Theorem 2.1.1 satisfy the hypotheses of the above proposition, taking, e.g., $f(\varepsilon) = \varepsilon$ and m_ε equal to the minimum of F_ε in $\overline{B_\delta(u_0)}$;

(ii) Note that the hypothesis above is satisfied if there exist constants m_ε such that
(a) $\Gamma\text{-}\limsup\limits_{\varepsilon\to 0}(F_\varepsilon(u_0) - m_\varepsilon) = 0$;
(b) $\Gamma\text{-}\liminf\limits_{\varepsilon\to 0}(F_\varepsilon(u_0) - m_\varepsilon) > 0$.

Indeed condition (a) implies that we may change the constants m_ε so that the Γ-limit exists, is 0 at u_0, and we have a recovery sequence with $F_\varepsilon(u_\varepsilon) = m_\varepsilon$, while (b) is kept unchanged. At this point is suffices to chose, e.g., $f(\varepsilon) = \varepsilon$.

Proof. The proof follows that of Theorem 2.1.1. Again, let u_ε be a a minimizer of F_ε on $\overline{B_\delta(u_0)}$; we can suppose that $u_\varepsilon \to \overline{u} \in \overline{B_\delta(u_0)}$ we then have

$$\begin{aligned} 0 = \widetilde{F}_0(u_0) &\le \widetilde{F}_0(\overline{u}) \le \liminf_{\varepsilon\to 0} \widetilde{F}_\varepsilon(u_\varepsilon) = \liminf_{\varepsilon\to 0} \min_{\overline{B_\delta(u_0)}} \widetilde{F}_\varepsilon \\ &\le \limsup_{\varepsilon\to 0} \inf_{\overline{B_\delta(u_0)}} \widetilde{F}_\varepsilon \le \inf_{B_\delta(u_0)} \widetilde{F}_0 = 0, . \end{aligned} \tag{2.6}$$

so that $\overline{u} = u_0$ and $u_\varepsilon \in B_\delta(u_0)$ for ε small enough, which proves the thesis after remarking that (local) minimization of F_ε and $\widetilde{F}_\varepsilon$ are equivalent up to additive and multiplicative constants. □

Proposition 2.3.3 *Let F_ε satisfy the coerciveness and lower-semicontinuity assumptions of Theorem* 2.1.1. *Suppose furthermore that there exist a bounded positive function f :*

$(0,+\infty)\to(0,+\infty)$, constants m_ε and ρ_ε with $\rho_\varepsilon>0$ and $\rho_\varepsilon\to 0$, and $\widetilde{u}_\varepsilon\to u_0$ such that the scaled functionals

$$\widetilde{F}_\varepsilon(v)=\frac{F_\varepsilon(\widetilde{u}_\varepsilon+\rho_\varepsilon v)-m_\varepsilon}{f(\varepsilon)} \tag{2.7}$$

are equicoercive and Γ-converge on $\overline{B_\delta(v_0)}$ to $\widetilde{F}_0$ with v_0 an isolated local minimum. Then there exists a sequence (u_ε) converging to u_0 of local minimizers of F_ε.

Proof. We can apply Theorem 2.1.1 to the functionals $\widetilde{F}_\varepsilon(v)$ concluding that there exist local minimizers v_ε of $\widetilde{F}_\varepsilon$ converging to v_0. The corresponding $u_\varepsilon=\widetilde{u}_\varepsilon+\rho_\varepsilon v_\varepsilon$ are local minimizers for F_ε converging to u_0. □

Example 2.3.4 We illustrate the proposition with the simple example

$$F_\varepsilon(x)=\sin\Big(\frac{x}{\varepsilon}\Big)+x,$$

whose Γ-limit $F(x)=x-1$ has no local (or global) minimizers. Take any $x_0\in\mathbb{R}$, $x_\varepsilon\to x_0$ any sequence with $\sin(x_\varepsilon/\varepsilon)=-1$, $m_\varepsilon=x_\varepsilon-1$, $\rho_\varepsilon=\varepsilon^\beta$ with $\beta\geq 1$, and $f(\varepsilon)=\varepsilon^\alpha$ with $\alpha\geq 0$, so that

$$\begin{aligned}\widetilde{F}_\varepsilon(t) &= \frac{\sin\Big(\frac{x_\varepsilon+\varepsilon^\beta t}{\varepsilon}\Big)+1}{\varepsilon^\alpha}+\varepsilon^{\beta-\alpha}t\\ &= \frac{\sin\Big(\varepsilon^{\beta-1}t-\frac{\pi}{2}\Big)+1}{\varepsilon^\alpha}+\varepsilon^{\beta-\alpha}t=\frac{1-\cos(\varepsilon^{\beta-1}t)}{\varepsilon^\alpha}+\varepsilon^{\beta-\alpha}t.\end{aligned}$$

In this case the Γ-limit $\widetilde{F}$ coincides with the pointwise limit of $\widetilde{F}_\varepsilon$. If $\beta=1$ and $0\leq\alpha\leq 1$ then we have (local) minimizers of $\widetilde{F}$ at all points of $2\pi\mathbb{Z}$; indeed if $\alpha=0$ then the sequence converges to $\widetilde{F}(x)=1-\cos x$, if $0<\alpha<1$ we have

$$\widetilde{F}(x)=\begin{cases}0 & \text{if } x\in 2\pi\mathbb{Z}\\ +\infty & \text{otherwise,}\end{cases}$$

and if $\alpha=1$

$$\widetilde{F}(x)=\begin{cases}x & \text{if } x\in 2\pi\mathbb{Z}\\ +\infty & \text{otherwise.}\end{cases}$$

In the case $2>\beta>1$ we have two possibilities: if $\alpha=2\beta-2$ then $\widetilde{F}(x)=\frac{1}{2}t^2$; if $\beta\geq\alpha>2\beta-2$ then

$$\widetilde{F}(x)=\begin{cases}0 & \text{if } x=0\\ +\infty & \text{otherwise.}\end{cases}$$

If $\alpha=\beta=2$ then $\widetilde{F}(x)=\frac{1}{2}t^2+t$. In all these cases we have isolated local minimizers in the limit.

Example 2.3.5 (density of local minima for oscillating metrics) We consider an infinite-dimensional example with a behaviour similar to the example above. Let

$$F_\varepsilon(u) = \int_0^1 a\Big(\frac{u}{\varepsilon}\Big)|u'|\,dt$$

defined on

$$X = \{u \in W^{1,1}((0,1);\mathbb{R}^2), u(0) = v_0,\ u(1) = v_1\}$$

equipped with the L^1-convergence. Here, the coefficient a is defined as

$$a(v) = \begin{cases} 1 & \text{if either } v_1 \text{ or } v_2 \in \mathbb{Z} \\ 2 & \text{otherwise.} \end{cases}$$

The Γ-limit is

$$F(u) = \int_0^1 \|u'\|_1 dt,$$

where

$$\|z\|_1 = |z_1| + |z_2|.$$

This is easily checked after remarking that recovery sequences (u_ε) are such that $a(u_\varepsilon(t)/\varepsilon) = 1$ a.e. (except possibly close to 0 and 1 if $a(v_0)/\varepsilon) \neq 1$ or $a(v_1)/\varepsilon) \neq 1$) and then that $|u'_\varepsilon| = |(u_\varepsilon)'_1| + |(u_\varepsilon)'_2|$. For example, if both components of (u_ε) are monotone, then

$$\begin{aligned} F_\varepsilon(u_\varepsilon) &= \int_0^1 a\Big(\frac{u_\varepsilon}{\varepsilon}\Big)|u'|\,dt = \int_0^1 |u'_\varepsilon|\,dt + o(1) \\ &= \int_0^1 (|(u_\varepsilon)'_1| + |(u_\varepsilon)'_2|)\,dt + o(1) \\ &= \Big|(z_1)_1 - (z_0)_1\Big| + \Big|(z_1)_2 - (z_0)_2\Big| + o(1) \\ &= \int_0^1 (|u'_1| + |u'_2|)\,dt + o(1) = F(u) + o(1). \end{aligned}$$

Note moreover, since the integrals are independent with respect to reparameterization, we can consider only target functions in

$$X^1 = \{u \in X : \ \|u'\|_1 \text{ constant a.e.}\}.$$

For simplicity we consider the case of u_0 with both components non-decreasing and suppose that $z_0, z_1 \in \varepsilon\mathbb{Z}^2$. Let $\widetilde{u}_\varepsilon \in X$ be non decreasing, with $a(\widetilde{u}_\varepsilon/\varepsilon) = a(\widetilde{u}_\varepsilon/\varepsilon) = 1$ a.e. and $\widetilde{u}_\varepsilon \to u_0$ in L^∞. We consider the energies

$$\widetilde{F}_\varepsilon(v) = \frac{F_\varepsilon(\widetilde{u}_\varepsilon + \varepsilon^2 v) - F(u_0)}{\varepsilon^2} = \frac{F_\varepsilon(v) - F_\varepsilon(\widetilde{u}_\varepsilon)}{\varepsilon^2}$$

(if $a(z_0/\varepsilon) = a(z_1/\varepsilon) = 1$ is not satisfied then we have to suitably define $\widetilde{u}_\varepsilon$ close to the endpoints, and take the rightmost term as the definition of $\widetilde{F}_\varepsilon$).

Consider $u_\varepsilon = \widetilde{u}_\varepsilon + \varepsilon^2 v \in X$ with $\|v_\varepsilon\|_{L^1} = C_\varepsilon \to C$ and with $\widetilde{F}_\varepsilon(v_\varepsilon) \leq C'$. We then have

$$\left|\left\{a\Big(\frac{u_\varepsilon}{\varepsilon}\Big) \neq 1\right\}\right| \leq C'\varepsilon^2.$$

Note that we can suppose that both components of u_ε be non-decreasing. Upon slightly modifying u_ε we can then suppose that $a\Big(\frac{u_\varepsilon}{\varepsilon}\Big) = 1$ a.e. If, after reparameterization of u_ε with constant speed, we have $u_\varepsilon \neq \widetilde{u}_\varepsilon$ then

$$\|u_\varepsilon - \widetilde{u}_\varepsilon\|_{L^1} \geq C''\varepsilon^2$$

This shows that the Γ-limit of $\widetilde{F}_\varepsilon(v)$ is $+\infty$ if $\|v\|_{L^1} \leq C''$ and $v \neq 0$.

We can apply Proposition 2.3.3 with $\rho_\varepsilon = f(\varepsilon) = \varepsilon^2$, to deduce that for every (increasing) u there exist local minimizers u_ε of F_ε converging to u.

Chapter 3

Stability

The notion of local minimizer is ‘scale-independent’; i.e., it does not depend on the rate at which energies converge, so that it does not discriminate, e.g., between energies

$$F_\varepsilon(x) = x^2 + \sin^2\Big(\frac{x}{\varepsilon}\Big) \qquad \text{or} \qquad F_\varepsilon(x) = x^2 + \sqrt{\varepsilon}\,\sin^2\Big(\frac{x}{\varepsilon}\Big).$$

We now examine a notion of *stability* such that, loosely speaking, a point is *stable* if it is not possible to reach a lower energy state from that point without crossing an energy barrier of a specified height. In this case the local minimizers in the first of the two sequence of energies are stable as $\varepsilon \to 0$, while those in the second sequence are not.

3.1 Stable points

We first introduce a notion of stability that often can be related to notions of local minimality.

Definition 3.1.1 (slide) *Let $F : X \to [0, +\infty]$ and $\delta > 0$. A continuous function $\phi : [0, 1] \to X$ is a* δ-slide *for F at $u_0 \in X$ if*

- *$\phi(0) = u_0$ and $F(\phi(1)) < F(\phi(0)) = F(u_0)$;*
- *there exits $\delta' < \delta$ such that $E(\phi(t)) \leq E(\phi(s)) + \delta'$ if $0 \leq s \leq t \leq 1$.*

Definition 3.1.2 (stability) *Let $F : X \to [0, +\infty]$ and $\delta > 0$. A point $u_0 \in X$ is* δ-stable *for F if no δ-slide exists for F at u_0.*

A point $u_0 \in X$ is stable *for F if it is δ-stable for some $\delta > 0$ (and hence for all δ small enough).*

Let $F_\varepsilon : X \to [0, +\infty]$. A sequence of points (u_ε) in X is uniformly stable *for (F_ε) if there exists $\delta > 0$ such that all u_ε are δ-stable for ε small.*

Example 3.1.3 (1) $F(x) = \begin{cases} 0 & x = 0 \\ \sin\Big(\dfrac{1}{x}\Big) & \text{otherwise.} \end{cases}$ The point 0 is not a local minimizer but it is δ-stable for $\delta < 1$;

(2) Similarly for $F(x) = \begin{cases} 0 & x = 0 \\ -x^2 + \sin^2\Big(\dfrac{1}{x}\Big) & \text{otherwise;} \end{cases}$

(3) Let $X = \mathbb{C}$ and $F(z) = F(\rho e^{i\theta}) = \begin{cases} \theta\rho & |z| \le 1 \\ -1 & \text{otherwise.} \end{cases}$

Then 0 is an isolated local minimum, but it is not stable; e.g., taking $\phi(t) = 2te^{i\delta/2}$. Note in fact that $\phi(0) = 0$, $F(\phi(1)) = -1 < 0$, and $\sup F(\phi(t)) = F(\phi(1/2)) = \delta/2$;

(4) We can generalize example (3) to an infinite-dimensional example. Take $X = L^2(-\pi, \pi)$ and

$$F(u) = \begin{cases} \displaystyle\sum_k \frac{1}{k^2}|c_k|^2 & \text{if } u = \sum_k c_k e^{ikx} \text{ and } \|u\|_{L^2} < 1 \\ -1 & \text{otherwise.} \end{cases}$$

The constant 0 is an isolated minimum point. F is lower semicontinuous, and continuous in $\{\|u\|_{L^2} < 1\}$. Note that $F(e_j) = \frac{1}{j^2}$ so that $\phi_j(t) = 2te_j$ is a δ-slide for $\delta > 1/j^2$;

(5) $F_\varepsilon(x) = x^2 + \sin^2\Big(\dfrac{x}{\varepsilon}\Big)$. Each bounded sequence of local minimizers is uniformly stable;

(6) $F_\varepsilon(x) = x^2 + \varepsilon^\alpha \sin^2\Big(\dfrac{x}{\varepsilon}\Big)$ with $0 < \alpha < 1$. No bounded sequence of local minimizers is uniformly stable (except those converging to 0).

Remark 3.1.4 (local minimality and stability) (i) If $F : X \to \mathbb{R}$ is continuous and u stable; then u is a local minimizer;

(ii) Let F be lower semicontinuous and coercive. Then every isolated local minimizer of F is stable.

(iii) if u is just a local minimizer then u may not be stable.

To check (i) suppose that u is not a local minimum for F. Then let ρ be such that $|F(v) - F(w)| < \delta$ if $u, w \in B_\rho(u)$, and let $u_\rho \in B_\rho(u)$ be such that $F(u_\rho) < F(u)$. Then it suffices to take $\phi(t) = u + t(u_\rho - u)$. Claim (ii) is immediately proved noting that if ρ is such that u is the absolute minimizer of F

To check (ii), let $\eta > 0$ be such that u_0 is an isolated minimum point in $B_\eta(u_0)$. If u_0 is not stable then there exist $1/k$ slides ϕ_k with final point outside $B_\eta(u_0)$. This implies that there exist $u_k = \phi_k(t_k)$ for some t_k with $u_k \in \partial B_\eta(u_0)$, so that $F(u_k) \le F(u_0) + 1/k$. By coerciveness, upon extraction of a subsequence $u_\varepsilon \to \overline{u} \in \partial B_\eta(u_0)$, and by lower semicontinuity $F(\overline{u}) \le \liminf_k F(u_k) \le F(u_0)$, which is a contradiction.

For (iii) take for example $u = 0$ for $F(u) = (1 - |u|) \wedge 0$ on $\mathbb{R}$.

3.2 Stable sequences of functionals

We now give a notion of stability of parameterized functionals.

Definition 3.2.1 (relative (sub)stability) *We say that a sequence (F_ε) is* (sub)stable relative to F *if the following holds*

• *if u_0 has a δ-slide for F and $u_\varepsilon \to u_0$, then each u_ε has a δ-slide for F_ε (for ε small enough).*

Remark 3.2.2 (relative (super)stability) The condition of sub-stability above can be compared to the lower bound for Γ-convergence. With this parallel in mind we can introduce a notion of *(super)stability relative to F* by requiring that

• if u_0 is an isolated local minimum for F then there exists $u_\varepsilon \to u$ such that (u_ε) is uniformly stable for F_ε.

Remark 3.2.3 (i) Note that if F is a constant then all (F_ε) are stable relative to F;

(ii) In general if $F_\varepsilon = F$ for all ε then (F_ε) may not be stable relative to F. Take for example

$$F_\varepsilon(x) = F(x) = \begin{cases} \sin\Big(\dfrac{1}{x}\Big) & \text{if } x > 0 \\ x & \text{if } x \leq 0; \end{cases}$$

then $x_0 = 0$ has δ-slides for all $\delta > 0$, while taking $x_\varepsilon = (2\pi\lfloor\frac{1}{\varepsilon}\rfloor - \frac{\pi}{2})^{-1}$ we have $x_\varepsilon \to x_0$ and x_ε has no δ-slide for $\delta < 2$.

The following proposition is in a sense the converse of Theorem 2.1.1 with Γ-convergence substituted with stability.

Proposition 3.2.4 *Let (F_ε) be (sub)stable relative to F and u_ε be a sequence of uniformly stable points for F_ε with $u_\varepsilon \to u$. Then u is stable for F.*

Proof. If $u_\varepsilon \to u$ and u_ε is uniformly stable then it is stable for some $\delta > 0$. By the (sub)stability of (F_ε) then u is δ' stable for all $0 < \delta' < \delta$; i.e., it is stable. □

Remark 3.2.5 The main drawback of the notion of stability of energies is that it is not in general compatible with the addition of (continuous) perturbations. Take for example $F_\varepsilon(x) = \sin^2\Big(\dfrac{x}{\varepsilon}\Big)$ and $F = 0$. Then F_ε is stable relative to F, but $G_\varepsilon(x) = F_\varepsilon(x) + x$ is not stable with respect to $G(x) = x$: each x has a δ-slide for all $\delta > 0$, but if $x_\varepsilon \to x$ is a sequence of local minimizers of G_ε then they are δ-stable for $\delta < 1$.

3.3 Stability and Γ-convergence

In this section we will couple stability with Γ-convergence, and try to derive some criteria in order to guarantee the compatibility with respect to the addition of continuous perturbations. The main issue is to construct δ-slides for the approximating functionals starting from δ-slides for the Γ-limit.

Example 3.3.1 We consider the one-dimensional energies

$$F_\varepsilon(u) = \int_0^1 a\Big(\frac{x}{\varepsilon}\Big)|u'|^2\,dx,$$

where a is a 1-periodic function with $0 < \inf a < \sup a < +\infty$, so that F_ε Γ-converge to the Dirichlet integral

$$F(u) = \underline{a}\int_0^1 |u'|^2\,dx\,.$$

We will also consider a perturbation of F_ε with

$$G(u) = \int_0^1 g(x,u)\,dx,$$

where g is a Carathéodory function with $|g(x,u)| \le C(1+|u|^2)$ (this guarantees that G is L^2-continuous).

We want to check that $F_\varepsilon + G$ is stable relative to $F+G$. To this end consider a point u_0 such that a δ-slide ϕ for $F+G$ exists at u_0, and points $u_\varepsilon \to u_0$. We wish to construct a δ-slide for $F_\varepsilon + G$ at u_0.

With fixed $K \in \mathbb{N}$ we consider points $x_i^K = i/K$ for $i = 0, \dots K$ and denote for every t with $\phi^K(t)$ the piecewise affine interpolation of $\phi(t)$ on the points x_i^K. Note that we have

- for all K we have $F(\phi^K(t)) \le F(\phi(t))$ by Jensen's inequality;
- $F(\phi^K(t)) \to F(\phi(t))$ as $K \to +\infty$;
- for fixed K the map $t \mapsto \phi^K(t)$ is continuous with respect to the *strong* H^1-convergence.

Indeed its gradient is piecewise constant and is weakly continuous in t, hence it is strongly continuous.

We fix $\delta' < \delta$ such that

$$F(\phi(t)) + G(\phi(t)) \le F(\phi(s)) + G(\phi(s)) + \delta' \text{ if } 0 \le s \le t \le 1,$$

choose $\delta'' > 0$ such that $\delta' + 2\delta'' < \delta$ and

$$F(\phi(1)) + G(\phi(1)) < F(u_0) + G(u_0) - 2\delta''.$$

Let K be large enough so that (if $u_0^K = \phi^K(0)$ denotes the interpolation of u_0)

$$F(u_0^K) + G(u_0^K) \ge F(u_0) + G(u_0) - \delta''$$

and

$$|G(\phi^K(t)) - G(\phi(t))| < \delta''$$

for all t. We then have

$$F(\phi^K(t)) + G(\phi^K(t)) \le F(\phi(t)) + G(\phi(t)) + \delta''\,.$$

We then claim that, up to a reparameterization, ϕ^K is a δ-slide for $F+G$ from u_0^K.

Indeed, let $M = \inf\{t : F(\phi^K(t)) + G(\phi^K(t)) < F(u_0^K) + G(u_0^K)\}$. This set is not empty since it contains the point 1. If $0 \le s \le t \le M$ then we have

$$\begin{aligned}
& F(\phi^K(t)) + G(\phi^K(t)) - F(\phi^K(s)) + G(\phi^K(s)) \\
\le\ & \sup\{F(\phi^K(r)) + G(\phi^K(r)) : 0 \le r \le M\} - F(u_0^K) + G(u_0^K) \\
\le\ & \sup\{F(\phi(r)) + G(\phi(r)) : 0 \le r \le M\} + 2\delta'' - F(u_0) + G(u_0) \\
\le\ & \delta' + 2\delta'' < \delta
\end{aligned}$$

By the continuity of $t \mapsto F(\phi^K(t)) + G(\phi^K(t))$ we can then find $\bar t > M$ such that $F(\phi^K(\bar t)) + G(\phi^K(\bar t)) < F(u_0^K) + G(u_0^K)$ and $s \mapsto \Phi^K(s\bar t)$ is a δ-slide. For the following, we suppose that $\bar t = 1$, so that we do not need any reparameterization.

