Abstracts of courses

Giovanni ALBERTI

"Concentration phenomena for functionals of Ginzburg-Landau type. A variational approach"

In these lectures I will describe an approach by Γ-convergence to certain concentration phenomena for minimizers of functionals of Ginzburg-Landau type, as developped in [ABO2] and [JS]. Consider the functionals

$$F_\varepsilon(u) := \int_\Omega \frac{1}{k}|Du|^k + \frac{1}{\varepsilon^2}W(u)$$

where ε is a positive parameter, u a map from the $(n+k)$-dimensional domain Ω into $\mathbf{R}^k$ ($n \geq 1$, $k \geq 2$) and W a positive potential that vanishes only on the unit sphere S^{k-1}. Let u_ε be minimizers of F_ε with suitable boundary condition.

As ε tends to 0, the second term in the integral prevails, and the minimizers u_ε take values close to S^{k-1} in larger and larger sets. However, the imposed boundary datum may force each u_ε to take all values in the unit ball. Accordingly, the Jacobians determinants of u_ε do not vanish, but concentrate on smaller and smaller sets. Indeed, they converge in a suitable sense to a minimal surface M (more precisely, a mass-minimzing rectifiable current) of codimension k and prescribed boundary, while the corresponding energy densities $e_\varepsilon(u_\varepsilon)$, suitably renormalized, converge in the sense of measures to the area of M.

This result can be made rigorous in the framework of Γ-convergence, by finding the limit of the functionals F_ε. In practice, we have to prove a suitable compactness result, and sharp upper and lower bounds for F_ε. The proof of compactness and lower bound for $n = 0$ (domain and target with same dimensions) relies on a fundamental estimate due to R. Jerrard [J], and will be explained in detail in the lectures of Mete Soner. I will briefly show how to extend these results to higher n, and then move to the proof of the upper bound. This is essentially based on the construction of maps with prescribed (singular) Jacobian described in [ABO1].

[ABO1] G. Alberti, S. Baldo, G. Orlandi: Functions with prescribed Jacobians. *J. European Math. Soc.*, to appear.
[ABO2] G. Alberti, S. Baldo, G. Orlandi: Variational convergence for functionals of Ginzburg-Landau type. Preprint 2002.
[J] R.L. Jerrard: Lower bound for generalized Ginzburg-Landau functionals. *SIAM J. Math. Anal.* 30 (1999), 721-746.
[JS] R.L. Jerrard, H.M. Soner: The Jacobian and the Ginzburg-Landau energy. *Calc. Var. Partial Differential Equations* 14 (2002), 151-191.

Adriana GARRONI

"Γ-convergence for concentration problems with critical growth"

We will consider a large class of variational problems where a concentration phenomenon occurs, and can be stated as the asymptotic analysis of maximum problems of the general form

$$\max\Big\{\int_\Omega f(u)\,dx \ : \ u \in H_0^1(\Omega)\ ,\ \int_\Omega |\nabla u|^2\,dx \leq \varepsilon^2\Big\}, \qquad \text{with } \ 0 \leq f(t) \leq c|t|^{2^*}.$$

Examples, with a proper choice of f, include non-linear variational problems with critical growth related to the Sobolev inequality, free-boundary problems, as the so-called Bernoulli problem, related to the isoperimetric inequality for the capacity, etc.

These problems can be treated using a proper version of the concentration-compactness alternative of PL Lions. We will give a formulation of the problem in terms of Gamma-convergence. This requires to define rescaled functionals on a space larger than the Sobolev space where $|\nabla u|^2$ is interpreted as a measure and the concentration phenomenon is the convergence of the maximizing sequence to a Dirac mass.

Using techniques of potential theory the concentration points can be characterized by means of the minima of the Robin function. Equivalently, the first order Gamma-limit can be expressed in terms of the Robin function.

M. Amar, A. Garroni: Γ-convergence of concentration problems with critical growth, *Ann. Scuola Norm. Sup. Pisa*, in print.

M. Flucher, *Variational Problems with Concentration*, Birkhäuser, Boston, 1999.

M. Flucher, A. Garroni, S. Müller: Concentration of low energy extremals: Identification of concentration points, *Calc. Var. Partial Differential Equations*, Vol. 14 (2002), no. 4, 483–516.

Didier SMETS

"Shape and evolution of concentration sets in some Ginzburg-Landau type equations"

The lectures will be devoted mainly to questions arising in the study of the Ginzburg-Landau equation in both the scalar (valued) case (often called the Allen-Cahn equation) and the complex case, and related to concentration phenomena. We will start by reviewing some important historical results in the context, and present the main ideas that have been used there. Next, we will focus on geometric properties of the concentration sets. Roughly speaking, in the static case one obtain a curvature equation for the concentration set, whereas in the time dependent case(s) some geometric flows arise naturally. Specific questions regarding convergence to these flows will be addressed in more details.

H. Mete SONER

"Jacobian and the Ginzburg-Landau energy"

Variational and dynamic problems related to the Ginzburg-Landau functional received considerable attention in the past decade. These types of functionals arise in many applications modeling phase transitions. The GL functional contains a small parameter that is loosely related to the thickness of the transition region. As this small parameter approaches to zero, the solutions become singular on an interface. Asymptotic analysis of the solutions as the parameter tends to zero is the crucial mathematical question. In these lectures, I will concentrate on models for which the singular set has codimension two. I will first describe a method developed by Jerrard for lower energy bounds. Then, I will prove an estimate bounding the Jacobian in terms of the GL energy. Finally, I will show how to use this estimate in variational and dynamic problems.