Next, we construct a δ-slide for $F_\varepsilon + G$. To this end, for the sake of simplicity, we assume that $N = \frac{1}{\varepsilon K} \in \mathbb{N}$. Let v be a function in $H_0^1(0,1)$ such that

$$\int_0^1 a(y)|v'+1|^2\,dy = \min\Big\{\int_0^1 a(y)|w'+1|^2\,dy : w \in H_0^1(0,1)\Big\} = \underline{a}.$$

Note that we also have

$$\int_0^N a(y)|v'+1|^2\,dy = \min\Big\{\int_0^N a(y)|w'+1|^2\,dy : w \in H_0^1(0,1)\Big\} = N\underline{a}.$$

We then define the function $\phi_\varepsilon^K(t)$ by setting on $[x_i^K, x_{i+1}^K]$

$$\phi_\varepsilon^K(t)(x_i^K + s) = \phi(t)(x_i^K) + K(\phi(t)(x_{i+1}^K) - \phi(t)(x_i^K))\Big(s + \varepsilon v(\frac{s}{\varepsilon})\Big), \quad 0 \le s \le \frac{1}{K}\,,$$

so that

$$F_\varepsilon(\phi_\varepsilon^K(t)) = F(\phi^K(t)).$$

Note again that we may suppose ε small enough so that $|G(\phi_\varepsilon^K(t)) - G(\phi^K(t))| = o(1)$ uniformly in t so that ϕ_ε^K is a δ-slide for $F_\varepsilon + G$ at $\phi_\varepsilon^K(0)$.

It now remains to construct a L^2-continuous function $\psi_\varepsilon : [0,1] \to H^1(0,1)$ with $\psi_\varepsilon(0) = u_\varepsilon$ and $\psi_\varepsilon(1) = \phi_\varepsilon^K(0)$ such that concatenating ψ_ε with ϕ_ε^K we have a δ-slide. This is achieved by taking the affine interpolation (in t) of u_ε and u_ε^K defined by setting on $[x_i^K, x_{i+1}^K]$

$$u_\varepsilon^K(x_i^K + s) = u_\varepsilon(x_i^K) + K(u_\varepsilon(x_{i+1}^K) - u_\varepsilon(x_i^K))\Big(s + \varepsilon v(\frac{s}{\varepsilon})\Big), \quad 0 \le s \le \frac{1}{K}\,,$$

on $(0,1/2)$ and of u_ε^K and $\phi_\varepsilon^K(0)$ on $(1/2,1)$.

Example 3.3.2 We consider the function $a : \mathbb{Z}^2 \to \{1,2\}$

$$a(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 \in \mathbb{Z} \text{ or } x_2 \in \mathbb{Z} \\ 2 & \text{otherwise,} \end{cases}$$

and the related scaled-perimeter functionals

$$F_\varepsilon(A) = \int_{\partial A} a\Big(\frac{x}{\varepsilon}\Big)\, d\mathcal{H}^1$$

defined on Lipschitz sets A. The energies F_ε Γ-converge, with respect to the convergence $A_\varepsilon \to A$, understood as the L^1 convergence of the corresponding characteristic functions, to an energy of the form

$$F(A) = \int_{\partial^* A} g(\nu) d\mathcal{H}^1$$

defined on all sets of finite perimeter (ν denotes the normal to $\partial^* A$). A direct computation (following for example the corresponding homogenization problem for curves) shows that actually

$$g(\nu) = \|\nu\|_1 = |\nu_1| + |\nu_2|.$$

Furthermore, it is easily seen that the same F is equivalently the Γ-limit of

$$\widetilde{F}_\varepsilon(A) = \mathcal{H}^1(\partial A),$$

defined on A which are the union of cubes $Q_i^\varepsilon := \varepsilon(i + (0,1)^2)$ with $i \in \mathbb{Z}^2$. We denote by $\mathcal{A}_\varepsilon$ the family of such A. Note that $\widetilde{F}_\varepsilon$ is the restriction of F_ε to $\mathcal{A}_\varepsilon$.

We now show that if A has a δ-slide for F and $A_\varepsilon \to A$, then each A_ε has a $(\delta + o(1))$-slide for F_ε (and so a δ-slide for ε sufficiently small). If is easily checked that the same argument can be used if we add to F_ε a continuous perturbation

$$G(A) = \int_A f(x)\, dx,$$

where f is a (smooth) bounded function, so that the stability can be used also for $F_\varepsilon + G$.

We first observe that an arbitrary sequence A_ε of Lipschitz sets converging to a set A can be substituted by a sequence in $\mathcal{A}_\varepsilon$ with the same limit. To check this, consider a connected component of ∂A_ε. Note that for ε small enough every portion of ∂A_ε parameterized by a curve $\gamma : [0,1] \to \mathbb{R}^2$ such that $a(\gamma(0)/\varepsilon) = a(\gamma(1)/\varepsilon) = 1$ and $a(\gamma(t)/\varepsilon) = 2$ for $0 < t < 1$ can be deformed continuously to a curve lying on $\varepsilon a^{-1}(1)$ and with the same endpoints. If otherwise a portion of ∂A_ε lies completely inside a cube Q_i^ε it can be shrunk to a point or expanded to the whole cube Q_i^ε. In both cases this process can be obtained by a $O(\varepsilon)$-slide, since either the lengths of the curves are bounded by 2ε, or the deformation can be performed so that the lengths are decreasing.

We can therefore assume that $A_\varepsilon \in \mathcal{A}_\varepsilon$ and that there exist a δ-slide for E at A obtained by continuous family $A(t)$ with $0 \le t \le 1$.

We fix $N \in \mathbb{N}$ and set $t_j^N = j/N$. For all $j \in \{1, \dots, N\}$ let $A_\varepsilon^{N,j}$ be a recovery sequence in $\mathcal{A}_\varepsilon$ for $A(t_j^N)$. Furthermore we set $A_n^{N,0} = A_\varepsilon$. Note that, since $A_\varepsilon^{N,j} \to A(t_j^N)$ and $A(t)$ is continuous, we have $|A_\varepsilon^{N,j} \triangle A_\varepsilon^{N,j+1}| = o(1)$ as $N \to +\infty$. We may suppose that the set $A_\varepsilon^{N,j+1}$ is the union of $A_\varepsilon^{N,j}$ and a family of cubes $Q_i^{N,j}$. We may order the indices i and construct a continuous family of sets $A^{N,j,i}(t)$ such that $A^{N,j,i}(0) = A_\varepsilon^{N,j} \cup \bigcup_{k<i} Q_k^{N,j}$, $A^{N,j,i}(1) = A_\varepsilon^{N,j} \cup \bigcup_{k \le i} Q_k^{N,j}$,

$$\Big(\mathcal{H}^1(A_\varepsilon^{N,j}) \wedge \mathcal{H}^1(A_\varepsilon^{N,j+1})\Big) - C\varepsilon \le \mathcal{H}^1(A^{N,j,i}(t)) \le \Big(\mathcal{H}^1(A_\varepsilon^{N,j}) \vee \mathcal{H}^1(A_\varepsilon^{N,j+1})\Big) + C\varepsilon.$$

Since also $|A^{N,j,i}(t)|$ differs from $|A_\varepsilon^{N,j}|$ and $|A_\varepsilon^{N,j+1}|$ by at most $o(1)$ as $N \to +\infty$, by concatenating all these families, upon reparametrization we obtain a family $A_n^N(t)$ such that $A_n^N(0) = A_\varepsilon$, $A_n^N(1) = A_\varepsilon(1)$, and, if $s < t$ then we have, for some $j < k$

$$F_\varepsilon(A_n^N(s)) \ge F(A(t_j^N)) - C\varepsilon - o(1),$$

$$F_\varepsilon(A_n^N(t)) \le F(A(t_k^N)) + C\varepsilon + o(1).$$

Since $A(t)$ is a δ-slide for E we have

$$F(A(t_k^N)) \le F(A(t_j^N)) + \varepsilon,$$

so that

$$F_\varepsilon(A_n^N(t)) \le F_\varepsilon(A_n^N(s)) + \delta + C\varepsilon + o(1).$$

By choosing N large enough and ε small enough we obtain the desired $(\delta + o(1))$-slide.

The previous example suggests a criterion for ‘strong’ stability (i.e., compatible with continuous perturbations), which is sometimes satisfied by Γ-converging sequences. We have constructed δ-slides for the approximating functionals in two steps: one in which we have transformed a limit δ-slide $\phi(t)$ considering recovery sequences (essentially, setting $\phi_\varepsilon(t) = u_\varepsilon^t$, where (u_ε^t) is a recovery sequence for $\phi(t)$), another where we have constructed an “almost-decreasing” path from u_ε to $\phi_\varepsilon(0)$. Note that this step, conversely, is possible thanks to the liminf inequality.

Theorem 3.3.3 (a criterion of strong stability) *Suppose that F_ε and F satisfy the following requirements:*

if ϕ is a path from u (i.e., $\phi : [0,1] \to X$, $\phi(0) = u$, and ϕ is continuous) and $u_\varepsilon \to u$, then there exist paths ψ_ε from u_ε and ϕ_ε from $\psi_\varepsilon(1)$ such that

(i) *$\tau \mapsto F_\varepsilon(\psi_\varepsilon(\tau))$ is decreasing up to $o(1)$ as $n \to +\infty$; i.e.,*

$$\sup_{0 \le \tau_1 < \tau_2 \le 1} \Big(F_\varepsilon(\psi_\varepsilon(\tau_2)) - F_\varepsilon(\psi_\varepsilon(\tau_1))\Big) \to 0 \text{ as } n \to \infty$$

(ii) $\sup_{\tau\in[0,1]}\mathrm{dist}(\phi_\varepsilon(\tau),\phi(\tau)) = o(1)$

(iii) *there exist* $0 = \tau_1^\varepsilon < \tau_2^\varepsilon < \ldots < \tau_\varepsilon^\varepsilon = 1$ *with* $\max_i[\tau_i^\varepsilon - \tau_{i-1}^\varepsilon] = o(1)$ *such that* $\max_i |F_\varepsilon(\phi_\varepsilon(\tau_i^\varepsilon)) - F(\phi(\tau_i^\varepsilon))| = o(1)$ *and* $F_\varepsilon(\phi_\varepsilon(\tau))$ *is between* $F_\varepsilon(\phi_\varepsilon(\tau_i^\varepsilon))$ *and* $F_\varepsilon(\phi_\varepsilon(\tau_{i+1}^\varepsilon))$ *for* $\tau \in (\tau_i^\varepsilon, \tau_{i+1}^\varepsilon)$, *up to* $o(1)$*; i.e., there exist infinitesimal* $\beta_n > 0$ *such that*

$$\min\Big\{F_\varepsilon(\phi_\varepsilon(\tau_i^\varepsilon)), F_\varepsilon(\phi_\varepsilon(\tau_{i+1}^\varepsilon))\Big\} - \beta_n \le F_\varepsilon(\phi_\varepsilon(\tau)) \le \max\Big\{F_\varepsilon(\phi_\varepsilon(\tau_i^\varepsilon)), F_\varepsilon(\phi_\varepsilon(\tau_{i+1}^\varepsilon))\Big\} + \beta_n$$

Then $(F_\varepsilon + G)$ *is stable relative to* $(E + G)$ *for every continuous* G *such that* $(F_\varepsilon + G)$ *is coercive.*

Proof. Suppose that u has a δ-slide ϕ for $F + G$ (and therefore a $(\delta - \delta')$-slide for some $\delta' > 0$) and $u_\varepsilon \to u$. Then we choose $\psi_\varepsilon, \phi_\varepsilon$ as in (i)–(iii) above and set $\phi'_\varepsilon(\tau) := \psi_\varepsilon(\tau)$ for $\tau \in [0,1]$, and $\phi'_\varepsilon(\tau) := \phi_\varepsilon(\tau - 1)$ for $\tau > 1$. We then consider $\tau_1 < \tau_2 \in [0,T]$. If $\tau_1, \tau_2 \in [0,1]$, then

$$F_\varepsilon(\phi'_\varepsilon(\tau_2)) - F_\varepsilon(\phi'_\varepsilon(\tau_1)) = F_\varepsilon(\psi_\varepsilon(\tau_2)) - F_\varepsilon(\psi_\varepsilon(\tau_1)) \le o(1).$$

If $\tau_1, \tau_2 > 1$, then

$$F_\varepsilon(\phi'_\varepsilon(\tau_2)) - F_\varepsilon(\phi'_\varepsilon(\tau_1)) = F_\varepsilon(\phi_\varepsilon(\tau_2)) - F_\varepsilon(\phi_\varepsilon(\tau_1)) \le E(\phi(\tau_j^\varepsilon)) - E(\phi(\tau_i^\varepsilon)) + o(1)$$

for some $\tau_i^\varepsilon \le \tau_j^\varepsilon$. If $\tau_1 < 1 < \tau_2$, then

$$F_\varepsilon(\phi'_\varepsilon(\tau_2)) - F_\varepsilon(\phi'_\varepsilon(\tau_1)) = F_\varepsilon(\phi_\varepsilon(\tau_2)) - F_\varepsilon(\psi_\varepsilon(\tau_1)) \le E(\phi(\tau_i^\varepsilon)) - E(\phi(0)) + o(1)$$

for some τ_i^ε, so that in any case

$$\begin{aligned}
&(F_\varepsilon(\phi'_\varepsilon(\tau_2)) + G(\phi'_\varepsilon(\tau_2))) - (F_\varepsilon(\phi'_\varepsilon(\tau_1)) + G(\phi'_\varepsilon(\tau_1)))\\
&\qquad \le (E(\phi(\tau_j)) + G(\phi(\tau_j))) - (E(\phi(\tau_i)) + G(\phi(\tau_i))) + o(1) \qquad (3.1)\\
&\qquad < \delta - \delta' + o(1)
\end{aligned}$$

for some $\tau_i \le \tau_j$, where we used the continuity of G together with (ii) and (iii), as well as the fact that ϕ is a δ-slide for u. The same argument gives

$$(F_\varepsilon + G)(\phi'_\varepsilon(1)) - (F_\varepsilon + G)(\phi'_\varepsilon(0)) \le (E + G)(\phi(1)) - (E + G)(\phi(0)) + o(1),$$

so that ϕ'_ε is a δ-slide for $F_\varepsilon + G$, for ε sufficiently small. □

Chapter 4

Local minimization as a selection criterion

In this chapter we use the fidelity of the description of local minimizers as a means of 'correcting' Γ-limits.

4.1 Equivalence by Γ-convergence

Definition 4.1.1 *Let (F_ε) and (G_ε) be sequences of functionals on a separable metric space X. We say that they are* equivalent by Γ-convergence *(or Γ-equivalent) if there exists a sequence (m_ε) of real numbers such that if $(F_{\varepsilon_j} - m_{\varepsilon_j})$ and $(G_{\varepsilon_j} - m_{\varepsilon_j})$ are Γ-converging sequences, their Γ-limits coincide and are proper (i.e., not identically $+\infty$ and not taking the value $-\infty$).*

Remark 4.1.2 (i) since Γ-convergence is sequentially compact (i.e,. every sequence has a Γ-converging subsequence), the condition in the definition is never empty. On the set of proper lower-semicontinuous functionals the definition above is indeed an equivalence relation (in particular a sequence (F_ε) is equivalent to itself);

(ii) note that if F_ε Γ-converge to F and G_ε Γ-converge to G then equivalence amounts to $F = G$ and F proper, and (F_ε) is equivalent to the constant sequence F;

(iii) the addition of the constants m_ε allows to consider and discriminate among diverging sequences (whose limit is not proper). For example the sequence of constants $F_\varepsilon = 1/\varepsilon$ and $G_\varepsilon = 1/\varepsilon^2$ are not equivalent, even though they diverge to $+\infty$. Note instead that $F_\varepsilon(x) = x^2/\varepsilon$ and $G_\varepsilon(x) = x^2/\varepsilon^2$ are equivalent.

Definition 4.1.3 (parameterized and uniform equivalence) *For all $\lambda \in \Lambda$ let (F^λ_ε) and (G^λ_ε) be sequences of functionals on a separable metric space X. We say that they are equivalent by Γ-convergence if for all λ they are equivalent according to the definition* *above. If Λ is a metric space we say that they are* uniformly Γ-equivalent *if there exist (m_ε^λ) such that*

$$\Gamma\text{-}\lim_j (F_{\varepsilon_j}^{\lambda_j} - m_{\varepsilon_j}^{\lambda_j}) = \Gamma\text{-}\lim_j (G_{\varepsilon_j}^{\lambda_j} - m_{\varepsilon_j}^{\lambda_j})$$

and are proper for all $\lambda_j \to \lambda$ and $\varepsilon_j \to 0$.

Remark 4.1.4 Suppose that F_ε^λ Γ-converges to F^λ and (F_ε^λ) and (F^λ) are uniformly Γ-equivalent as above, and that all functionals are equi-coercive and Λ is compact. Then we have

$$\sup_\Lambda |\inf F_\varepsilon^\lambda - \min F^\lambda| = o(1)$$

or, equivalently, that $f_\varepsilon(\lambda) = \inf F_\varepsilon^\lambda$ converges uniformly to $f(\lambda) = \min F^\lambda$ on Λ. This follows immediately from the fundamental theorem of Γ-convergence and the compactness of Λ.

Example 4.1.5 Take $\Lambda = [-1, 1]$

$$F_\varepsilon^\lambda(u) = \int_0^1 \Big(\frac{W(u)}{\varepsilon} + \varepsilon|u'|^2\Big)\,dt, \qquad \int_0^1 u\,dt = \lambda$$

with W as in Example 1.5.4. Then we have for fixed λ the Γ-limit

$$F^\lambda(u) = \begin{cases} 0 & \text{if } u(x) = \lambda \\ +\infty & \text{otherwise} \end{cases}$$

if $\lambda = \pm 1$ and

$$F^\lambda(u) = \begin{cases} c_W \#(S(u)) & \text{if } u \in BV((0,1); \{\pm 1\}) \text{ and } \int_0^1 u\,dt = \lambda \\ +\infty & \text{otherwise.} \end{cases}$$

Note that $f_\varepsilon(\lambda) = \inf F_\varepsilon^\lambda$ is a continuous function, while

$$f(\lambda) = \min F^\lambda = \begin{cases} 0 & \text{if } |\lambda| = 1 \\ 1 & \text{otherwise} \end{cases}$$

is not continuous; hence, the convergence $f_\varepsilon \to f$ is not uniform, which implies that (F_ε^λ) and (F^λ) are not uniformly Γ-equivalent.

4.2 A selection criterion

We use the concept of equivalence as above to formalize a problem of the form: given F_ε find "simpler" G_ε equivalent to F_ε such that local minimizers/minima of G_ε are "close" to those of F_ε.

We will proceed as follows:

• compute the Γ-limit F of F_ε. This suggests a limit domain and a class of energies (e.g., energies with sharp interfaces in place of diffuse ones; convex homogeneous functionals in place of oscillating integrals, etc.);

• if the description given by F is not satisfactory, then "perturb" F so as to obtain a family (G_ε) Γ-equivalent to (F_ε).

4.3 A 'quantitative' example: phase transitions

We consider

$$F_\varepsilon(u) = \int_0^1 \Big(\frac{W(u)}{\varepsilon} + \varepsilon|u'|^2\Big)\,dt$$

with W a double-well potential with wells in ± 1 as in Example 1.5.4, and u restricted to 1-periodic functions (i.e., $u(1) = u(0)$). This constraint is compatible with the Γ-limit, which is then given by

$$F(u) = c_W \#(S(u) \cap [0,1)) \qquad u \in BV((0,1); \{\pm 1\})$$

(again, u is extended to a periodic function, so that it may have a jump at 0, which then is taken into account in the limit energy).

• Note that all functions in $BV((0,1);\{\pm 1\})$ are L^1-local minimizers (even though not isolated). This is a general fact when we have a lower-semicontinuous function taking discrete values.

• We now show that F_ε has no non-trivial L^1-local minimizer. We consider the simplified case

$$W(u) = (|u| - 1)^2\,.$$

In this case $c_W = 2$. Suppose otherwise that u is a local minimizer. If $u \geq 0$ (equivalently, $u \leq 0$) then

$$F_\varepsilon(u) = \int_0^1 \Big(\frac{(u-1)^2}{\varepsilon} + \varepsilon|u'|^2\Big)\,dt.$$

Since this functional is convex, its only local minimizer is the global minimizer $u = 1$. Otherwise, we can suppose, up to a translation, that there exists $L \in (0,1)$ such that $u(\pm L/2) = 0$ and $u(x) > 0$ for $|x| < L/2$. Again, using the convexity of

$$F^L_\varepsilon(u) = \int_{-L/2}^{L/2} \Big(\frac{(u-1)^2}{\varepsilon} + \varepsilon|u'|^2\Big)\,dt$$

we conclude that u must be the global minimizer of F_L with zero boundary data; i.e., the solution of

$$\begin{cases} u'' = \dfrac{1}{\varepsilon^2}(u-1) \\[2ex] u(\pm\frac{L}{2}) = 0. \end{cases}$$

This gives

$$u(x) = 1 - \Big(\cosh\Big(\frac{L}{2\varepsilon}\Big)\Big)^{-1}\cosh\Big(\frac{x}{\varepsilon}\Big)$$

and

$$F^L_\varepsilon(u) = 2\frac{\sinh\Big(\frac{L}{\varepsilon}\Big)}{\Big(\cosh\Big(\frac{L}{2\varepsilon}\Big)\Big)^2}\,.$$

Note that

$$\frac{d^2}{dL^2}F^L_\varepsilon(u) = -\frac{2}{\varepsilon^2}\frac{\sinh\Big(\frac{L}{2\varepsilon}\Big)}{\Big(\cosh\Big(\frac{L}{2\varepsilon}\Big)\Big)^3};$$

i.e., this minimum value is a concave function of L. This immediately implies that no local minimizer may exist with changing sign; in fact, such a minimizer would be a local minimizer of the function

$$f(L_1,\dots,L_K) = 2\sum_{k=1}^{K}\frac{\sinh\Big(\frac{L_k}{\varepsilon}\Big)}{\Big(\cosh\Big(\frac{L_k}{2\varepsilon}\Big)\Big)^2}, \tag{4.1}$$

for some $K>0$ under the constraint $L_k>0$ and $\sum_k L_k = 1$, which is forbidden by the negative definiteness of its Hessian matrix. Note moreover that

$$F^L_\varepsilon(u) = 2 - 4e^{-\frac{L}{\varepsilon}} + O(e^{-\frac{2L}{\varepsilon}})$$

and that $-4e^{-\frac{L}{\varepsilon}}$ is still a concave function of L.

• We can now propose a 'correction' to F by considering in its place

$$G_\varepsilon(u) = c_W\#(S(u)) - \sum_{x\in S(u)\cap[0,1)} 4e^{-\frac{1}{\varepsilon}|x-\max(S(u)\cap(-\infty,x)|}$$

defined on periodic functions with $u\in BV((0,1);\{\pm1\})$. It is easily seen that G_ε Γ-converges to F, and is hence equivalent to F_ε; thanks to the concavity of the second term the same argument as above shows that we have no non-trivial local minimizers. As a side remark note that this approximation also maintains the stationary points of F_ε, which are functions with K jumps at distance $1/K$. This is easily seen after remarking that the distances between consecutive points must be a stationary point for (4.1). Moreover, the additional terms can also be computed as a development by Γ-convergence, which extends this equivalence to 'higher order'.

4.4 A ‘qualitative’ example: Lennard-Jones atomistic systems

We consider a scaled systems of one-dimensional nearest-neighbour atomistic interactions through a Lennard-Jones type interaction. Let J be a C^2 potential as in Figure 4.1, with domain $(-1, +\infty)$ (we set $J(w) = +\infty$ for $w \le -1$), minimum in 0 with $J''(0) > 0$, convex in $(-1, w_0)$, concave in $(w_0, +\infty)$ and tending to $J(\infty) < +\infty$ at $+\infty$. We consider the

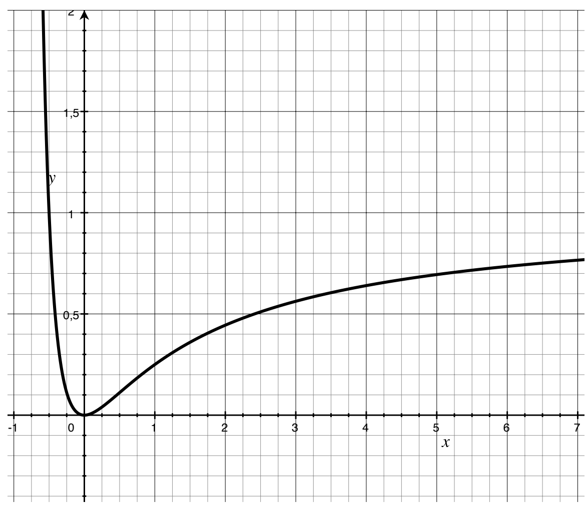

Figure 4.1: a (translation of a) Lennard-Jones potential

energy

$$F_\varepsilon^\lambda(u) = \sum_{i=1}^{N} J\Big(\frac{u_i - u_{i-1}}{\sqrt{\varepsilon}}\Big)$$

with the boundary conditions $u_0 = 0$ and $u_N = \lambda \ge 0$. Here $\varepsilon = 1/N$ with $N \in \mathbb{N}$. The vector $(u_0, \dots, u_N)$ is identified with a discrete function defined on $\varepsilon\mathbb{Z} \cap [0,1]$ or with its piecewise-affine interpolation. With this last identification, F_ε^λ can be viewed as functionals in $L^1(0,1)$, and their Γ-limit computed with respect to that topology.

It must be noted that for all $\overline{w} > 0$ we have

$$\#\Big\{i : \frac{u_i - u_{i-1}}{\sqrt{\varepsilon}} > \overline{w}\Big\} \le \frac{1}{J(\overline{w})} F_\varepsilon^\lambda(u),$$

so that this number of indices is equi-bounded along sequences with equibounded energy. We may therefore suppose that the corresponding points εi converge to a finite set $S \subset [0,1]$. For fixed $\overline{w}$, we have $J(w) \ge \overline{c}|w|^2$ on $(-\infty, \overline{w}]$ for some $\overline{c} > 0$; this gives, if A is compactly contained in $(0,1)\setminus S$, that

$$F^\lambda_\varepsilon(u)\geq \overline{c}\sum_i\Big(\frac{u_i-u_{i-1}}{\sqrt{\varepsilon}}\Big)^2=\overline{c}\sum_i\varepsilon\Big(\frac{u_i-u_{i-1}}{\varepsilon}\Big)^2\geq\overline{c}\int_A|u'|^2\,dt$$

(the sum extended to i such that $\frac{u_i-u_{i-1}}{\sqrt{\varepsilon}}\leq\overline{w}$). By the arbitrariness of A in this estimate we then have that if $u_\varepsilon\to u$ and $F^\lambda_\varepsilon(u_\varepsilon)\leq C<+\infty$ then u is piecewise-H^1; i.e., there exists a finite set $S\subset(0,1)$ such that $u\in H^1((0,1)\setminus S)$. Taking into account the boundary conditions, we can extend all functions to $u(x)=0$ for $x\leq 0$ and $u(x)=\lambda$ for $x\geq\lambda$. and denote by $S(u)$ (set of discontinuity points of u) the minimal set such that $u\in H^1((-s,1+s)\setminus S(u))$ for $s>0$. The reasoning above also shows that

$$\overline{c}\int_0^1|u'|^2\,dt+J(\overline{w})\#(S(u))$$

is a lower bound for the Γ-limit of F^λ_ε.

The Γ-limit on piecewise-$H^1(0,1)$ functions can be computed by optimizing the choice of $\overline{w}$ and $\overline{c}$, and can be shown to be

$$F^\lambda(u)=\frac{1}{2}J''(0)\int_0^1|u'|^2\,dt+J(\infty)\#(S(u)\cap[0,1])$$

with the constraint that $u^+>u^-$ on $S(u)$ and the boundary conditions $u^-(0)=0$, $u^+(1)=\lambda$ (so that $S(u)$ is understood to contain also 0 or 1 if $u^+(0)>0$ or $u-(1)<\lambda$). For simplicity of notation we suppose

$$\frac{1}{2}J''(0)=J(\infty)=1.$$

• *Local minimizers of* F^λ. By the strict convexity of $\int_0^1|u'|^2\,dt$ this part of the energy is minimized, given the average $z=\int_0^1u'\,dt$, by the piecewise-constant gradient $u'=z$ From now on we tacitly assume that u' is constant. We then have two cases depending on the number of jumps:

(i) if $S(u)=\emptyset$ then $z=\lambda$, and this is a strict local minimizer since any L^1 perturbation with a jump of size w and (average) gradient z has energy z^2+1 independent of w, which is strictly larger than λ^2 if the perturbation is small;

(ii) if $\#S(u)\geq 1$ then L^1 local minimizers are all functions with $u'=0$ (since otherwise we can strictly decrease the energy by taking a small perturbation v with the same set of discontinuity points and $v'=su'$ with $s<1$).

The energy of the local minima in dependence of λ is pictured in Figure 4.2.

• *Local minimizers of* F^λ_ε. This is a finite-dimensional problem, whose stationarity condition is

$$J'\Big(\frac{u_i-u_{i-1}}{\sqrt{\varepsilon}}\Big)=\sigma\qquad\text{for all } i,$$

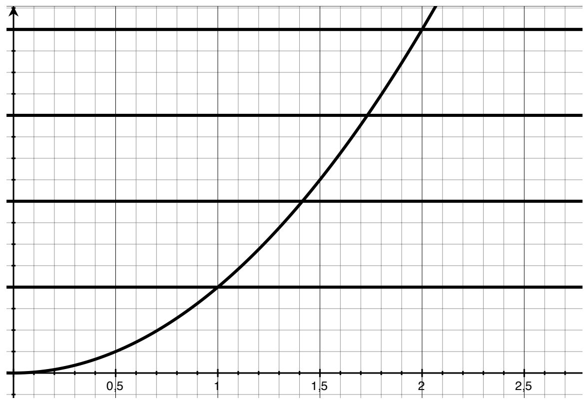

Figure 4.2: local minima for F^{λ}

for some $\sigma > 0$. The shape of J' is pictured in Figure 4.3; its maximum is achieved for $w = w_0$. Note that for all $0 < \sigma < J'(w_0)$ we have two solutions of $J'(w) = \sigma$, while we have no solution for $\sigma > J'(w_0)$.

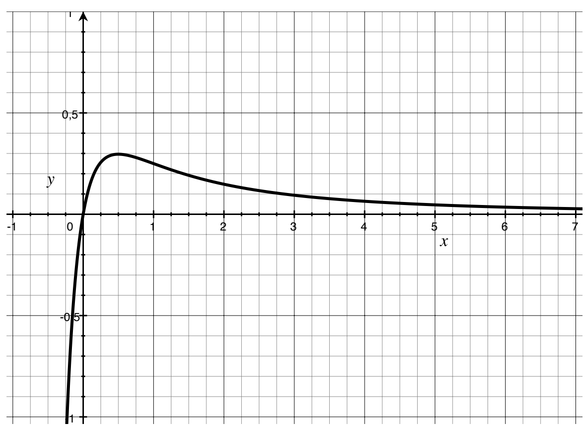

Figure 4.3: derivative of J

We have three cases:
(i) we have

$$\frac{u_i - u_{i-1}}{\sqrt{\varepsilon}} \leq w_0 \tag{4.2}$$

for all i. In this case the boundary condition gives $\frac{u_i - u_{i-1}}{\varepsilon} = \lambda$ for all i, so that we have the constraint.

$$\lambda \leq \frac{w_0}{\sqrt{\varepsilon}}. \tag{4.3}$$

This solution is a local minimum. This is easily checked when $\lambda < \frac{w_0}{\sqrt{\varepsilon}}$ since small perturbations maintain the condition (4.2). In the limit case $\lambda = \frac{w_0}{\sqrt{\varepsilon}}$ we may consider only perturbations where (4.2) is violated at exactly one index (see (ii) below), to which there corresponds an energy

$$J(w_0 + t) + (N-1)J\Big(w_0 - \frac{t}{N-1}\Big),$$

for $t \geq 0$, which has a local minimum at 0.

(ii) condition (4.2) is violated by two (or more) indices j and k. Let $\overline{w}$ be such that

$$\frac{u_j - u_{j-1}}{\sqrt{\varepsilon}} = \frac{u_k - u_{k-1}}{\sqrt{\varepsilon}} = \overline{w} > w_0.$$

We may perturb $u_i - u_{i-1}$ only for $i = j, k$, so that the energy varies by

$$f(s) := J(\overline{w} + s) + J(\overline{w} - s) - 2J(\overline{w})\,. \tag{4.4}$$

We have $f'(0) = 0$ and $f''(0) = 2J''(\overline{w}) < 0$, which contradicts the minimality of u.

(iii) condition (4.2) is violated exactly by one index. The value of $w = \frac{u_i - u_{i-1}}{\sqrt{\varepsilon}}$ for the $N-1$ indices satisfying (4.2) is obtained by computing local minimizers of the energy on such functions, which is

$$f_\varepsilon^\lambda(w) := (N-1)J(w) + J\Big(\frac{\lambda}{\sqrt{\varepsilon}} - (N-1)w\Big)$$

defined for $0 \leq w \leq \min\Big\{w_0, \frac{1}{N-1}\Big(\frac{\lambda}{\sqrt{\varepsilon}} - w_0\Big)\Big\}$. We compute

$$(f_\varepsilon^\lambda)'(w) := (N-1)\Big(J'(w) - J'\Big(\frac{\lambda}{\sqrt{\varepsilon}} - (N-1)w\Big)\Big)\,.$$

Note that

$$f_\varepsilon^\lambda(0) = J\Big(\frac{\lambda}{\sqrt{\varepsilon}}\Big) = 1 - o(1)$$

and $(f_\varepsilon^\lambda)'(0) < 0$. If $\lambda > w_0/\sqrt{\varepsilon}$ then $(f_\varepsilon^\lambda)'(w) = 0$ has a unique solution, which is a local minimizer, while if $\lambda \leq w_0/\sqrt{\varepsilon}$ we have two solutions $w_1 < w_2$, of which the first one is a local minimizer. We then have a unique curve of local minimizers with one jump.

The energy of the local minima in dependence of λ is schematically pictured in Figure 4.4.

• *A qualitative comparison of local minimization.* First, the local minimizer for F_ε^λ which never exceed the convexity threshold (corresponding to the minimizer with $S(u) = \emptyset$ for F^λ) exists only for $\lambda \leq w_0/\sqrt{\varepsilon}$; second, we only have one curve of local minimizers for F_ε^λ which exceed the convexity threshold for only one index (corresponding to the minimizers with $\#S(u) = 1$ for F^λ).

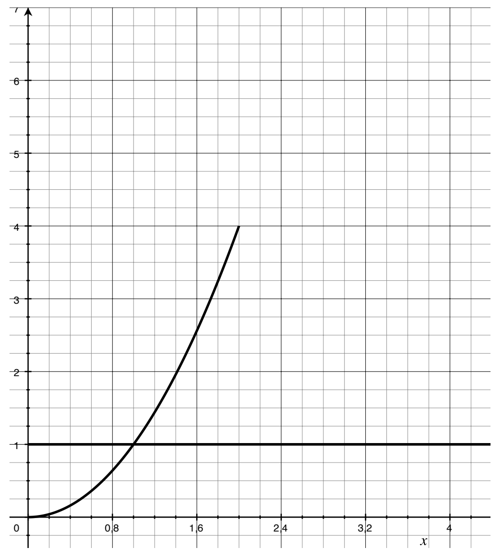

Figure 4.4: local minima for F_ε^λ

• Γ-*equivalent energies.* We choose energies defined on piecewise-H^1 functions of the form

$$G_\varepsilon^\lambda(u) = \int_0^1 |u'|^2\,dt + \sum_{t\in S(u)} g\Big(\frac{u^+ - u^-}{\sqrt{\varepsilon}}\Big),$$

again with the constraint that $u^+ > u^-$ on $S(u)$ and the boundary conditions $u^-(0) = 0$, $u^+(1) = \lambda$. In order that local minimizers satisfy $\#(S(u)) \le 1$ we require that $g : (0,+\infty) \to (0,+\infty)$ be strictly concave. In fact, with this condition the existence of two points in $S(u)$ is ruled out by noticing that given $w_1, w_2 > 0$ the function $t \mapsto g(w_1 + t) + g(w_2 - t)$ is concave. Moreover, we also require that g satisfy

$$\lim_{w\to+\infty} g(w) = 1.$$

With this condition is is easily seen that we have the Γ-convergence of G_ε^λ to F^λ.

In order to make a comparison with the local minimizers of F_ε^λ we first consider local minimizers with $S(u) = \emptyset$; i.e., $u(t) = \lambda t$. Such a function is a local minimizer if it is not energetically favourable to introduce a small jump of size w; i.e., if 0 is a local minimizer for

$$g_\varepsilon^\lambda(w) := (\lambda - w)^2 + g\Big(\frac{w}{\sqrt{\varepsilon}}\Big),$$

where we have extended the definition of g by setting $g(0) = 0$. Note that if g is not continuous in 0 then 0 is a strict local minimizer for g_ε^λ for all λ. Otherwise, we can compute the derivative, and obtain that

$$\frac{d}{dw} g_\varepsilon^\lambda(0) = -2\lambda + \frac{1}{\sqrt{\varepsilon}} g'(0).$$

For ε small enough, 0 is a (isolated) local minimizer if and only if $\frac{d}{dw} g_\varepsilon^\lambda(0) > 0$; i.e.,

$$\lambda < \frac{1}{2\sqrt{\varepsilon}} g'(0)\,.$$

If we choose

$$g'(0) = 2w_0$$

we obtain the desired constraint on this type of local minimizers. A possible simple choice of g is

$$g(w) = \frac{2w_0 w}{1 + 2w_0 w}.$$

We finally consider local minimizers with $\#(S(u)) = 1$. If w denotes the size of the jump then again computing the derivative of the energy, we conclude the existence of a single local minimizer w with

$$2(\lambda - w) = \frac{1}{\sqrt{\varepsilon}} g'\Big(\frac{w}{\sqrt{\varepsilon}}\Big),$$

and energy approaching 1 as $\varepsilon \to 0$.

• With the choice above the pictures of the local minimizers for G_ε^λ and for F_ε^λ are of the same type, but may vary in quantitative details. We have not addressed the problem of the uniformity of this description, for which a refinement of the choice of g could be necessary.

• As a conclusion, we remark that this example has some modeling implications. The functional F^λ can be seen as a one-dimensional version of the energy of a brittle elastic medium according to Griffith's theory of Fracture ($S(u)$ represents the fracture site in the reference configuration), which is then interpreted as a continuum approximation of an atomistic model with Lennard Jones interactions. The requirement that also local minima may be reproduced by the limit theory has made us modify our functional F^λ obtaining another sequence of energies, which maintain an internal parameter ε. Energies of the form G_ε^λ are present in the literature, and are related to Barenblatt's theory of ductile Fracture. Note that in all these considerations the parameter λ appears in the functionals only as a boundary condition, and does not influence the form of the energy.

Chapter 5

Minimizing movements

5.1 An energy-driven implicit-time discretization

We now introduce a notion of energy-based motion which generalizes an implicit-time scheme for the approximation of solutions of gradient flows to general (also non differentiable) energies. We will use the terminology of *minimizing movements*, introduced by De Giorgi, even though we will not use the precise notation used in the literature.

Definition 5.1.1 (minimizing movements) *Let X be a separable Hilbert space, $F : X \to [0, +\infty]$ coercive and lower semicontinuous. Given x_0 and $\tau > 0$ we define recursively x_k as a minimizer for the problem*

$$\min\Big\{F(x) + \frac{1}{2\tau}\|x - x_{k-1}\|^2\Big\}, \tag{5.1}$$

and the piecewise-constant trajectory $u^\tau : [0, +\infty) \to X$ given by

$$u^\tau(t) = x_{\lfloor t/\tau \rfloor}\,. \tag{5.2}$$

A minimizing movement *for F from x_0 is any limit of a subsequence u^{τ_j} uniform on compact sets of $[0, +\infty)$.*

In this definiton we have taken $F \geq 0$ and X Hilbert for the sale of simplicity. In particular we can X a metric space and the (power of the) distance in place of the squared norm.

Remark 5.1.2 A heuristic explanation of the definition above is given when F is smooth. In this case, with the due notation, a minimizer for (5.1) solves the equation

$$\frac{x_k - x_{k-1}}{\tau} = -\nabla F(u_k); \tag{5.3}$$

i.e., u^τ solves the equation

$$\frac{u^\tau(t) - u^\tau(t-\tau)}{\tau} = -\nabla F(u^\tau(t)). \tag{5.4}$$

If we may pass to the limit in this equation as $u^\tau \to u$ then

$$\frac{\partial u}{\partial t} = -\nabla F(u). \tag{5.5}$$

This is easily shown if $X = \mathbb{R}^n$ and $F \in C^2(\mathbb{R}^n)$. In this case by taking any $\varphi \in C_0^\infty((0,T);\mathbb{R}^n)$ we have

$$-\int_0^T \langle \nabla F(u^\tau), \varphi\rangle dt = \int_0^T \Big\langle \frac{u^\tau(t) - u^\tau(t-\tau)}{\tau}, \varphi\Big\rangle dt = -\int_0^T \Big\langle u^\tau(t), \frac{\varphi(t) - \varphi(t+\tau)}{\tau}\Big\rangle dt,$$

from which, passing to the limit

$$\int_0^T \langle \nabla F(u), \varphi\rangle dt = \int_0^T \langle u, \varphi'\rangle dt;$$

i.e., (5.5) is satisfied in the sense of distributions, and hence in the classical sense.

Remark 5.1.3 (stationary solutions) Let x_0 be a local minimizer for F, then the only minimizing movement for F from x_0 is the constant function $u(t) = x_0$.

Indeed, if x_0 is a minimizer for F when $\|x - x_0\| \le \delta$ by the positiveness of F it is the only minimizer of $F(x) + \frac{1}{2\tau}\|x - x_0\|^2$ for $\tau \le \delta^2/F(x_0)$ if $F(x_0) > 0$ (any τ if $F(x_0) = 0$). So that $x_k = x_0$ for all k for these τ.

Proposition 5.1.4 (existence of minimizing movements) *For all F and x_0 as above there exists a minimizing movement $u \in C^{1/2}([0,+\infty);X)$.*

Proof. By the coerciveness and lower semicontinuity of F we obtain that u_k are well defined for all k. Moreover, since

$$F(x_k) + \frac{1}{2\tau}\|x_k - x_{k-1}\|^2 \le F(x_{k-1}),$$

we have $F(x_k) \le F(x_{k-1})$ and

$$\|x_k - x_{k-1}\|^2 \le 2\tau(F(x_{k-1}) - F(x_k)), \tag{5.6}$$

so that for $t > s$

$$\|u^\tau(t) - u^\tau(s)\| \quad \le \quad \sum_{k=\lfloor s/\tau\rfloor+1}^{\lfloor t/\tau\rfloor} \|x_k - x_{k-1}\|$$

$$
\begin{aligned}
&\leq \sqrt{\lfloor t/\tau \rfloor - \lfloor s/\tau \rfloor} \sqrt{\sum_{k=\lfloor s/\tau \rfloor+1}^{\lfloor t/\tau \rfloor} \|x_k - x_{k-1}\|^2} \\
&\leq \sqrt{\lfloor t/\tau \rfloor - \lfloor s/\tau \rfloor} \sqrt{2\tau \sum_{k=\lfloor s/\tau \rfloor+1}^{\lfloor t/\tau \rfloor} (F(x_{k-1}) - F(x_k))} \\
&= \sqrt{\lfloor t/\tau \rfloor - \lfloor s/\tau \rfloor} \sqrt{2\tau(F(x_{\lfloor s/\tau \rfloor}) - F(x_{\lfloor t/\tau \rfloor}))} \\
&\leq \sqrt{2F(x_0)}\sqrt{\tau(\lfloor t/\tau \rfloor - \lfloor s/\tau \rfloor)} \\
&\leq \sqrt{2F(x_0)}\sqrt{t - s + \tau}
\end{aligned}
$$

This shows that the functions u^τ are (almost) equicontinuous and equibounded in $C([0,+\infty);X)$. Hence, they converge uniformly. Moreover, passing to the limit we obtain

$$\|u(t) - u(s)\| \leq \sqrt{2F(x_0)}\sqrt{|t-s|}$$

so that $u \in C^{1/2}([0,+\infty);X)$. □

Example 5.1.5 (non-uniqueness of minimizing movements) If F is not C^2 we may have more than one minimizing movement.

(i) *Bifurcation at times with multiple minimizers.* A simple example is $F(x) = -\frac{1}{\alpha}|x|^\alpha$ with $0 < \alpha < 2$, which is not C^2 at $x = 0$. In this case, for $x_0 = 0$ we have a double choice for minimum problem (5.1); i.e.,

$$x_1 = \pm\tau^{1/(2-\alpha)}.$$

Once x_1 is chosen all other valued are determined, and it can be seen that either $x_k > 0$ for all k or $x_k < 0$ for all k (for $\alpha = 1$, e.g., we have $x_k = \pm k\tau$), and that in the limit we have the two solutions of

$$\begin{cases} u' = |u|^{(\alpha-2)}u \\ u(0) = 0 \end{cases}$$

with $u(t) \neq 0$ for $t > 0$. Note in particular that we do not have the trivial solution $u = 0$. In this example we do not have to pass to a subsequence of τ.

(i) *Different movements depending on subsequences of τ.* Discrete trajectories can be different depending on the time step τ. We give an explicit example, close in spirit to the previous one. In this example the function F is asymmetric, so that x_1 is unique but may take positive or negative values depending on τ.

We define F as the Lipschitz function taking value 0 at $x = 0$, for $x > 0$

$$F'(x) = \begin{cases} -1 & \text{if } 2^{-2k-1} < x < 2^{-2k},\ k \in \mathbb{N} \\ -2 & \text{otherwise for } x > 0 \end{cases}$$

and $F'(x) = 3 + F'(-x)$ for $x < 0$. It is easily seen that for $x_0 = 0$ we may have a unique minimizer x_1 with $x_1 > 0$ or $x_1 < 0$ depending on τ. In particular we have $x_1 = -2^{-2k} < 0$ for $\tau = 2^{-2k-1}$ and $x_1 = 2^{-2k+1} > 0$ for $\tau = 2^{-2k}$. In the two cases we then have again the solutions to

$$\begin{cases} u' = -F(u) \\ u(0) = 0 \end{cases}$$

with $u(t) < 0$ for all $t > 0$ or $u(t) > 0$ for all $t > 0$, respectively.

Remark 5.1.6 Gradient flows, and hence minimizing movements, trivially do not commute even with uniform convergence. As a simple example, take $X = \mathbb{R}$ and

$$F_\varepsilon(x) = x^2 - \rho \sin\Bigl(\frac{x}{\varepsilon}\Bigr),$$

with $\rho = \rho_\varepsilon \to 0$ as $\varepsilon \to 0$, uniformly converging to $F(x) = x^2$. If also

$$\varepsilon << \rho,$$

then for fixed x_0 the solutions u_ε to the equation

$$\begin{cases} u_\varepsilon' = -2u_\varepsilon + \dfrac{\rho}{\varepsilon} \cos\Bigl(\dfrac{u_\varepsilon}{\varepsilon}\Bigr) \\ u_\varepsilon(0) = x_0 \end{cases}$$

converge to the constant function $u_0(t) = x_0$ as $\varepsilon \to 0$, which does not solve

$$\begin{cases} u' = -2u \\ u(0) = x_0\,. \end{cases}$$

This is easily seen by studying the stationary solutions of

$$-2x + \frac{\rho}{\varepsilon} \cos\Bigl(\frac{x}{\varepsilon}\Bigr) = 0\,.$$

5.2 Minimizing movements along a sequence of functionals

With Remark 5.1.6 above in mind, in order to give a meaningful limit for the energy-driven motion along a sequence of functionals it may be useful to vary the definition of minimizing movement as follows.

Definition 5.2.1 (minimizing movements along a sequence) *Let X be a separable Hilbert space, $F_\varepsilon : X \to [0, +\infty]$ equicoercive and lower semicontinuous and $x_0^\varepsilon \to x_0$ with*

$$F_\varepsilon(x_0^\varepsilon) \le C < +\infty, \tag{5.7}$$

and $\tau_\varepsilon > 0$ converging to 0 as $\varepsilon \to 0$. Fixed $\varepsilon > 0$ we define recursively x_k^ε as a minimizer for the problem

$$\min\Big\{F_\varepsilon(x) + \frac{1}{2\tau}\|x - x_{k-1}^\varepsilon\|^2\Big\}, \tag{5.8}$$

and the piecewise-constant trajectory $u^\varepsilon : [0,+\infty) \to X$ given by

$$u^\varepsilon(t) = x_{\lfloor t/\tau_\varepsilon \rfloor}\,. \tag{5.9}$$

A minimizing movement *for F_ε from x_0^ε is any limit of a subsequence u^{ε_j} uniform on compact sets of $[0,+\infty)$.*

With the same proof as Proposition 5.1.4 we can show the following result.

Proposition 5.2.2 *For every F_ε and x_0^ε as above there exist minimizing movements for F_ε from x_0^ε in $C^{1/2}([0,+\infty);X)$.*

5.3 An example: "overdamped dynamics" of Lennard-Jones interactions

Let J be as in Section 4.4 and $\frac{1}{\varepsilon} = N \in \mathbb{N}$. We consider the energies

$$F_\varepsilon(u) = \sum_{i=1}^N J\Big(\frac{u_i - u_{i-1}}{\sqrt{\varepsilon}}\Big)$$

with the periodic boundary condition $u_N = u_0$. As proved in Section 4.4, after identification of u with a piecewise-constant function on $[0,1]$, these energies Γ-converge to the energy

$$F(u) = \int_0^1 |u'|^2\,dt + \#(S(u)\cap[0,1)), \qquad u^+ > u^-,$$

defined on piecewise-H^1 functions, in this case extended 1-periodically on the whole real line.

In this section we apply the minimizing movements scheme to F_ε as a sequence of functionals in $L^2(0,1)$. In order to have initial data u_0^ε with equibounded energy, we may suppose that these are the discretization of a single piecewise-H^1 function u_0 (with a slight abuse of notation we will continue to denote all these discrete functions by u_0).

With fixed ε and τ, the time-discretization scheme consists in defining recursively u^k as a minimizer of

$$u \mapsto \sum_{i=1}^N J\Big(\frac{u_i - u_{i-1}}{\sqrt{\varepsilon}}\Big) + \frac{1}{2\tau}\sum_{i=1}^N \varepsilon|u_i - u_i^{k-1}|^2. \tag{5.10}$$

By Proposition 5.2.2, upon extraction of a subsequence, the functions $u^\tau(t) = u_{\lfloor t/\tau\rfloor}$ converge uniformly in L^2 to a function $u \in C^{1/2}([0,+\infty); L^2(0,1))$. Moreover, since we have $F(u(t)) \leq F(u_0) < +\infty$, $u(t)$ is a piecewise-H^1 function for all t.

We now describe the motion of the limit u. For the sake of simplicity we suppose that u_0 is a piecewise-Lipschitz function and that $S(u_0) \cap \{\varepsilon i : i \in \{1,\dots,N\} = \emptyset$ (so that we do not have any ambiguity in the definition of the interpolations of u_0)..

We first write down the Euler-Lagrange equations for u^k, which amount to a simple N-dimensional system of equations obtained by deriving (5.10) with respect to u_i

$$\frac{1}{\sqrt{\varepsilon}}\Big(J'\Big(\frac{u_i^k - u_{i-1}^k}{\sqrt{\varepsilon}}\Big) - J'\Big(\frac{u_{i+1}^k - u_i^k}{\sqrt{\varepsilon}}\Big)\Big) + \frac{\varepsilon}{\tau}(u_i^k - u_i^{k-1}) = 0. \tag{5.11}$$

- With fixed $i \in \{1,\dots,N\}$ let v_k be defined by

$$v_k = \frac{u_i^k - u_{i-1}^k}{\varepsilon}\,.$$

For simplicity of notation we set

$$J_\varepsilon(w) = \frac{1}{\varepsilon} J(\sqrt{\varepsilon}\, w).$$

By (5.11) and the corresponding equation for $i-1$, which can be rewritten as

$$J'_\varepsilon\Big(\frac{u_{i-1}^k - u_{i-2}^k}{\varepsilon}\Big) - J'_\varepsilon\Big(\frac{u_i^k - u_{i-1}^k}{\varepsilon}\Big) + \frac{\varepsilon}{\tau}(u_{i-1}^k - u_{i-1}^{k-1}) = 0.$$

we have

$$\begin{aligned}
\frac{v_k - v_{k-1}}{\tau} &= \frac{1}{\tau}\Big(\frac{u_i^k - u_{i-1}^k}{\varepsilon} - \frac{u_i^{k-1} - u_{i-1}^{k-1}}{\varepsilon}\Big)\\
&= \frac{1}{\varepsilon}\Big(\frac{u_i^k - u_i^{k-1}}{\tau} - \frac{u_{i-1}^k - u_{i-1}^{k-1}}{\tau}\Big)\\
&= \frac{1}{\varepsilon^2}\Big(\Big(J'_\varepsilon\Big(\frac{u_{i-1}^k - u_{i-2}^k}{\varepsilon}\Big) - J'\Big(\frac{u_i^k - u_{i-1}^k}{\varepsilon}\Big)\Big)\\
&\qquad -\Big(J'_\varepsilon\Big(\frac{u_i^k - u_{i-1}^k}{\varepsilon}\Big) - J'_\varepsilon\Big(\frac{u_{i+1}^k - u_i^k}{\varepsilon}\Big)\Big),
\end{aligned}$$

so that

$$\frac{v_k - v_{k-1}}{\tau} - \frac{2}{\varepsilon^2} J'_\varepsilon(v_k) = -\frac{1}{\varepsilon^2}\Big(\Big(J'_\varepsilon\Big(\frac{u_{i-1}^k - u_{i-2}^k}{\varepsilon}\Big) + J'\Big(\frac{u_{i+1}^k - u_i^k}{\varepsilon}\Big)\Big) \geq -\frac{2}{\varepsilon^2} J'_\varepsilon(\frac{w_0}{\sqrt{\varepsilon}}) \tag{5.12}$$

We recall that we denote by w_0 the maximum point of J'.

We can interpret (5.12) as an inequality for the difference system

$$\frac{v_k - v_{k-1}}{\eta} - 2J'_\varepsilon(v_k) \geq -2J'_\varepsilon(\frac{w_0}{\sqrt{\varepsilon}}),$$

where $\eta = \tau/\varepsilon^2$ is interpreted as a discretization step. Note that $v_k = w_0/\sqrt{\varepsilon}$ for all k is a stationary solution of the equation

$$\frac{v_k - v_{k-1}}{\eta} - 2J'_\varepsilon(v_k) = -2J'_\varepsilon(\frac{w_0}{\sqrt{\varepsilon}})$$

and that J'_ε are equi-Lipschitz functions on $[0, +\infty)$. If $\eta << 1$ this implies that if $v_{k_0} \leq w_0/\sqrt{\varepsilon}$ for some k_0 then

$$v_k \leq \frac{w_0}{\sqrt{\varepsilon}} \qquad \text{for } k \geq k_0,$$

or, equivalently, that if $\tau << \varepsilon^2$ the set

$$S^k_\varepsilon = \Big\{i \in \{1, \dots, N\} : \frac{u^k_i - u^k_{i-1}}{\varepsilon} \geq \frac{w_0}{\sqrt{\varepsilon}}\Big\}$$

is decreasing with k. By our assumption on u_0, for ε small enough we then have

$$S^0_\varepsilon = \Big\{i \in \{1, \dots, N\} : [\varepsilon(i-1), \varepsilon i] \cap S(u_0) \neq \emptyset\Big\},$$

so that, passing to the limit

$$S(u(t)) \subseteq S(u_0) \qquad \text{for all } t \geq 0. \tag{5.13}$$

- Taking into account that we may define

$$u^\tau(t, x) = u^{\lfloor t/\tau \rfloor}_{\lfloor x/\varepsilon \rfloor},$$

we may choose functions $\phi \in C^\infty_0(0, T)$ and $\psi \in C^\infty_0(x_1, x_2)$, with $(x_1, x_2) \cap S(u_0) = \emptyset$, and obtain from (5.11)

$$\begin{aligned} &\int_0^T \int_{x_1}^{x_2} u^\tau(t, x)\Big(\frac{\phi(t) - \phi(t+\tau)}{\tau}\Big)\psi(x)\, dx\, dt \\ = \ & -\int_0^T \int_{x_1}^{x_2} \Big(\frac{1}{\sqrt{\varepsilon}} J'\Big(\sqrt{\varepsilon}\frac{u^\tau(t, x) - u^\tau(t, x-\varepsilon)}{\varepsilon}\Big)\Big)\phi(t)\Big(\frac{\psi(x) - \psi(x+\varepsilon)}{\varepsilon}\Big)\, dx\, dt\,. \end{aligned}$$

Taking into account that

$$\lim_{\varepsilon \to 0} \frac{1}{\sqrt{\varepsilon}} J'(\sqrt{\varepsilon} w) = 2w,$$

we can pass to the limit and obtain that

$$-\int_0^T \int_{x_1}^{x_2} u(t,x)\phi'(t)\psi(x)\,dx\,dt = \int_0^T \int_{x_1}^{x_2} 2\frac{\partial u}{\partial x}\phi(t)\psi'(x)\,dx\,dt\,;$$

i.e., that

$$\frac{\partial u}{\partial t} = -2\frac{\partial^2 u}{\partial x^2} \tag{5.14}$$

in the sense of distributions (and hence also classically) in $(0,T)\times(x_1,x_2)$. By the arbitrariness of the interval (x_1,x_2) we have that equation (5.14) is satisfied for x in $(0,1)\setminus S(u_0)$.

• We now derive boundary conditions on $S(u(t))$. Let i_0+1 belong to S^0_ε, and suppose that $u^+(t,x)-u^-(t,x)\geq c>0$. Then we have

$$\lim_{\tau\to 0}\frac{1}{\sqrt{\varepsilon}}J'\Big(\frac{u_{i_0}^{\lfloor t/\tau\rfloor}-u_{i_0-1}^{\lfloor t/\tau\rfloor}}{\sqrt{\varepsilon}}\Big)=0\,.$$

If $i<i_0$, from (5.11) it follows, after summing up the indices from i to i_0, that

$$\sum_{j=i}^{i_0}\frac{\varepsilon}{\tau}(u_j^k-u_j^{k-1}) = -\frac{1}{\sqrt{\varepsilon}}J'\Big(\frac{u_i^k-u_{i-1}^k}{\sqrt{\varepsilon}}\Big)\,. \tag{5.15}$$

We may choose $i=i_\varepsilon$ such that $\varepsilon i_\varepsilon\to\overline{x}$ and we may deduce from (5.15) that

$$\int_{\overline{x}}^{x_0}\frac{\partial u}{\partial t}\,dx = -2\frac{\partial u}{\partial x}(\overline{x}),$$

where $x_0\in S(u(t))$ is the limit of εi_0. Letting $\overline{x}\to x_0^-$ we obtain

$$\frac{\partial u}{\partial x}(x_0^-)=0.$$

Similarly we obtain the homogeneous Neumann condition at x_0^+.

Summarizing, the minimizing movement of the scaled Lennard-Jones energies F_ε from a piecewise-H^1 function consists in a piecewise-H^1 motion, following the heat equation on $(0,1)\setminus S(u_0)$, with homogeneous Neumann boundary conditions on $S(u_0)$ (as long as $u(t)$ has a discontinuity at the corresponding point of $S(u_0)$).

Chapter 6

Geometric minimizing movements

We now examine some minimizing movements describing the motion of sets. Such a motion can be framed in the setting of the previous chapter after identification of a set A with its characteristic function $u = \chi_A$. The energies we are going to consider are of perimeter type; i.e., with

$$F(A) = \mathcal{H}^{n-1}(\partial A) \tag{6.1}$$

as a prototype in the notation of the previous section. A heuristic arguments suggests that the variation of the perimeter be linked to the notion of curvature; hence, we expect to be able to obtain motion by mean curvature as a minimizing movement; i.e., for a smooth set, its motion in the normal direction with velocity proportional to its curvature. In the simplest case of initial datum a ball $A_0 = B_{R_0}(0)$ in $\mathbb{R}^2$ the motion is given by concentric balls with radii satisfying

$$\begin{cases} R' = -\dfrac{c}{R} \\ \\ R(0) = R_0; \end{cases} \tag{6.2}$$

i.e., $R(t) = \sqrt{R_0^2 - 2ct}$, valid until the *extinction time* $t = R_0^2/2c$, when the radius vanishes. We will see that in order to obtain geometric motions as minimizing movements we will have to modify the procedure described in the previous chapter.

Example 6.0.1 (pinning for the perimeter motion) Let $n = 2$. We apply the minimizing-movement procedure to the perimeter functional (6.1) and the initial datum $A_0 = B_{R_0}(0)$ in $\mathbb{R}^2$.

With fixed τ, since

$$\int_{\mathbb{R}^2} |\chi_A - \chi_B|^2\, dx = |A \triangle B|,$$

the minimization to determine A_1 is

$$\min\Big\{\mathcal{H}^1(\partial A) + \frac{1}{2\tau}|A \triangle A_0|\Big\}. \tag{6.3}$$

We note that we can restrict our attention to sets A contained in A_0, since otherwise taking $A \cap A_0$ as test sets in their place would decrease both terms in the minimization. Once this is observed, we also note that, given $A \subset A_0$, if $B_R(x) \subset A_0$ has the same measure as A then it decreases the perimeter part of the energy (strictly, if A itself is not a ball) while keeping the second term fixed. Hence, we can limit our analysis to balls $B_R(x) \subset A_0$, for which the energy depends only on R. The incremental problem is then given by

$$\min\Big\{2\pi R + \frac{\pi}{2\tau}(R_0^2 - R^2) : 0 \leq R \leq R_0\Big\}, \tag{6.4}$$

whose minimizer is either $R = 0$ (with value $\frac{\pi}{2\tau}R_0^2$) or $R = R_0$ (with value $2\pi R_0$) since in (6.4) we are minimizing a concave function of R. For τ small the minimizer is then R_0. This means that the motion is trivial: $A_k = A_0$ for all k, and hence also the resulting minimizing movement is trivial.

6.1 A first (unsuccessful) generalization

We may generalize the scheme of the minimizing movements by taking a more general distance term in the minimization; e.g., considering x_k as a minimizer of

$$\min\Big\{F(x) + \frac{1}{\tau}\Phi(\|x - x_{k-1}\|)\Big\}, \tag{6.5}$$

where Φ is a continuous increasing function with $\Phi(0) = 0$. As an example, we can consider

$$\Phi(z) = \frac{1}{p}|z|^p\,.$$

Note that in this case we obtain the estimate

$$\|x_k - x_{k-1}\|^p \leq p\,\tau(F(x_{k-1} - F(x_k))$$

for the minimizer x_k. Using Hölder's inequality as in the case $p = 2$, we end up with (for $j > h$)

$$\begin{aligned}
\|x_j - x_h\| &\leq (j-h)^{(p-1)/p}\Big(\sum_{k=h+1}^{j}\|x_k - x_{k-1}\|^p\Big)^{1/p} \\
&\leq (p\,F(x_0))^{1/p}(\tau^{1/(p-1)}(j-h))^{(p-1)/p}.
\end{aligned}$$

In order to obtain a $(1 - \frac{1}{p})$ Hölder continuity for the interpolated function u^τ we have to define it as

$$u^\tau(t) = u_{\lfloor t/\tau^{1/(p-1)}\rfloor}.$$

Note that we may use the previous definition $u^\tau(t) = u_{\lfloor t/\tau \rfloor}$ for the interpolated function if we change the parameter τ in (6.5) and consider instead the problem

$$\min\Big\{F(x) + \frac{1}{\tau^{p-1}}\Phi(\|x - x_{k-1}\|)\Big\} \tag{6.6}$$

to define x_k.

Example 6.1.1 ((non-)geometric minimizing movements) We use the scheme above, with a slight variation in the exponents since we will be interested in the description of the motion in terms of the radius of a ball in $\mathbb{R}^2$ (which is the square root of the L^2-norm and not the norm itself). As in the previous example, we take the initial datum $A_0 = B_{R_0} = B_{R_0}(0)$, and consider A_k defined recursively as a minimizer of

$$\min\Big\{\mathcal{H}^1(\partial A) + \frac{1}{p\tau^{p-1}}|A \triangle A_0|^p\Big\}, \tag{6.7}$$

with $p > 1$. As above, at each step the minimizer is given by balls

$$B_{R_k}(x_k) \subset B_{R_{k-1}}(x_{k-1}). \tag{6.8}$$

The value of R_k is determined by solving

$$\min\Big\{2\pi R + \frac{\pi^p}{p\tau^{p-1}}(R_{k-1}^2 - R^2)^p : 0 \le R \le R_{k-1}\Big\}, \tag{6.9}$$

which gives

$$\frac{R_k - R_{k-1}}{\tau} = -\frac{1}{\pi R_k^{1/(p-1)}(R_k + R_{k-1})}. \tag{6.10}$$

Note that in this case the minimum value is not taken at $R_k = R_{k-1}$ (this can be checked, e.g., by checking that the derivative of the function to be minimized in (6.9) is positive at R_{k-1}). By passing to the limit in (6.10) we deduce the equation

$$R' = -\frac{1}{2\pi R^{p/(p-1)}} \tag{6.11}$$

(valid until the extinction time).

Despite having obtained an equation for R we notice that this approach is not satisfactory, since

• **(non-geometric motion)** in (6.8) we have infinitely many solutions; namely, all balls centered in x_k with

$$|x_{k-1} - x_k| \le R_{k-1} - R_k.$$

This implies that we may have moving centres $x(t)$ provided that $|x'| \le R'$ and $x(0) = 0$; in particular we may choose $x(t) = (R_0 - R(t))z$ for any $z \in B_1(0)$ which converges to $R_0 z$;

i.e., the point where the sets concentrate at the vanishing time may be any point in $\overline{B}_{R_0}$ at the extinction time. This implies that the motion is not a geometric one: sets do not move according to geometric quantities.

• **(failure to obtain mean-curvature motion)** even if we obtain an equation for R we never obtain the mean curvature flow since $p/(p-1) > 1$.

6.2 A variational approach to curvature-driven motion

In order to obtain motion by curvature Almgren, Taylor and Wang have introduced a variation of the implicit-time scheme described above, where the term $|A \triangle A_k|$ is substituted by an integral term which favours variations which are 'uniformly distant' to the boundary of A_k. The problem defining A_k is then

$$\min\Big\{\mathcal{H}^1(\partial A) + \frac{1}{\tau}\int_{A\triangle A_{k-1}} \mathrm{dist}(x, \partial A_{k-1})\, dx\Big\}\,. \tag{6.12}$$

We will not prove a general convergence result for an arbitrary initial datum A_0, but we will check the convergence to mean-curvature motion for $A = B_{R_0}$ in $\mathbb{R}^2$.

In this case we note that if A_{k-1} is a ball centered in 0 then we have

• A_k is contained in A_{k-1}. To check this note that, given a test set A, considering $A \cap A_{k-1}$ as a test set in its place decreases the energy in (6.12), strictly if $A \setminus A_{k-1} \neq \emptyset$;

• A_k is convex and with baricenter in 0. To check this, first, note that each connected component of A_k is convex. Otherwise, considering the convex envelopes decreases the energy (strictly, if one of the connected components if not convex). Then note that if 0 is not the baricenter of a connected component of A_k then a small translation towards 0 strictly decreases the energy (this follows by computing the derivative of the volume term along the translation). In particular, we only have one (convex) connected component;

From these properties we can conclude that A_k is indeed a ball centered in 0. Were it not so, there would be a line through 0 such that the boundary of A_k intersects transversally this line. By a reflection argument we then obtain a non-convex set $\widetilde{A}_k$ with energy not greater than the one of A_k. Its convexification would then strictly decrease the energy. This shows that each A_k is of the form

$$A_k = B_{R_k} = B_{R_k}(0).$$

We can now compute the equation satisfied by R_k, by minimizing

$$\min\Big\{2\pi R + \frac{2\pi}{\tau}\int_R^{R_{k-1}} (R_{k-1} - \rho)\rho\, d\rho\Big\}\,, \tag{6.13}$$

which gives

$$\frac{R_k - R_{k-1}}{\tau} = -\frac{1}{R_k}\,. \tag{6.14}$$

Passing to the limit gives the desired mean curvature equation (6.2).

Chapter 7

Homogenization of minimizing movements

We now examine minimizing movements along oscillating sequences (with many local minima). We first treat a model case, and subsequently compute minimizing movements for geometric motions.

7.1 An example in the real line

We apply the minimizing-movement scheme to the functions

$$F_\varepsilon(x) = -\Big\lfloor \frac{x}{\varepsilon} \Big\rfloor \varepsilon$$

converging to $F(x) = -x$ (see Fig. 7.1). This is a prototype of a function with many local

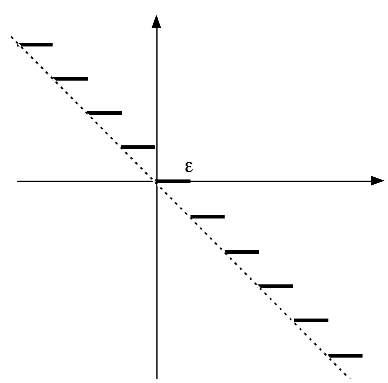

Figure 7.1: the function F_ε

minimizers (actually, in this case all points are local minimizers) converging to a function with few local minimizers (actually, none).

Note that, with fixed ε, for any initial datum x_0 the minimizing movement for F_ε is trivial: $u(t) = x_0$, since all points are local minimizers. Conversely the corresponding minimizing movement for the limit is $u(t) = x_0 + t$.

We now fix an initial datum x_0, the space scale ε and the time scale τ, and examine the successive-minimization scheme from x_0. Note that it is not restrictive to suppose that $0 \le x_0 < 1$ up to a translation in $\varepsilon\mathbb{Z}$.

The first minimization, giving x_1 is

$$\min\Big\{F_\varepsilon(x) + \frac{1}{2\tau}(x - x_0)^2\Big\}. \tag{7.1}$$

The function to minimize is pictured in Figure 7.2 in normalized coordinates ($\varepsilon = 1$); note that it equals $-x + \frac{1}{2\tau}(x - x_0)^2$ if $x \in \varepsilon\mathbb{Z}$.

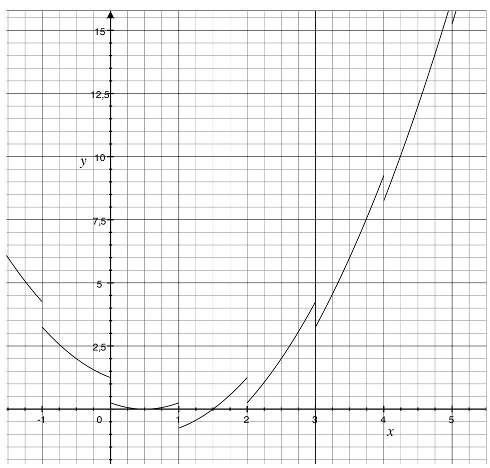

Figure 7.2: the function in the minimization problem (7.1)

We have two possibilities:

(i) the minimizer belongs to $[0, \varepsilon)$. This occurs exactly if $F_\varepsilon(\varepsilon) > 0$; i.e.,

$$\tau < \frac{(x_0 - \varepsilon)^2}{2\varepsilon}. \tag{7.2}$$

In this case the only minimizer is still x_0. This implies that for all k we have $x_k = x_0$. Note however that condition (7.2) is asymptotically empty as $\varepsilon \to 0$ if $x_0 \neq 0$. Hence, we may suppose that $x_0 = 0$, so that condition (7.2) becomes

$$\frac{\tau}{\varepsilon} < \frac{1}{2};$$

(ii) if $\frac{\tau}{\varepsilon} > \frac{1}{2}$ then for ε small the minimum is taken on $\varepsilon\mathbb{Z}$. So that again we may suppose that $x_0 = 0$.

Note that if $x_0 = 0$ and $\frac{\tau}{\varepsilon} = \frac{1}{2}$ then we have a double choice for the minimizer. These cases will be examined separately.

If $x_0 = 0$ then x_1 is computed by solving

$$\min\Big\{F_\varepsilon(x) + \frac{1}{2\tau}x^2 : x \in \varepsilon\mathbb{Z}\Big\}, \tag{7.3}$$

and is characterized by

$$x_1 - \frac{1}{2}\varepsilon \le \tau \le x_1 + \frac{1}{2}\varepsilon.$$

We then have

$$x_1 = \Big\lfloor \frac{\tau}{\varepsilon} + \frac{1}{2}\Big\rfloor \varepsilon \qquad \text{if } \frac{\tau}{\varepsilon} + \frac{1}{2} \notin \mathbb{Z}$$

(note again that we have two solutions for $\frac{\tau}{\varepsilon} + \frac{1}{2} \in \mathbb{Z}$, and we examine this case separately). The same computation is repeated at each k giving

$$\frac{x_k - x_{k-1}}{\tau} = \Big\lfloor \frac{\tau}{\varepsilon} + \frac{1}{2}\Big\rfloor \frac{\varepsilon}{\tau}.$$

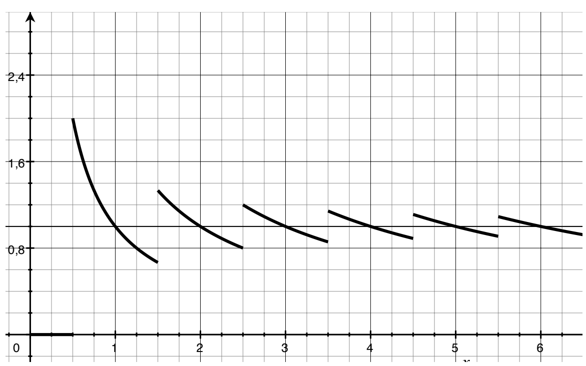

Figure 7.3: the velocity v in terms of w

We can now choose τ and ε tending to 0 simultaneously and pass to the limit. The behaviour of the limit minimizing movements is governed by the quantity

$$w = \lim_{\varepsilon\to 0} \frac{\tau}{\varepsilon}, \tag{7.4}$$

which we may suppose exists up to subsequences. If $w + \frac{1}{2} \notin \mathbb{Z}$ then the minimizing movement along F_ε from x_0 is uniquely defined by

$$u(t) = x_0 + vt, \text{ with } v = \Big\lfloor w + \frac{1}{2}\Big\rfloor \frac{1}{w},$$

so that the whole sequence converges if the limit in (7.4) exists. Note that

• **(pinning)** we have $v = 0$ exactly when $\frac{\tau}{\varepsilon} < \frac{1}{2}$ for ε small. In particular this holds for $\tau << \varepsilon$ (i.e., for $w = 0$);

• **(limit motion for slow times)** if $\varepsilon << \tau$ then the motion coincides with the gradient flow of the limit, with velocity 1;

• **(discontinuous dependence of the velocity)** the velocity is a discontinuous function of w at points of $\frac{1}{2} + \mathbb{Z}$. Note moreover that it may be actually greater than the limit velocity 1. The graph of v is pictured in Figure 7.3

• **(non-uniqueness at $w \in \frac{1}{2} + \mathbb{Z}$)** in these exceptional cases we may have either of the two velocities $1 + \frac{1}{2w}$ or $1 - \frac{1}{2w}$ in the cases $\frac{\varepsilon}{\tau} + \frac{1}{2} > w$ or $\frac{\varepsilon}{\tau} + \frac{1}{2} < w$ for all ε small respectively, but we may also have any $u(t)$ with

$$1 - \frac{1}{2w} \leq u'(t) \leq 1 + \frac{1}{2w}$$

if we have precisely $\frac{\varepsilon}{\tau} + \frac{1}{2} = w$ for all ε small, since in this case at every time step we may choose any of the two minimizers giving the extremal velocities. Note therefore that in this case the limit is not determined only by w, and in particular it mat depend on the subsequence even if the limit (7.4) exists.

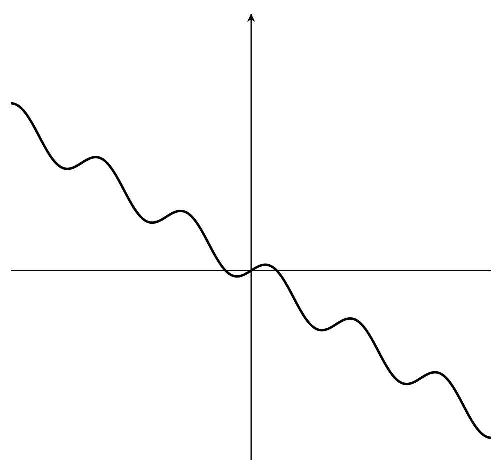

Figure 7.4: other potentials giving the same homogenization pattern

We remark that the functions F_ε above can be substituted by functions with isolated local minimizers; e.g. by taking ($\alpha > 0$)

$$F_\varepsilon(x) = -\Big\lfloor \frac{x}{\varepsilon} \Big\rfloor \varepsilon + \alpha \Big(x - \Big\lfloor \frac{x}{\varepsilon} \Big\rfloor \varepsilon \Big),$$

with isolated local minimizers at $\varepsilon\mathbb{Z}$ (for which the computations run exactly as above), or

$$F_\varepsilon(x) = -x + (1 + \alpha)\varepsilon \sin\Big(\frac{x}{\varepsilon} \Big) .$$

Note that the presence of an energy barrier between local minimizers does not influence the velocity of the final minimizing movement, that can always be larger than 1 (the velocity as $\varepsilon << \tau$).

7.2 Homogenization of flat flows

We now consider geometric functionals with many local minimizers, which give a more refined example of homogenization. The functionals we consider are defined on (sufficiently regular) subsets of $\mathbb{R}^2$ by

$$F_\varepsilon(A) = \int_{\partial A} a\Big(\frac{x}{\varepsilon}\Big) d\mathcal{H}^1, \tag{7.5}$$

where

$$a(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 \in \mathbb{Z} \text{ or } x_2 \in \mathbb{Z} \\ 4 & \text{otherwise.} \end{cases}$$

Here, for simplicity we consider the value 4, so that arguments are clearer, but this value is not optimal and can be lowered. This is however not relevant for our purposes. The Γ-limit of the energies F_ε is the *crystalline energy*

$$F(A) = \int_{\partial A} \|\nu\|_1 d\mathcal{H}^1, \tag{7.6}$$

with $\|(\nu_1, \nu_2)\|_1 = |\nu_1| + |\nu_2|$. The minimizing movement for F is called a *flat flow*. We will first briefly describe it, and then compare it with the minimizing movements for F_ε.

7.2.1 Motion by crystalline curvature

The incremental problems for the minimizing-movement scheme for F are of the form

$$\min\Big\{ F(A) + \frac{1}{\tau} \int_{A \triangle A_{k-1}} \mathrm{dist}_\infty(x, \partial A_{k-1})\, dx \Big\}, \tag{7.7}$$

where for technical reasons we consider the ∞-distance

$$\mathrm{dist}_\infty(x, B) = \inf\{\|x-y\|_\infty : y \in B\}.$$

However, again this will not be relevant in our computations.

We only consider the case of an initial datum A_0 a rectangle, which plays the role played by a ball for motion by mean curvature. Note that, as in Section 6.2, we can prove that if A_{k-1} is a rectangle, then we can limit the computation in (7.7) to

- A contained in A_{k-1} (otherwise $A \cap A_{k-1}$ strictly decreases the energy)
- A with each connected component a rectangle (otherwise taking the least rectangle containing a given component would decrease the energy, strictly if A is not a rectangle);

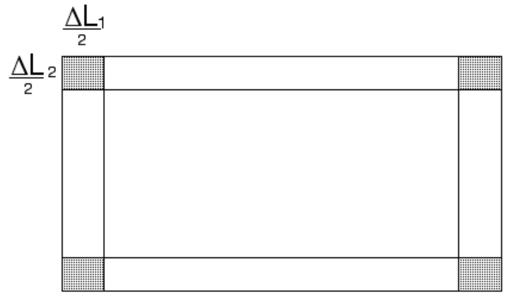

Figure 7.5: incremental crystalline minimization

• A connected and with the same baricenter as A_0 (since translating the baricenter towards 0 decreases the energy).

Hence, we may suppose that

$$A_k = \Big[-\frac{L_{k,1}}{2}, \frac{L_{k,1}}{2}\Big] \times \Big[-\frac{L_{k,2}}{2}, \frac{L_{k,2}}{2}\Big]$$

for all k. In order to iteratively determine L_k we have to minimize the energy

$$\min\Big\{2(L_{k,1} + \Delta L_1) + 2(L_{k,2} + \Delta L_2) + \frac{1}{\tau}\int_{A\triangle A_{k-1}} \mathrm{dist}_\infty(x, \partial A_{k-1})\, dx\Big\}. \qquad (7.8)$$

In this computation it is easily seen that for τ small the integral term can be substituted by

$$\frac{L_{k,1}}{4}(\Delta L_2)^2 + \frac{L_{k,2}}{4}(\Delta L_1)^2.$$

This argument amounts to noticing that the contribution of the small rectangles at the corners highlighted in Figure 7.5 is negligible as $\tau \to 0$. The optimal increments (more precisely, decrements) ΔL_j are then determined by the conditions

$$\begin{cases} 1 + \dfrac{L_{k,2}}{4\tau}\Delta L_1 = 0 \\ 1 + \dfrac{L_{k,1}}{4\tau}\Delta L_2 = 0. \end{cases} \qquad (7.9)$$

Hence, we have the difference equations

$$\frac{\Delta L_1}{\tau} = -\frac{4}{L_{k,2}}, \qquad \frac{\Delta L_2}{\tau} = -\frac{4}{L_{k,1}}, \qquad (7.10)$$

which finally gives the system of ODEs for the limit rectangles, with edges of length $L_1(t)$ and $L_2(t)$ respectively,

$$\begin{cases} L_1' = -\dfrac{4}{L_2} \\ L_2' = -\dfrac{4}{L_1}. \end{cases} \tag{7.11}$$

Geometrically, each edge of the rectangle moves inwards with velocity inversely proportional to its length; more precisely, equal to twice the inverse of its length (so that the other edge contracts with twice this velocity). Hence, the inverse of the length of an edge plays the role of the curvature in this context (crystalline curvature).

It is worth noticing that by (7.11) all rectangles are homothetic, since $\frac{d}{dt}\frac{L_1}{L_2} = 0$, and with area satisfying

$$\frac{d}{dt}L_1L_2 = -8,$$

so that $L_1(t)L_2(t) = L_{0,1}L_{0,2} - 8t$, which gives the extinction time $t = L_{0,1}L_{0,2}/8$. In the case of an initial datum a square of side length L_0, the sets are squares whose side length at time t is given by $L(t) = \sqrt{L_0^2 - 8t}$ in analogy with the evolution of balls by mean curvature flow.

7.2.2 Homogenization of oscillating perimeters

We consider the sequence F_ε in (7.5). Note that for any (sufficiently regular) initial datum A_0 we have that $A'_\varepsilon \subset A_0 \subset A''_\varepsilon$, where A'_ε and A''_ε are such that $F_\varepsilon(A'_\varepsilon) = \mathcal{H}^1(\partial A'_\varepsilon)$ and $F_\varepsilon(A''_\varepsilon) = \mathcal{H}^1(\partial A''_\varepsilon)$ and $|A''_\varepsilon \setminus A'_\varepsilon| = O(\varepsilon)$. Such sets are local minimizers for F_ε and hence the minimizing movement of F_ε from either of them is trivial. As a consequence, if $A_\varepsilon(t)$ is a minimizing movement for F_ε from A_0 we have

$$A'_\varepsilon \subset A_\varepsilon(t) \subset A''_\varepsilon$$

This shows that for any set A_0 the only limit $\lim_{\varepsilon\to 0} A_\varepsilon(t)$ of minimizing movements for F_ε from A_0 is the trivial motion $A(t) = A_0$.

We now compute the minimizing movements along the sequence F_ε with initial datum a rectangle, and compare it with the flat flow described in the previous section.

For simplicity of computation we deal with a constrained case, when

• for every ε the initial datum $A_0 = A_0^\varepsilon$ is a rectangle centered in 0 such that $F_\varepsilon(A) = \mathcal{H}^1(\partial A)$ (i.e., its edge lengths $L_{0,j}$ belong to $2\varepsilon\mathbb{Z}$). In analogy with x_0 in the example in Section 7.1, if this does not hold then either it does after one iteration or we have a pinned state $A_k = A_0$ for all k;

• all competing A are rectangles with $F_\varepsilon(A) = \mathcal{H}^1(\partial A)$ centered in 0. The fact that all competing sets are rectangles follows as for the flat flow in the previous section. The fact that $F_\varepsilon(A_k) \le F_\varepsilon(A_{k-1})$ then implies that the minimal rectangles satisfy $F_\varepsilon(A_k) = \mathcal{H}^1(\partial A_k)$. The only real assumption at this point is that they are centered in 0. This hypothesis can be removed, upon a slightly more complex computation, which would only make the arguments less clear.

After this simplifications, the incremental problem is exactly as in (7.7) since for competing sets we have $F_\varepsilon(A) = F(A)$, the only difference being that now $L_{k,1}, L_{k,2} \in 2\varepsilon\mathbb{Z}$. The problem in terms of $\triangle L_j$, using the same simplification for (7.8) as in the previous section, is then

$$\min\Big\{2(L_{k,1}+\Delta L_1)+2(L_{k,2}+\Delta L_2)+\frac{L_{k,1}}{4\tau}(\Delta L_2)^2+\frac{L_{k,2}}{4\tau}(\Delta L_1)^2 : \Delta L_j \in 2\varepsilon\mathbb{Z}\Big\}. \quad (7.12)$$

This is a minimization problem for a parabola as the ones in Section 7.1 that gives

$$\Delta L_1 = -\Big\lfloor \frac{4\tau}{\varepsilon L_{k,2}} + \frac{1}{2}\Big\rfloor \varepsilon \text{ if } \frac{4\tau}{\varepsilon L_{k,2}} + \frac{1}{2} \notin \mathbb{Z} \quad (7.13)$$

(the other cases giving two solutions), and an analogous equation for ΔL_2. Passing to the limit we have the system of ODEs, governed by the parameter

$$w = \lim_{\varepsilon\to 0}\frac{\tau}{\varepsilon}$$

(which we may suppose up to subsequences), which reads as

$$\begin{cases} L_1' = -\dfrac{1}{w}\Big\lfloor \dfrac{4w}{L_2} + \dfrac{1}{2}\Big\rfloor \\ L_2' = -\dfrac{1}{w}\Big\lfloor \dfrac{4w}{L_1} + \dfrac{1}{2}\Big\rfloor . \end{cases} \quad (7.14)$$

Note that the right-hand side is a discontinuous function of L_j, so some care must be taken at times t when $\frac{4w}{L_j(t)} + \frac{1}{2} \in \mathbb{Z}$. However, apart some exceptional cases, this condition holds only for a countable number of t, and is therefore negligible.

We can compare the resulting minimizing movements with the crystalline curvature flow, related to F.

• **(total pinning)** if $\tau << \varepsilon$ ($w = 0$) then we have $A(t) = A_0$;

• **(crystalline curvature flow)** if $\varepsilon << \tau$ then we have the minimizing movements described in the previous section;

• **(partial pinning/asymmetric curvature flow)** if $0 < w < +\infty$ then we have

(i) (*total pinning*) if both $L_{0,j} > 8w$ then the motion is trivial $A(t) = A_0$;

(ii) (*partial pinning*) if $L_{0,1} > 8w$, $L_{0,2} < 8w$ and $\frac{4w}{L_{0,2}} + \frac{1}{2} \notin \mathbb{Z}$ then the horizontal edges do not move, but they contract with constant velocity until $L_1(t) = 8w$;

(iii) (*asymmetric curvature flow*) if $L_{0,1} \le 8w$ and $L_{0,2} < 8w$ then we have a unique motion with $A(t) \subset\subset A(s)$ if $t > s$, up to a finite extinction time. Note however that the sets $A(s)$ are nor homothetic, except for the trivial case when A_0 is a square.

Some cases are not considered above, namely those when we do not have uniqueness of minimizers in the incremental problem. This may lead to a multiplicity of minimizing movements, as remarked in Section 7.1.

It is worthwhile to highlight that we may rewrite the equations for L'_j as a variation of the crystalline curvature flow; e.g., for L'_1 we can write it as

$$L'_1 = -f\Big(\frac{L_2}{w}\Big)\frac{4}{L_2}, \qquad \text{with } f(z) = \frac{z}{4}\Big\lfloor\frac{4}{z}+\frac{1}{2}\Big\rfloor .$$

This suggests that the 'relevant' homogenized problem is the one obtained for $\frac{\tau}{\varepsilon}=1$, as all the others can be obtained from this one by a scaling argument.

We note that the scheme can be applied to the evolution of more general sets, but the analysis of the rectangular case already highlights the new features deriving from the microscopic geometry.

7.3 Flat flow with oscillating forcing term

We now consider another minimizing-movement scheme linked to the functional F in (7.6). In this case the oscillations are given by a lower-order forcing term. We consider, in $\mathbb{R}^2$,

$$G_\varepsilon(A) = \int_{\partial A}\|\nu\|_1 d\mathcal{H}^1 + \int_A g\Big(\frac{x_1}{\varepsilon}\Big)dx, \tag{7.15}$$

where g is 1-periodic and

$$g(s) = \begin{cases}\alpha & \text{if } 0<x<\frac12\\ \beta & \text{if } \frac12<x<1\end{cases}$$

with $\alpha,\beta\in\mathbb{R}$ and $\alpha<\beta$. Note that the additional term may be negative, so that this functional is not positive; however, the minimizing-movement scheme can be applied unchanged.

Since the additional term converges continuously in L^1 as $\varepsilon\to 0$, the Γ-limit is simply

$$G(A) = \int_{\partial A}\|\nu\|_1 d\mathcal{H}^1 + \frac{\alpha+\beta}{2}|A|. \tag{7.16}$$

7.3.1 Flat flow with forcing term

We now consider minimizing movements for G. As in Section 7.2.1 we only deal with a constrained problem, when both the initial datum and the competing sets are rectangles centered in 0. With the notation of Section 7.2.1 we are led to the minimum problem

$$\min\Big\{2(L_{k,1}+\Delta L_1+L_{k,2}+\Delta L_2)+\frac{L_{k,1}}{4\tau}(\Delta L_2)^2+\frac{L_{k,2}}{4\tau}(\Delta L_1)^2+\frac{\alpha+\beta}{2}(L_{k,1}+\Delta L_1)(L_{k,2}+\Delta L_2)\Big\}. \tag{7.17}$$

The minimizing pair $(\Delta L_1, \Delta L_2)$ satisfies

$$\frac{\Delta L_1}{\tau} = -\Big(\frac{4}{L_{k,2}} + (\alpha+\beta)\Big(1 + \frac{\Delta L_2}{L_{k,2}}\Big)\Big) \tag{7.18}$$

and the analogous equation for $\frac{\Delta L_2}{\tau}$. Passing to the limit we have

$$\begin{cases} L_1' = -\Big(\frac{4}{L_2} + \alpha + \beta\Big) \\ L_2' = -\Big(\frac{4}{L_1} + \alpha + \beta\Big), \end{cases} \tag{7.19}$$

so that each edge moves with velocity $\frac{2}{L_2} + \frac{\alpha+\beta}{2}$, with the convention that it moves inwards if this number is positive, outwards if it is negative.

Note that if $\alpha + \beta \geq 0$ then L_1 and L_2 are always decreasing and we have finite-time extinction, while if $\alpha + \beta < 0$ then we have an equilibrium for $L_j = \frac{4}{|\alpha+\beta|}$, and we have expanding rectangles, with an asymptotic velocity of each side of $\frac{|\alpha+\beta|}{2}$ as the side length diverges.

7.3.2 Homogenization of forcing terms

In order to highlight new homogenization phenomena, we treat the case $\tau << \varepsilon$ only. Again, we consider the constrained case when both the initial datum and the competing sets are rectangles centered in 0 and adopt the notation of Section 7.2.1.

Taking into account that $\tau << \varepsilon$ the incremental minimum problem can be approximated by

$$\begin{aligned} \min\Big\{ & 2(L_{k,1} + \Delta L_1 + L_{k,2} + \Delta L_2) + \frac{L_{k,1}}{4\tau}(\Delta L_2)^2 + \frac{L_{k,2}}{4\tau}(\Delta L_1)^2 \\ & + \frac{\alpha+\beta}{2} L_{k,1} L_{k,2} + \frac{\alpha+\beta}{2} L_{k,1} \Delta L_2 + g\Big(\frac{L_{k,1}}{2\varepsilon}\Big) L_{k,2} \Delta L_1 \Big\}. \end{aligned} \tag{7.20}$$

In considering the term $g\Big(\frac{L_{k,1}}{2\varepsilon}\Big)$ we assume implicitly that τ is so small that both $\frac{L_{k,1}}{2\varepsilon}$ and $\frac{L_{k,1}+\Delta L_1}{2\varepsilon}$ belong to the same interval where g is constant. This can be assumed up to a number of k that is negligible as $\tau \to 0$.

For the minimizing pair of (7.20) we have

$$\begin{cases} 2 + \dfrac{L_{k,2}}{2\tau} \Delta L_1 + g\Big(\dfrac{L_{k,1}}{2\varepsilon}\Big) L_{k,2} = 0 \\ 2 + \dfrac{L_{k,1}}{2\tau} \Delta L_2 + \dfrac{\alpha+\beta}{2} L_{k,1} = 0; \end{cases} \tag{7.21}$$

that is,

$$\begin{cases} \dfrac{\Delta L_1}{\tau} = -\Big(\dfrac{4}{L_{k,2}} + 2g\Big(\dfrac{L_{k,1}}{2\varepsilon}\Big)\Big) \\ \dfrac{\Delta L_2}{\tau} = -\Big(\dfrac{4}{L_{k,1}} + (\alpha+\beta)\Big). \end{cases} \tag{7.22}$$

This systems shows that the horizontal edges move with velocity $\frac{2}{L_{k,1}} + \frac{\alpha+\beta}{2}$, while the velocity of the vertical edges depends on the location of the edge and is

$$\frac{2}{L_{k,2}} + g\Big(\frac{L_{k,1}}{2\varepsilon}\Big).$$

We then deduce that the limit velocity for the horizontal edges of length L_1 is

$$\frac{2}{L_1} + \frac{\alpha+\beta}{2} \tag{7.23}$$

As for the vertical edges, we have:

• **(mesoscopic pinning)** if L_2 is such that

$$\Big(\frac{2}{L_2} + \alpha\Big)\Big(\frac{2}{L_2} + \beta\Big) < 0$$

then the vertical edge is eventually pinned in the minimizing-movement scheme. This pinning is not due to the equality $L_{k+1,1} = L_{k,1}$ in the incremental problem, but to the fact that the vertical edge move in different directions depending on the value of g;

• **(homogenized velocity)** if on the contrary the vertical edge length satisfies

$$\Big(\frac{2}{L_2} + \alpha\Big)\Big(\frac{2}{L_2} + \beta\Big) > 0$$

then we have a limit *effective velocity* of the vertical edge given by the harmonic mean of the two velocities $\frac{2}{L_2} + \alpha$ and $\frac{2}{L_2} + \beta$; namely,

$$\frac{(2+\alpha L_2)(2+\beta L_2)}{L_2\Big(2 + \frac{\alpha+\beta}{2}L_2\Big)}\,. \tag{7.24}$$

We finally examine some cases explicitly.

(i) Let $\alpha = -\beta$. Then we have

$$\begin{cases} L_2' = -\dfrac{4}{L_1} \\ L_1' = -2\dfrac{(2-\beta L_2)\vee 0}{L_2}\,; \end{cases}$$

i.e., the vertical edges are pinned if their length is larger than $2/\beta$. In this case, the horizontal edges move inwards with constant velocity $\frac{2}{L_{0,1}}$. In this way the vertical edges shrink with rate $\frac{4}{L_{0,1}}$ until their length is $2/\beta$. After this, the whole rectangle shrinks in all directions.

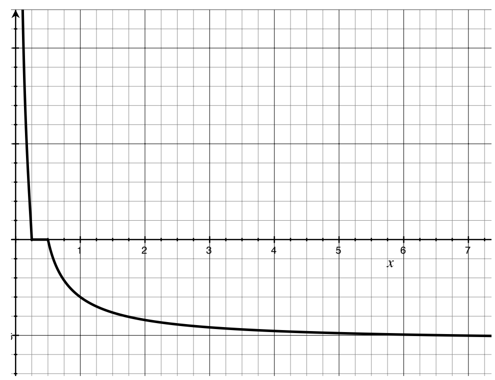

Figure 7.6: velocity with an interval of mesoscopic pinning

(ii) Let $\alpha < \beta < 0$. Then for the vertical edges we have an interval of "mesoscopic pinning" corresponding to

$$\frac{2}{|\beta|} \leq L_2 \leq \frac{2}{|\alpha|} \tag{7.25}$$

The velocity of the vertical edges in dependence of their length is then given by

$$v = \begin{cases} 0 & \text{if (7.25) holds} \\ \frac{(2+\alpha L_2)(2+\beta L_2)}{L_2\left(2+\frac{\alpha+\beta}{2}L_2\right)} & \text{otherwise} \end{cases}$$

and is pictured in Figure 7.6. Instead, the velocity of the horizontal edges is given by (7.23), so that they move inwards if

$$L_1 < \frac{4}{|\alpha+\beta|},$$

and outwards if $L_1 > \frac{4}{|\alpha+\beta|}$.

In this case we can consider as initial datum a square of side length L_0.

If $L_0 \leq \frac{2}{|\beta|}$ then all edges move inwards until a finite extinction time;

if $\frac{2}{|\beta|} < L_0 < \frac{4}{|\alpha+\beta|}$ then first only the horizontal edges move inwards until the vertical edge reaches the length $\frac{2}{|\beta|}$, after which all edges move inwards;

if $\frac{4}{|\alpha+\beta|} < L_0 < \frac{2}{|\alpha|}$ then first only the horizontal edges move outwards until the vertical edge reaches the length $\frac{2}{|\alpha|}$, after which all edges move outwards;

if $L_0 \geq \frac{2}{|\alpha|}$ then all edges move outwards, and the motion is defined for all times. The asymptotic velocity of the vertical edges as the length of the edges diverges is

$$\left|\frac{2\alpha\beta}{\alpha+\beta}\right|,$$

lower than $\left|\frac{\alpha+\beta}{2}\right|$ (the asymptotic velocity for the horizontal edges).

Chapter 8

Different time scales

In this chapter we treat some variations on the minimizing-movement scheme motivated by some time-scaling argument.

8.1 Long-time behaviour

We will introduce a new parameter λ and follow the iterative minimizing scheme from an initial datum x_0 by considering x_k defined recursively as a minimizer of

$$\min\Big\{F_\varepsilon(x) + \frac{\lambda}{2\tau}\|x - x_{k-1}\|^2\Big\}, \tag{8.1}$$

and setting $u^\tau(t) = u^{\tau,\lambda}(t) = x_{\lfloor t/\tau \rfloor}$. Equivalently, we may view this as applying the minimizing-movement scheme to

$$\min\Big\{\frac{1}{\lambda}F_\varepsilon(x) + \frac{1}{2\tau}\|x - x_{k-1}\|^2\Big\}. \tag{8.2}$$

Note that we may compare this scheme with the usual one where x_i are defined as minimizers of the minimizing-movement scheme with time scale $\eta = \tau/\lambda$ giving u^η as a discretization with lattice step η. Then we have

$$u^\tau(t) = x_{\lfloor t/\tau \rfloor} = x_{\lfloor t/\lambda\eta \rfloor} = u^\eta\Big(\frac{t}{\lambda}\Big).$$

Hence, the introduction of λ corresponds to a scaling of time.

We now first give some simple examples which motivate the study of time-scaled problems.

Example 8.1.1 Consider in $\mathbb{R}^2$ the energy

$$F_\varepsilon(x, y) = \frac{1}{2}(x^2 + \varepsilon y^2).$$

The corresponding gradient flow is then

$$\begin{cases} x' = -x \\ y' = -\varepsilon y, \end{cases}$$

with solutions of the form

$$(x_\varepsilon(t), y_\varepsilon(t)) = (x_0 e^{-t}, y_0 e^{-\varepsilon t}).$$

These solutions converge to $(x(t), y(t)) = (x_0 e^{-t}, y_0)$, solving

$$\begin{cases} x' = -x \\ y' = 0, \end{cases}$$

which is the gradient flow of the limit $F(x, y) = \frac{1}{2}x^2$. Note that

$$\lim_{t\to+\infty}(x_\varepsilon(t), y_\varepsilon(t)) = (0, 0) \neq (0, y_0) = \lim_{t\to+\infty}(x(t), y(t)).$$

The trajectories of the solutions $(x_\varepsilon, y_\varepsilon)$ lie on the curves

$$\frac{y}{y_0} = \left(\frac{x}{x_0}\right)^\varepsilon$$

and are pictured in Fig. 8.1.

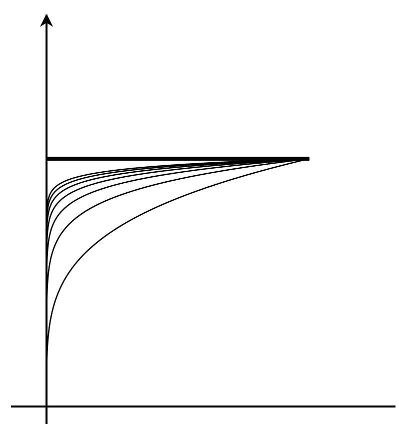

Figure 8.1: trajectories of the solutions, and their pointwise limit

The solutions can be seen as superposition of $(x(t), y(t))$ and $\varepsilon(x_\infty(t), y_\infty(t))$, where

$$(x_\infty(t), y_\infty(t)) := (0, e^{-t})$$

is the solution of

$$\begin{cases} x' = 0 \\ y' = -y \\ (x(0), y(0)) = (0, y_0). \end{cases}$$

The solution (x_∞, y_∞) can be obtained by scaling $(x_\varepsilon, y_\varepsilon)$; namely,

$$(x_\infty(t), y_\infty(t)) = \lim_{\varepsilon\to 0}(x_\varepsilon(t/\varepsilon), y_\varepsilon(t/\varepsilon)).$$

In this case the scaled time-scale is $\lambda = \varepsilon$. Note that the limit of the scaled solutions does not satisfy the original initial condition, but its "projection" on the set of (local) minimizers of the limit energy F (or, in other words, the domain of the limit of the energies $\frac{1}{\varepsilon}F_\varepsilon$).

Example 8.1.2 A similar example can be constructed in one dimension, taking, e.g.,

$$F_\varepsilon(x) = \frac{\varepsilon}{2}x^2 + \frac{1}{2}((|x|-1)\vee 0)^2\,.$$

If $x_0 < -1$ then the corresponding solutions x_ε satisfy:

- the limit $x(t) = \lim_{\varepsilon\to 0} x_\varepsilon(t)$ solves

$$\begin{cases} x' = -x-1 \\ x(0) = x_0, \end{cases}$$

which corresponds to the gradient flow of the energy

$$F(x) = \frac{1}{2}((|x|-1)\vee 0)^2\,.$$

- the scaled limit $x_\infty(t) = \lim_{\varepsilon\to 0} x_\varepsilon(t/\varepsilon)$ solves

$$\begin{cases} x' = -x \\ x(0) = -1, \end{cases}$$

which corresponds to the gradient flow of the energy

$$F_\infty(x) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}F_\varepsilon(x)\,.$$

In this case the initial datum is the projection of x_0 on the domain of F_∞.

We now give more examples with families of energies F_ε Γ-converging to a limit F. Since we are mainly interested in highlighting the existence of a time scale at which the scaled motion is not trivial, we will make some simplifying assumptions, one of which is that the initial datum be a local minimizer for F, so that the minimizing movement for the limit from that point is trivial.

Example 8.1.3 We take as F the 1D Mumford-Shah functional on $(0,1)$ defined by

$$F(u) = \int_0^1 |u'|^2 dt + \#(S(u)),$$

with domain the set of piecewise-H^1 functions. We take

$$F_\varepsilon(u) = \int_0^1 |u'|^2 dt + \sum_{(S(u))} g\Big(\frac{|u^+ - u^-|}{\varepsilon}\Big),$$

where g is a positive concave function with

$$\lim_{z\to+\infty} g(z) = 1.$$

We also consider the boundary conditions

$$u(0-) = 0, \qquad u(1+) = 1.$$

We suppose that

- u_0 is a local minimizer for F; i.e., it is piecewise constant;
- $\#(S(u_0)) = \{x_0, x_1\}$ (the simplest non-trivial local minimizer) with $0 \le x_0 < x_1 \le 1$;
- competing functions are also piecewise constant.

With these conditions, it is immediately seen that all minimizers u_k obtained by iterative minimization satisfy:

- $S(u_k) \subset \{x_0, x_1\}$.

We may use as a one-dimensional parameter the constant value z_k of u_k on (x_0, x_1). The minimum problem defining z_k is then (supposing that $z_0 > 0$ so that all $z_k > 0$)

$$\min\Big\{\frac{1}{\lambda}\Big(g\Big(\frac{z}{\varepsilon}\Big) + g\Big(\frac{1-z}{\varepsilon}\Big)\Big) + \frac{1}{2\tau}(x_1 - x_0)(z - z_{k-1})^2\Big\},$$

which gives

$$(x_1 - x_0)\frac{z_k - z_{k-1}}{\tau} = -\frac{1}{\varepsilon\lambda}\Big(g'\Big(\frac{z_k}{\varepsilon}\Big) - g'\Big(\frac{1-z_k}{\varepsilon}\Big)\Big).$$

As an example, we may take

$$g(z) = \frac{z}{1+z},$$

so that the equation for z_k becomes

$$(x_1 - x_0)\frac{z_k - z_{k-1}}{\tau} = -\frac{\varepsilon}{\lambda}\Big(\frac{1}{\varepsilon^2 + z_k^2} - \frac{1}{\varepsilon^2 + (z_k - 1)^2}\Big).$$

This suggests the scale

$$\lambda = \varepsilon,$$

and with this choice gives the limit equation for $z(t)$

$$z' = -\frac{1 - 2z}{(x_1 - x_0)z^2(z-1)^2}.$$

In this time scale, unless we are in the equilibrium $z = \frac{1}{2}$ the middle value moves towards the closest value between 0 and 1.

Example 8.1.4 We consider another approximation of the Mumford-Shah functional: the (scaled) Perona-Malik functional. In the notation for discrete functionals (see Section 4.4), we may define

$$F_\varepsilon(u) = \sum_{i=1}^{N} \frac{1}{|\log\varepsilon|} \log\Big(1 + \varepsilon|\log\varepsilon|\Big|\frac{u_i - u_{i-1}}{\varepsilon}\Big|^2\Big).$$

Note that also the pointwise limit on piecewise-H^1 functions gives the Mumford-Shah functional since

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon|\log\varepsilon|} \log\Big(1 + \varepsilon|\log\varepsilon|z^2\Big) = z^2$$

and

$$\lim_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|} \log\Big(1 + |\log\varepsilon|\frac{w^2}{\varepsilon}\Big) = 1$$

for all $w \neq 0$.

As in the previous example, we consider the case when competing functions are non-negative piecewise constants with $S(u) \subset S(u_0) = \{x_0, x_1\}$ and with the boundary conditions $u(0-) = 0, u(1+) = 1$. The computation is then reduced to a one-dimensional problem with unknown the constant value z_k defined by the minimization

$$\min\Big\{\frac{1}{\lambda|\log\varepsilon|}\Big(\log\Big(1 + |\log\varepsilon|\frac{z^2}{\varepsilon}\Big) + \log\Big(1 + |\log\varepsilon|\frac{(z-1)^2}{\varepsilon}\Big)\Big) + \frac{1}{2\tau}(x_1 - x_0)(z - z_{k-1})^2\Big\},$$

which gives the equation

$$(x_1 - x_0)\frac{z_k - z_{k-1}}{\tau} = -\frac{2}{\lambda}\Big(\frac{z}{\varepsilon + |\log\varepsilon|z^2} + \frac{z-1}{\varepsilon + |\log\varepsilon|(z-1)^2}\Big).$$

This suggests the time scale

$$\lambda = \frac{1}{|\log\varepsilon|},$$

and gives the equation for $z(t)$

$$z' = -\frac{2}{(x_1 - x_0)} \cdot \frac{1 - 2z}{z(1-z)},$$

which provides a qualitative behaviour of z similar to the previous example.

Example 8.1.5 We now consider the sharp-interface model with

$$F(u) = \#(S(u) \cap [0, 1))$$

defined on all piecewise-constant 1-periodic functions with values in ± 1. For F all functions are local minimizers.

We take

$$F_\varepsilon(u) = \#(S(u) \cap [0,1)) - \sum_{x_i \in [0,1) \cap S(u)} e^{-\frac{x_{i+1}-x_i}{\varepsilon}},$$

where $\{x_i\} = S(u)$ is a numbering of $S(u)$ with $x_i < x_{i+1}$.

We take as initial datum u_0 with $\#(S(u_0)) = 2$; hence, $S(u_0) = \{x_0, y_0\}$, and apply the Almgren-Taylor-Wang variant of the iterative minimization process, after identifying u_0 with $A_0 = [x_0, y_0]$. The computation of $A_1 = [x_1, y_1]$ is obtained by the minimization problem

$$\min\Big\{-\frac{1}{\lambda}\Big(e^{-\frac{(y-x)}{\varepsilon}} + e^{-\frac{(1+x-y)}{\varepsilon}}\Big) + \frac{1}{2\tau}((x-x_0)^2 + (y-y_0)^2)\Big\},$$

which gives

$$\begin{aligned}
\frac{x_1 - x_0}{\tau} &= \frac{1}{\varepsilon\lambda}\Big(e^{-\frac{(y_1-x_1)}{\varepsilon}} - e^{-\frac{(1+x_1-y_1)}{\varepsilon}}\Big)\\
\frac{y_1 - y_0}{\tau} &= -\frac{1}{\varepsilon\lambda}\Big(e^{-\frac{(y_1-x_1)}{\varepsilon}} - e^{-\frac{(1+x_1-y_1)}{\varepsilon}}\Big).
\end{aligned}$$

Let $y_0 - x_0 < 1/2$; we argue that the scaled time scale is

$$\lambda = \frac{1}{\varepsilon} e^{-\frac{y_0-x_0}{\varepsilon}},$$

for which we have

$$\begin{aligned}
\frac{x_1 - x_0}{\tau} &= \Big(e^{-\frac{(y_1-y_0-x_1+x_0)}{\varepsilon}} - e^{-\frac{(1+x_1-x_0-y_1+y_0)}{\varepsilon}}\Big)\\
\frac{y_1 - y_0}{\tau} &= -\Big(e^{-\frac{(y_1-y_0-x_1+x_0)}{\varepsilon}} - e^{-\frac{(1+x_1-x_0-y_1+y_0)}{\varepsilon}}\Big).
\end{aligned}$$

In terms of $L_k = y_k - x_k$ this can be written as

$$\frac{L_1 - L_0}{\tau} = -2\Big(e^{-\frac{(L_1-L_0)}{\varepsilon}} - e^{-\frac{(1+L_0-L_1)}{\varepsilon}}\Big).$$

Under the assumption $\tau << \varepsilon$ we have in the limit

$$L' = -2\Big(e^{o(1)} - e^{-\frac{1}{\varepsilon}+o(1)}\Big) = -2\,,$$

which shows that the two closer interfaces move towards each other shortening linearly their distance.

8.2 Reversed time

In a finite-dimensional setting a condition to be able to define a minimizing movement for F is that

$$u \mapsto F(u) + \frac{1}{\tau}|u - \overline{u}|^2 \tag{8.3}$$

be lower semicontinuous and coercive for all $\overline{u}$ and for τ sufficiently small. This is not in contrast with requiring that also

$$u \mapsto -F(u) + \frac{1}{\tau}|u - \overline{u}|^2 \tag{8.4}$$

satisfy the same conditions; for example if F is continuous and of quadratic growth. If the iterative scheme gives a solution for the gradient flow, a minimizing movement u for the second scheme produces a solution $v(t) = u(-t)$ to the backward problem

$$\begin{cases} v'(t) = -F(v(t)) & \text{for } t \le 0 \\ v(0) = u_0 \end{cases}$$

In an infinite-dimensional setting the two requirements of being able to define both the minimizing movement (8.3) and (8.4) greatly limits the choice of F, and rules out all interesting cases. A possible approach to the definition of a backward minimizing movement is then to introduce a (finite-dimensional) approximation F_ε to F, for which we can define a minimizing motion along $-F_\varepsilon$. We now give an example in the context of crystalline motion.

Example 8.2.1 We consider in $\mathbb{R}^2$

$$F(A) = \int_{\partial A} \|\nu\|_1 d\mathcal{H}^1,$$

and F_ε the restriction of F to the sets of the form

$$\bigcup \Big\{ \varepsilon i + \Big(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\Big)^2 : i \in B \Big\},$$

where B is a subset of $\mathbb{Z}^2$. Hence, we may identify these union of ε-cubes with the corresponding B. Even though this is not a finite-dimensional space, we will be able to apply the Almgren-Taylor-Wang scheme.

We choose (with the identifications with subsets of $\mathbb{Z}^2$) as initial datum

$$A_0^\varepsilon = \{(0,0)\} = \Big(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\Big)^2,$$

and solve iteratively

$$\min \Big\{ -F_\varepsilon(A) + \frac{1}{\tau} \int_{A \setminus A_{k-1}^\varepsilon} \mathrm{dist}_\infty(x, \partial A_{k-1}^\varepsilon)\, dx \Big\}.$$

In the interpretattion as a reversed-time scheme, this means that we are solving a problem imposing the extinction at time 0.

Note that taking F in place of F_ε would immediately give the value $-\infty$ in the minimum problem above; e.g., by considering sets of the form (in polar coordinates)

$$A_j = \{(\rho,\theta) : \rho \leq 3\varepsilon + \varepsilon \sin(j\theta)\},$$

which contain A_0^ε, are contained in $B_{4\varepsilon}(0)$ and have a perimeter larger than $4j\varepsilon$ (see Fig. 8.2).

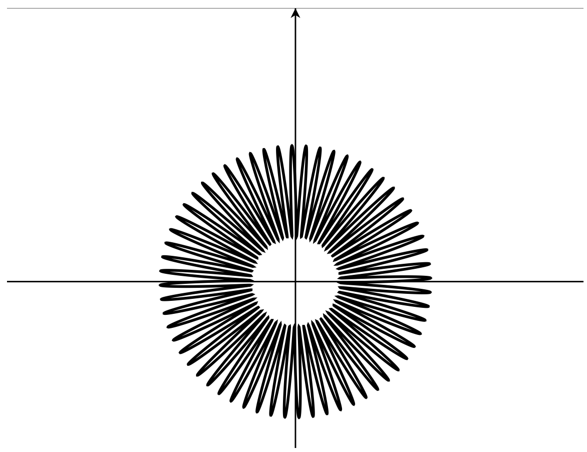

Figure 8.2: small sets with large perimeter

Under the assumption that $\varepsilon << \tau$ we easily see that all minimizing sets are the checkerboard structure corresponding to indices $i \in \mathbb{Z}^2$ with $i_1 + i_2$ even contained in a square Q_k centered in 0 (see Fig. 8.3). We may take the sides L_k of those squares as

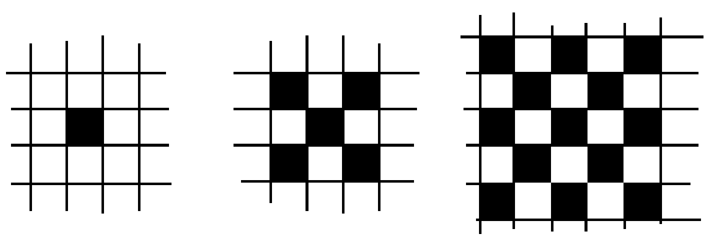

Figure 8.3: enucleating sets

unknown. The incremental problems can be rewritten as

$$\min\Big\{-\frac{2}{\varepsilon}((L_{k-1}+\Delta L)^2 - L_{k-1}^2) + \frac{1}{\tau}(L_{k-1}(\Delta L)^2 + r_k(\Delta L)^2)\Big\},$$

with r_k negligible as $\tau \to 0$. For the interfacial part, we have taken into account that for ε small the number of squares contained in a rectangle is equal to its area divided by $2\varepsilon^2$ and each of the squares gives an energy contribution of 4ε; for the distance part, we note that the integral can be equivalently taken on half of $Q_k \setminus Q_{k-1}$. Minimization in ΔL gives

$$\frac{\Delta L}{\tau} = \frac{2}{\varepsilon}\Big(1 + \frac{\Delta L}{L_{k-1}}\Big).$$

Letting $\tau \to 0$ this motion blows up. However, we may scale the time, formally considering $s = \varepsilon\tau$. In this slow time scale the growth is linear:

$$L(s) = 2s\,.$$

What we have obtained is the description of the structure of ε-squares (the checkerboard one) along which the increase of the perimeter is maximal (and, in a sense, the decrease of the perimeter is maximal for the reverse-time problem).

Chapter 9

Stability theorems

We now face the problem of determining conditions under which the minimizing-movement scheme commutes with Γ-convergence. Let F_ε Γ-converge to F with initial data x_ε converging to x_0. The first question is whether by choosing suitably $\varepsilon = \varepsilon(\tau)$ the minimizing movement along the sequence F_ε from x_ε converges to a minimizing movement for the limit F from x_0. The second issue is whether, by assuming some further properties on F_ε we may deduce that the same thing happens for any choice of ε. In order to give an answer we will use results from the theory of gradient flows recently elaborated by Ambrosio, Gigli and Savarè, and by Sandier and Serfaty.

9.1 Commutability along 'fast-converging' sequences

We consider an equi-coercive sequence F_ε of (non-negative) lower-semicontinuous functionals on a Hilbert space X Γ-converging to F. Note that if $y_\varepsilon \to y_0$ then the solutions of

$$\min\Big\{F_\varepsilon(x) + \frac{1}{2\tau}\|x - y_\varepsilon\|^2\Big\} \tag{9.1}$$

converge to solutions of

$$\min\Big\{F(x) + \frac{1}{2\tau}\|x - y_0\|^2\Big\} \tag{9.2}$$

since we have a continuously converging perturbation of a Γ-converging sequence. We want this convergence to hold along the sequences of minimum problems defining minimizing movements.

Let now $x_\varepsilon \to x_0$. Let τ be fixed. We consider the sequence $\{x_k^{\tau,\varepsilon}\}$ defined by iterated minimization of F_ε with initial point x_ε. Since $x_\varepsilon \to x_0$, up to subsequences we have $x_1^{\tau,\varepsilon} \to x_1^{\tau,0}$, which minimizes

$$\min\Big\{F(x) + \frac{1}{2\tau}\|x - x_0\|^2\Big\}\,. \tag{9.3}$$

The point $x_2^{\tau,\varepsilon}$ converge to $x_2^{\tau,0}$. Since they minimize

$$\min\Big\{F_\varepsilon(x) + \frac{1}{2\tau}\|x - x_1^{\tau,\varepsilon}\|^2\Big\} \tag{9.4}$$

and $x_1^{\tau,\varepsilon} \to x_1^{\tau,0}$ their limit is a minimizer of

$$\min\Big\{F(x) + \frac{1}{2\tau}\|x - x_1^{\tau,0}\|^2\Big\}. \tag{9.5}$$

This operation can be repeated iteratively, obtaining (upon subsequences) $x_k^{\tau,\varepsilon} \to x_k^{\tau,0}$, and $\{x_k^{\tau,0}\}$ iteratively minimizes F with initial point x_0.

With fixed $T > 0$, let $K = \lfloor T/\tau \rfloor + 1$. Then, we deduce that there exists $\varepsilon = \varepsilon(\tau)$ such that we have

$$\|x_k^{\tau,\varepsilon} - x_k^{\tau,0}\| \leq \tau \text{ for all } k = 1, \ldots, K.$$

Upon subsequences of τ these two schemes converge respectively to a minimizing movement along F_ε and a minimizing movement for F. We have then proved the following result.

Theorem 9.1.1 *Let F_ε be a equi-coercive sequence of (non-negative) lower-semicontinuous functionals on a Hilbert space X Γ-converging to F, let $x_\varepsilon \to x_0$. Then there exists a choice of $\varepsilon = \varepsilon(\tau)$ such that every minimizing movement along F_ε with initial data x_ε converge to a minimizing movement for F from x_0 on $[0, T]$ for all T.*

Remark 9.1.2 Note that, given x_ε and F_ε, if F has more than one minimizing movement from x_0 then the approximation gives a choice criterion. As an example take $F(x) = -|x|$, $F_\varepsilon(x) = -|x + \varepsilon|$ and $x_0 = x_\varepsilon = 0$.

9.2 Stability for convex energies

We now use the theory of gradient flows to deduce stability results if the functionals satisfy some convexity assumptions. For the sake of simplicity we will assume that X is a Hilbert space and all F_ε are convex.

9.2.1 Convergence estimates

We first recall some results on minimizing movements for a single convex functional F.

Proposition 9.2.1 *Let F be convex, $z \in X$ and let w be a minimizer of*

$$\min\Big\{F_\varepsilon(x) + \frac{1}{2\eta}\|x - z\|^2\Big\}. \tag{9.6}$$

Then

$$\|x - w\|^2 - \|x - z\|^2 \leq 2\eta(F(x) - F(w)) \tag{9.7}$$

for all $x \in X$.

Proof. We recall that the inequality

$$\|sx + (1-s)w - z\|^2 \le s\|x - z\|^2 + (1-s)\|w - z\|^2 - s(1-s)\|x - w\|^2 \tag{9.8}$$

holds for all $x, w, z \in X$ and $s \in [0,1]$. Using this property and the convexity of F, thanks to the minimality of w we have

$$\begin{aligned} F(w) + \frac{1}{2\eta}\|w - z\|^2 &\le F(sx + (1-s)w) + \frac{1}{2\eta}\|sx + (1-s)w - z\|^2 \\ &\le sF(x) + (1-s)F(w) \\ &\qquad + \frac{1}{2\eta}(s\|x - z\|^2 + (1-s)\|w - z\|^2 - s(1-s)\|x - w\|^2). \end{aligned}$$

After regrouping and dividing by s, from this we have

$$\frac{1}{2\eta}(\|w - z\|^2 + (1-s)\|x - w\|^2 - \|x - z\|^2) \le F(x) - F(w)$$

and then the desired (9.7) after letting $s \to 0$ and dropping the positive term $\|w - z\|^2$. □

Remark 9.2.2 Let $\{z_k\} = \{z_k^\eta\}$ be a minimizing scheme for F from z_0 with time-step η. Then (9.7) gives

$$\|x - z_{k+1}\|^2 - \|x - z_k\|^2 \le 2\eta(F(x) - F(z_{k+1})) \tag{9.9}$$

for all $x \in X$.

We now fix $\tau > 0$ and two initial data x_0 and y_0 and want to compare the resulting $\{x_k\} = \{x_k^\tau\}$ obtained by iterated minimization with time-step τ and initial datum x_0 and $\{y_k\} = \{y_k^{\tau/2}\}$ with time-step $\tau/2$ and initial datum y_0. Note that the corresponding continuous-time interpolations are

$$u^\tau(t) := x_{\lfloor t/\tau \rfloor}, \qquad v^{\tau/2}(t) = y_{\lfloor 2t/\tau \rfloor}, \tag{9.10}$$

so that the comparison must be performed between x_k and y_{2k}.

Proposition 9.2.3 *For all $j \in \mathbb{N}$ we have*

$$\|x_j - y_{2j}\|^2 - \|x_0 - y_0\|^2 \le 2\tau F(x_0)\,.$$

Proof. We first give an estimate between x_1 and y_2. We first apply (9.9) with $\eta = \tau$, $z_k = x_0$, $z_{k+1} = y_1$ and $x = y_2$ which gives

$$\|y_2 - x_1\|^2 - \|y_2 - x_0\|^2 \le 2\tau(F(y_2) - F(x_1))\,. \tag{9.11}$$

If instead we apply (9.9) with $\eta = \tau/2$, $z_k = y_0$, $z_{k+1} = y_1$ and $x = x_0$, or $z_k = y_1$, $z_{k+1} = y_2$ and $x = x_0$ we get, respectively,

$$\begin{aligned} \|x_0 - y_1\|^2 - \|x_0 - y_0\|^2 &\leq \tau(F(x_0) - F(y_1)) \\ \|x_0 - y_2\|^2 - \|x_0 - y_1\|^2 &\leq \tau(F(x_0) - F(y_2)), \end{aligned}$$

so that, summing up,

$$\|x_0 - y_2\|^2 - \|x_0 - y_0\|^2 \leq 2\tau F(x_0) - \tau F(y_1) - F(y_2) \leq 2\tau(F(x_0) - F(y_2)), \tag{9.12}$$

where we have used that $F(y_2) \leq F(y_1)$ in the last inequality. Summing up (9.11) and (9.12) we obtain

$$\|x_1 - y_2\|^2 - \|x_0 - y_0\|^2 \leq 2\tau(F(x_0) - F(x_1)). \tag{9.13}$$

We now compare the later indices. We can repeat the same argument with x_0 and y_0 substituted by x_1 and y_2, so that by (9.13) we get

$$\|x_2 - y_4\|^2 - \|x_1 - y_2\|^2 \leq 2\tau(F(x_1) - F(x_2)), \tag{9.14}$$

and, summing (9.13),

$$\|x_2 - y_4\|^2 - \|x_0 - y_0\|^2 \leq 2\tau(F(x_0) - F(x_2)). \tag{9.15}$$

Iterating this process we get

$$\|x_j - y_{2j}\|^2 - \|x_0 - y_0\|^2 \leq 2\tau(F(x_0) - F(x_j)) \leq 2\tau F(x_0) \tag{9.16}$$

as desired. □

Theorem 9.2.4 *Let F be convex and let $F(x_0) < +\infty$. Then there exists a unique minimizing movement u for F from x_0 such that, if u^τ is defined by (9.10), then*

$$\|u^\tau(t) - u(t)\| \leq 6\sqrt{F(x_0)}\sqrt{\tau}$$

for all $t \geq 0$.

Proof. With fixed τ we first prove the convergence of $u^{2^{-j}\tau}$ as $j \to +\infty$. By Proposition 9.2.3 applied with $y_0 = x_0$ and $2^{-j}\tau$ in place of τ we have

$$\|u^{2^{-j}\tau}(t) - u^{2^{-j-1}\tau}(t)\| \leq 2^{-j/2}\sqrt{2\tau}\sqrt{F(x_0)} \tag{9.17}$$

for all t. This shows the convergence to a limit $u_\tau(t)$, which in particular satisfies

$$\|u^\tau(t) - u_\tau(t)\| \quad \leq \quad \sqrt{2}\sum_{j=0}^{\infty} 2^{-j/2}\sqrt{\tau}\sqrt{F(x_0)} \leq 6\sqrt{F(x_0)}\sqrt{\tau}\,. \tag{9.18}$$

The limit u_τ can be characterized as follows: with fixed x, inequality (9.9) applied to $z_k = u^{2^{-j}\tau}((k-1)2^{-j}\tau)$ $(k \geq 1)$ can be seen as describing in the sense of distribution the derivative

$$\frac{d}{dt}\frac{1}{2}\|x - u^{2^{-j}\tau}(t)\|^2 \leq \sum_{k=1}^{\infty}\Big(F(x) - F\Big(u^{2^{-j}\tau}((k-1)2^{-j}\tau)\Big)\Big)2^{-j}\tau\,\delta_{k2^{-j}\tau}\,. \tag{9.19}$$

Note in fact that $x \mapsto \frac{1}{2}\|x - u^{2^{-j}\tau}\|^2$ is a piecewise-constant function with discontinuities in $2^{-j}\tau\mathbb{Z}$, whose size is controlled by (9.9). Since the measures

$$\mu_j = \sum_{k=1}^{\infty} 2^{-j}\tau\,\delta_{k2^{-j}\tau}$$

converge to the Lebesgue measure, and $u^{2^{-j}\tau}(t) \to u_\tau(t)$ for all t, so that by the lower semicontinuity of F

$$F(u_\tau(t)) \leq \liminf_{j\to+\infty} F\Big(u^{2^{-j}\tau}(t)\Big),$$

we deduce that

$$\frac{d}{dt}\frac{1}{2}\|x - u_\tau(t)\|^2 \leq F(x) - F(u_\tau(t)) \tag{9.20}$$

for all x. Equation (9.20) is sufficient to characterize u_τ. We only sketch the argument: suppose otherwise that (9.20) is satisfied by some other $v(t)$. Then we have

$$\langle x - u_\tau, \nabla u_\tau\rangle \leq F(x) - F(u_\tau) \qquad \text{and} \qquad \langle x - v, \nabla v\rangle \leq F(x) - F(v)$$

for all x. Inserting $x = v(t)$ and $x = u_\tau(t)$ respectively, and summing the two inequalities we have

$$\frac{d}{dt}\frac{1}{2}\|v(t) - u_\tau(t)\|^2 = \langle v - u_\tau, \nabla v - \nabla u_\tau\rangle \leq 0\,.$$

Since $v(0) = u_\tau(0)$ we then have $v = u_\tau$.

This argument shows that $u = u_\tau$ does not depend on τ. We then have the convergence of the whole sequence, and (9.18) gives the desired estimate of $\|u^\tau - u\|$. □

9.2.2 Stability along sequences of convex energies

From the estimates in the previous section, and the convergence argument in Section 9.1 we can deduce the following stability results.

Theorem 9.2.5 *Let F_ε be a sequence of lower-semicontinuous coercive positive convex energies Γ-converging to F, and let $x_0^\varepsilon \to x_0$ with $\sup_\varepsilon F_\varepsilon(x_0^\varepsilon) < +\infty$. Then*

(i) *for every choice of τ and ε converging to 0 the family u^ε introduced in Definition* 5.2.1 *converges to the unique u given by Theorem* 9.2.4*;*

(ii) *the sequence of minimizing movements u_ε for F_ε from x_0^ε (given by Theorem* 9.2.4 *with F_ε in place of F) also converge to the same minimizing movement u.*

Proof. We first show (ii). Indeed, by the estimate in Theorem 9.2.4 we have that, after defining u^τ_ε following the notation of that theorem,

$$\|u^\tau - u\|_\infty \le M\sqrt{\tau}, \qquad \|u^\tau_\varepsilon - u_\varepsilon\|_\infty \le M\sqrt{\tau},$$

where

$$M = 6\sup_\varepsilon F_\varepsilon(x^\varepsilon_0).$$

In order to show that $u_\varepsilon \to u$ it suffices to show that $u^\tau_\varepsilon \to u^\tau$ for fixed τ. That has already been noticed to hold in Section 9.1.

In order to prove (i) it suffices to use the triangular inequality

$$\|u^\tau_\varepsilon - u\| \le \|u^\tau_\varepsilon - u_\varepsilon\| + \|u_\varepsilon - u\| \le M\sqrt{\tau} + o(1)$$

by Theorem 9.2.4 and (ii). □

Example 9.2.6 (parabolic homogenization) We can consider $X = L^2(0,T)$,

$$F_\varepsilon(u) = \int_0^T a\Big(\frac{x}{\varepsilon}\Big)|u'|^2\,dx, \qquad F(u) = \underline{a}\int_0^T |u'|^2\,dx$$

with the notation of Section 1.4. We take as initial datum u_0 independent of ε. Since all functionals are convex, lower semicontinuous and coercive, from Theorem 9.2.5 we deduce the converge of the corresponding minimizing movements. From this we deduce the convergence of the solutions of the parabolic problem with oscillating coefficients

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} = \dfrac{\partial}{\partial x}\Big(a\Big(\dfrac{x}{\varepsilon}\Big)\dfrac{\partial u_\varepsilon}{\partial x}\Big) \\ u_\varepsilon(x,0) = u_0(x) \end{cases}$$

to the solution of the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \underline{a}\,\dfrac{\partial^2 u}{\partial x^2} \\ u_\varepsilon(x,0) = u_0(x)\,. \end{cases}$$

9.3 Sandier-Serfaty theory

We conclude this section by giving a brief (and simplified) account of another very fruitful approach to gradient flows that allows to prove the stability of certain solutions with respect to Γ-convergence.

We consider a family of Hilbert spaces X_ε and functionals $F_\varepsilon : X_\varepsilon \to (-\infty, +\infty]$, which are C^1 on their domain. We denote by $\nabla_{X_\varepsilon} F_\varepsilon$ the gradient of F_ε in X_ε.

Definition 9.3.1 *Let $T > 0$; we say that $u_\varepsilon \in H^1([0,T);X_\varepsilon)$ is a* solution for the gradient flow of F_ε *if*

$$\frac{\partial u_\varepsilon}{\partial t} = -\nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon)$$

almost everywhere on $(0,T)$. Such solution for the a gradient flow is conservative *if*

$$F_\varepsilon(u_\varepsilon(0)) - F_\varepsilon(u_\varepsilon(s)) = \int_0^s \Big\| \frac{\partial u_\varepsilon}{\partial t} \Big\|^2_{X_\varepsilon} dt$$

for all $\tau \in (0,T)$.

We suppose that there exists a Hilbert space X and a notion of metrizable convergence $x_\varepsilon \to x$ of families of elements of X_ε to an element of X. With respect to that convergence, we suppose that F_ε Γ-converge to a functional F, which is also C^1 on its domain.

Theorem 9.3.2 (Sandier-Serfaty Theorem) *Let F_ε and F be as above with F_ε Γ-converging to F, let u_ε be a family of conservative solutions for the gradient flow of F_ε with initial data $u_\varepsilon(0) = u^\varepsilon$ converging to u^0. Suppose furthermore that*

- (well-preparedness of initial data) *u^ε is a recovery sequence for $F(u^0)$;*
- (lower bound) *upon subsequences u_ε converges to some $u \in H^1((0,T);X)$ and*

$$\liminf_{\varepsilon\to 0} \int_0^s \Big\| \frac{\partial u_\varepsilon}{\partial t} \Big\|^2_{X_\varepsilon} dt \geq \int_0^s \Big\| \frac{\partial u}{\partial t} \Big\|^2_{X} dt \tag{9.21}$$

$$\liminf_{\varepsilon\to 0} \|\nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon(s))\|^2_{X_\varepsilon} \geq \|\nabla_X F(u(s))\|^2_X \tag{9.22}$$

for all $s \in (0,T)$.

Then u is a solution for the gradient flow of F with initial datum u^0, $u_\varepsilon(t)$ is a recovery sequence for $F(u(t)$ for all t and the inequalities in (9.21) and (9.22) are equalities.

Proof. Using the fact that u_ε is conservative and that for all t

$$-\Big\langle \nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon(t)), \frac{\partial u_\varepsilon}{\partial t} \Big\rangle = \frac{1}{2}\Big(\|\nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon(t))\|^2_{X_\varepsilon} + \Big\| \frac{\partial u_\varepsilon}{\partial t} \Big\|^2_{X_\varepsilon} \Big)$$

since

$$\Big\| \nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon) + \frac{\partial u_\varepsilon}{\partial t} \Big\|^2_{X_\varepsilon} = 0,$$

we get

$$\begin{aligned} F_\varepsilon(u_\varepsilon(0)) - F_\varepsilon(u_\varepsilon(t)) &= \int_0^t \Big\| \frac{\partial u_\varepsilon}{\partial t} \Big\|^2_{X_\varepsilon} ds \\ &= -\int_0^t \Big\langle \frac{\nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon), \partial u_\varepsilon}{\partial t} \Big\rangle_{X_\varepsilon} ds \end{aligned}$$

$$= \frac{1}{2}\int_0^t \left(\|\nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon)\|^2_{X_\varepsilon} + \left\|\frac{\partial u_\varepsilon}{\partial t}\right\|^2_{X_\varepsilon} \right) ds$$

By the lower-bound assumption then we have

$$\begin{aligned} \liminf_{\varepsilon\to 0}(F_\varepsilon(u_\varepsilon(0)) - F_\varepsilon(u_\varepsilon(t))) &\geq \frac{1}{2}\int_0^t \left(\|\nabla_X F(u)\|^2_X + \left\|\frac{\partial u}{\partial t}\right\|^2_X \right) ds \\ &\geq -\int_0^t \left\langle \nabla_X F(u), \frac{\partial u}{\partial t} \right\rangle_{X_\varepsilon} ds. \end{aligned} \tag{9.23}$$

The last term equals

$$-\int_0^t \frac{d}{dt} F(u)\, ds = F(u(0)) - F(u(t)),$$

so that we have

$$\liminf_{\varepsilon\to 0}(F_\varepsilon(u_\varepsilon(0)) - F_\varepsilon(u_\varepsilon(t))) \geq F(u(0)) - F(u(t)).$$

Since $u_\varepsilon(0)$ is a recovery sequence for $F(u(0))$ we then have

$$F(u(t)) \geq \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon(t)), \tag{9.24}$$

so that $u_\varepsilon(t)$ is a recovery sequence for $u(t)$ and indeed we have equality in (9.24) and hence both inequalities in (9.23) are equalities. The second one of those shows that

$$\left\|\nabla_X F(u) + \frac{\partial u}{\partial t}\right\|^2_X = 0,$$

for all t, and hence the thesis. □

Chapter 10

Parameterized motion driven by global minimization

A different type of "quasi-static" motion can be defined starting from a global-minimization criterion. The ingredients are

- a parameter-dependent energy;
- a dissipation satisfying a non-decreasing constraint;
- (time)-parameterized forcing condition.

An entire much more general theory (of rate-independent motion) can be developed starting from these ingredients. We will only deal with a simplified example, with the aim of examining its stability with respect to perturbations.

10.1 Damage

We consider a one-dimensional setting. Our functions will be parameterized on a fixed interval $(0,1)$. In this case we have

- the parameter space will be that of all measurable subsets A of $(0,1)$. The set A will be understood as the *damage set*;
- the energies depending on a set A will be

$$F_A(u) = \alpha \int_A |u'|^2\,dx + \beta \int_{(0,1)\setminus A} |u'|^2\,dx,$$

where $0 < \alpha < \beta$. In an elastic interpretation, u represents the deformation of a bar, whose elastic constant is β in the undamaged set and $\alpha < \beta$ in the damaged set;

- the dissipation is

$$D(A) = \gamma |A|,$$

with $\gamma > 0$. The work done to damage a portion A of the material is proportional to the measure of A;

• the condition that forces the solution to be parameter dependent ("time-dependent") is a boundary condition

$$u(0) = 0, \qquad u(1) = g(t),$$

where g is a continuous function with $g(0) = 0$. Here the parameter is $t \in [0, T]$.

Definition 10.1.1 *A* solution *to the evolution related to the energy, dissipation and boundary conditions above is a pair (u^t, A^t) with $u^t \in H^1(0,1)$, $A^t \subset (0,1)$, and such that*

- **(monotonicity)** *we have $A^s \subset A^t$ for all $s < t$*
- **(minimization)** *the pair (u^t, A^t) minimizes*

$$\min\Big\{ F_A(u) + D(A) : u(0) = 0, u(1) = g(t), A^s \subset A \text{ for all } s < t \Big\} \tag{10.1}$$

- **continuity** *the energy $\mathcal{E}(t) = F_{A^t}(u^t)$ is continuous*
- **homogeneous initial datum** *u^0 is the constant 0 and $A^0 = \emptyset$.*

The continuity assumption allows to rule out trivial solutions as those with $A^t = (0,1)$ for all $t > 0$. It is usually replaced by a more physical condition of energy conservation. In our context this assumption is not relevant.

Note that t acts only as a parameter (the motion is "rate independent"). Hence, for example if g is monotone increasing, it suffices to consider $g(t) = t$. We will construct by hand a solution in this simplified one-dimensional context.

Note that the value in the minimum problem

$$\min\Big\{ F_A(u) + D(A) : u(0) = 0, u(1) = t, \Big\} \tag{10.2}$$

depends on A only through $\lambda = |A|$, and its value is given by

$$\frac{\alpha\beta}{\lambda\beta + (1-\lambda)\alpha} t^2 + \gamma\lambda. \tag{10.3}$$

By minimizing over λ we obtain the optimal value of the measure of the damaged region

$$\lambda_{\min}(t) = \begin{cases} 0 & \text{if } |t| \le \sqrt{\frac{\alpha\gamma}{\beta(\beta-\alpha)}} \\ 1 & \text{if } |t| \ge \sqrt{\frac{\beta\gamma}{\alpha(\beta-\alpha)}} \\ t\sqrt{\frac{\alpha\beta}{\gamma(\beta-\alpha)}} - \frac{\alpha}{\beta-\alpha} & \text{otherwise} \end{cases} \tag{10.4}$$

the minimum value

$$m(t) = \begin{cases} \beta t^2 & \text{if } |t| \le \sqrt{\frac{\alpha\gamma}{\beta(\beta-\alpha)}} \\ \alpha t^2 + \gamma & \text{if } |t| \ge \sqrt{\frac{\beta\gamma}{\alpha(\beta-\alpha)}} \\ 2t\sqrt{\frac{\alpha\beta\gamma}{\beta-\alpha}} - \frac{\gamma\alpha}{\beta-\alpha} & \text{otherwise.} \end{cases} \tag{10.5}$$

The interpretation of this formula is as follows. For small values of the total displacement t the material remains undamaged, until it reaches a first threshold. Then a portion of size $\lambda_{\min}(t)$ of the material damages, lowering the elastic constant of the material and the overall value of the sum of the internal energy and the dissipation, until all the material is damaged when reaching the second threshold. Note that in this case $\mathcal{E}(t) = m(t)$.

The value in (10.3) is obtained by first minimizing in u. Conversely, we may first minimize in A. We then have

$$\min\Big\{\int_0^1 \min_A\{\chi_A(\alpha|u'|^2+\gamma), \chi_{(0,1)\setminus A}\beta|u'|^2\}\,dx : u(0)=0, u(1)=g(t)\Big\} \tag{10.6}$$

The lower-semicontinuous envelope of the integral energy is given by the integral with energy function the convex envelope of

$$f(z) = \min\{\alpha z^2, \beta z^2+\gamma\}, \tag{10.7}$$

which is exactly given by formula (10.5); i.e.,

$$m(t) = f^{**}(t)$$

(see Fig. 10.1)

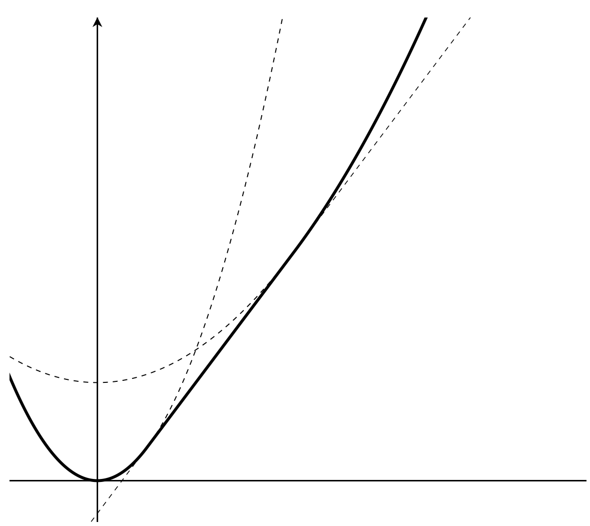

Figure 10.1: minimal value $m(t)$ for the damage problem

Irreversibility. An important feature of the monotonicity condition for A^t is irreversibility of damage, which implies that for non-increasing g the values of $m(g(t))$ will depend on the highest value taken by $\lambda_{\min}(g(t))$ on $[0,t]$. In particular, for a "loading-unloading" cycle with $g(t) = \frac{T}{2} - |t - \frac{T}{2}|$, the value of $\mathcal{E}(t)$ is given by

$$\mathcal{E}(t) = \begin{cases} m(t) & \text{for } 0 \le t \le T/2 \\ \frac{\alpha\beta}{\lambda_{\min}(T/2)\beta + (1-\lambda_{\min}(T/2))\alpha}(T-t)^2 + \gamma\lambda_{\min}(T/2) & \text{for } T/2 \le t \le T. \end{cases}$$

This formula highlights that once the maximal value $\lambda_{\min}(T/2)$ is reached, then the damaged region A^t remains fixed, so that the problem becomes a quadratic minimization (plus the constant value of the dissipation). We plot $m'(t)$ and draw a cycle in Fig. 10.2 In

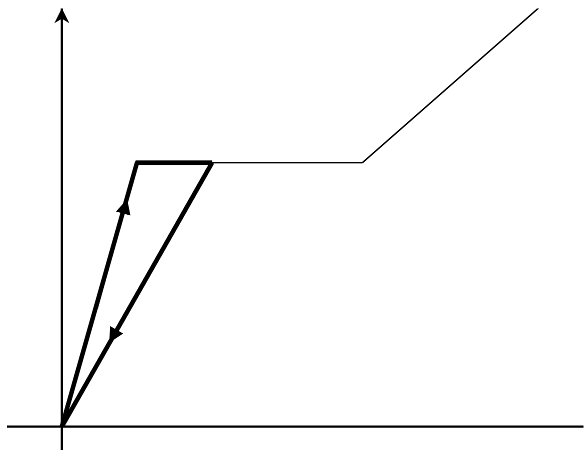

Figure 10.2: plot of $m'(t)$ and derivative of the energy $\mathcal{E}$ along a cycle

particular, if $\frac{T}{2} \ge \sqrt{\frac{\beta\gamma}{\alpha(\beta-\alpha)}}$ then the material is completely damaged in the "unloading" regime.

10.1.1 Homogenization of damage

We now examine the behaviour of the previous process with respect to Γ-convergence in the case of homogenization. To that end we introduce the energies

$$F_{\varepsilon,A}(u) = \int_{(0,1)\setminus A} \beta\Big(\frac{x}{\varepsilon}\Big)|u'|^2\,dx + \int_A \alpha\Big(\frac{x}{\varepsilon}\Big)|u'|^2\,dx, \tag{10.8}$$

where α and β are 1-periodic functions with

$$\alpha(y) = \begin{cases} \alpha_1 & \text{for } 0 \le y < \frac12 \\ \alpha_2 & \text{for } \frac12 \le y < 1 \end{cases} \qquad \beta(y) = \begin{cases} \beta_1 & \text{for } 0 \le y < \frac12 \\ \beta_2 & \text{for } \frac12 \le y < 1 \end{cases}$$

with $0 < \alpha_j < \beta_j$. Note that for fixed A the functionals $F_{\varepsilon,A}$ Γ-converge to

$$F_{\varepsilon,A}(u) = \underline{\beta} \int_{(0,1)\setminus A} |u'|^2\,dx + \underline{\alpha} \int_A \Big(\frac{x}{\varepsilon}\Big)|u'|^2\,dx, \tag{10.9}$$

with

$$\underline{\alpha} = \frac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2} < \frac{2\beta_1\beta_2}{\beta_1+\beta_2} = \underline{\beta}.$$

Hence, the Γ-limit is exactly of the form examined beforehand.

We now examine instead the damage process at fixed ε. For simplicity of computation we suppose that $\frac{1}{\varepsilon} \in \mathbb{N}$. The dissipation will be of the form

$$D_\varepsilon(A) = \int_A \gamma(\frac{x}{\varepsilon}\Big)|u'|^2\,dx,$$

where again γ is a 1-periodic function with

$$\gamma(y) = \begin{cases} \gamma_1 & \text{for } 0 \le y < \frac{1}{2} \\ \gamma_2 & \text{for } \frac{1}{2} \le y < 1 \end{cases}$$

with $\gamma_j > 0$. In the case $\gamma_1 = \gamma_2$ we obtain the same dissipation as above, independent of ε.

In order to compute the minimum value

$$m^\varepsilon(t) = \min\Big\{F_{\varepsilon,A}(u) + D_\varepsilon(A) : u(0) = 0, u(1) = t, A \subset (0,1)\Big\}$$

we first remark that, for the one-dimensional nature of the problem, this value is equal to $m(t) := m^1(t)$ independent of ε. This value can be obtained by minimizing separately on $(0, 1/2)$ and $(1/2, 1)$, so that

$$m(t) = \frac{1}{2}\min\Big\{m_1(t_1) + m_2(t_2) : \frac{t_1+t_2}{2} = t\Big\},$$

and m_1, m_2 are given by the damaging process in the two subintervals, so that by (10.5)

$$m_j(t) = \begin{cases} \beta_j t^2 & \text{if } |t| \le \sqrt{\frac{\alpha_j\gamma}{\beta_j(\beta_j-\alpha_j)}} \\ \alpha_j t^2 + \gamma & \text{if } |t| \ge \sqrt{\frac{\beta_j\gamma_j}{\alpha_j(\beta_j-\alpha_j)}} \\ 2t\sqrt{\frac{\alpha_j\beta_j\gamma_j}{\beta_j-\alpha_j}} - \frac{\gamma_j\alpha_j}{\beta_j-\alpha_j} & \text{otherwise.} \end{cases} \tag{10.10}$$

We can therefore easily compute $m(t)$. In the hypothesis, e.g, that

$$p_2 := \sqrt{\frac{\alpha_2\beta_2\gamma_2}{\beta_2-\alpha_2}} < \sqrt{\frac{\alpha_1\beta_1\gamma_1}{\beta_1-\alpha_1}} =: p_1,$$

we can easily compute $m'(t)$ as

$$m'(t) = \begin{cases} 2\underline{\beta} t & \text{if } |t| \leq \frac{p_2}{\underline{\beta}} \\ 2p_2 & \text{if } \frac{p_1}{\underline{\beta}} < |t| < \frac{p_2(\beta_1+\alpha_2)}{2\beta_1\alpha_2} \\ \frac{4\beta_1\alpha_2}{\beta_1+\alpha_2} t & \text{if } \frac{p_2(\beta_1+\alpha_2)}{2\beta_1\alpha_2} \leq |t| \leq \frac{p_1(\beta_1+\alpha_2)}{2\beta_1\alpha_2} \\ 2p_1 & \text{if } \frac{p_1(\beta_1+\alpha_2)}{2\beta_1\alpha_2} < |t| < \frac{p_1}{\underline{\alpha}} \\ 2\underline{\alpha} t & \text{if } |t| \geq \frac{p_1}{\underline{\alpha}}. \end{cases}$$

The outcome is pictured in Fig. 10.3. It highlights that the behaviour is different from the

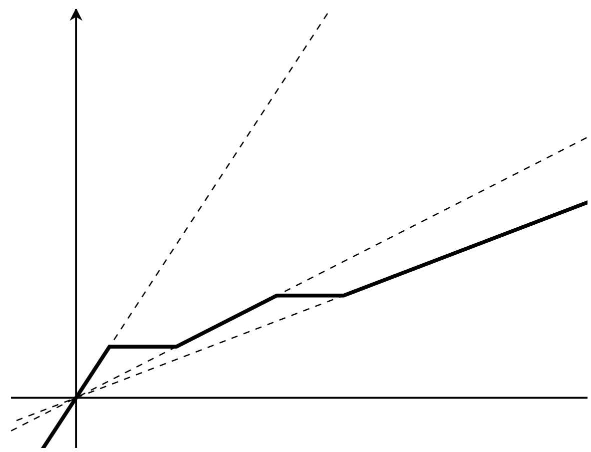

Figure 10.3: damage in a periodic microstructure

one computed above: for small values of the total displacement t the overall response is the same as the one of the homogenized behaviour of the two 'strong' materials. At a first critical value one (and only one) of the two materials starts to damage (this corresponds to the first constant value $2p_1$ of m') until it is completely damaged. Then the material behaves as a mixture of a strong and a damaged material, until also the second material starts to damage. After also this has completely damaged, the behaviour is that of the homogenized energy for two weak materials.

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Exam Problems

1. In Example 7.1 take the intervals where F is constant with variable length (e.g. periodic, or random).

2. In Example 7.3.2 consider the case $\varepsilon \approx \tau$ (in the spirit of Section 7.2.2).

3. In Example 8.1.3 compute the scaled motion
1) when the initial datum has three (or more) jump points
2) with Neumann boundary conditions instead of Dirichlet boundary conditions.

4. In Example 8.2.1:
1) consider the case $\varepsilon \approx \tau$.
2) compute the motion with different functions of the distance (as in Example 6.1.1).

5. In Example 10.1.1
1) describe what happens if $\alpha_1 < \beta_1$ but $\beta_2 < \alpha_2$;
2) consider the case when $D(A) = |A| + \#(\partial A)$ or when we assume that A is an interval.