Entanglement Measures in Quantum Field Theory

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based on joint work with K Sanders

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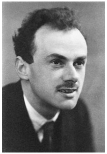
What is different in quantum theory?

Heisenberg: *"If there were to exist experiments allowing for a simultaneous measurement of* p *and* q *exceeding in precision what corresponds to the uncertainty relation, then quantum theory would be impossible."*

What is different in quantum theory?



Dirac: *"It's the phase."*

What is different in quantum theory?

Schrödinger: *"I would not call [entanglement]* **one** *but rather* **the** *characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought".* [THIS TALK]

What is entanglement?

Entanglement

Entanglement concerns subsystems (usually two, called A and B) of an ambient system. Roughly, one asks how much “information” one can extract about the state of the total system by performing **separately local, coordinated operations** in A and B.

What entanglement is **not**

Entanglement $\neq$ correlations

Entanglement is different in general from correlations between A and B which can exist with or without entanglement!

Example of correlations: We prepare an ensemble of pairs of cards. For each pair, both cards are either black or both are white. One card of each pair goes to A, the other to B. A knows that if he uncovers one of his cards at random, he will get black with probability p and white with probaility $1 - p$. But he knows with probability 1 that if the card uncovered is white, then so is the corresponding card of B! Ensembles of A and B are maximally correlated but **not** entangled!

$\Rightarrow$ "Classical correlations" but no entanglement

What is entanglement?

Standard "grammar" of quantum theory (w/o dynamics="semantics"):

- **observables:** operators a on Hilbert space $\mathcal{H}$
- **states:** $\omega \leftrightarrow$ statistical operator, $\omega(a) = \mathrm{Tr}(\rho a) =$ expectation value
- **pure state:** $\rho = |\Omega\rangle\langle\Omega|$. Cannot be written as convex combination of other states, otherwise mixed.
- **independent systems** A and B: $\mathcal{H}_A \otimes \mathcal{H}_B$, observables for A: $a \otimes 1_B$, observables for B: $1_A \otimes b$
- **measurement:** possible outcomes of a are its eigenvalues λ_n.

$$p_n = \text{Probability of measuring } \lambda_n = \mathrm{Tr}(P_n \rho P_n)$$

Here $P_n =$ eigenprojection of a corresponding to λ_n. Immediately afterwards, state $= \frac{1}{p_n} P_n \rho P_n$.

Separable states:

Convex combinations of product states (statistical operators $\rho_A \otimes \rho_B$).

What is entanglement?

Classically: State on bipartite system $\leftrightarrow$ probability density on phase space $\Gamma_A \times \Gamma_B$. Always separable! This motivates:

Entangled states

A state is called "entangled" if it is **not** separable.

Example: $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$ spin-1/2 systems, Bell state $\rho = |\Omega\rangle\langle\Omega|$

$$|\Omega\rangle \propto |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle.$$

is (maximally) entangled.

Example: n dimensions $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^n$:

$$|\Omega\rangle \propto \sum_j |j\rangle \otimes |j\rangle$$

Example: ∞ dimensions:

$$|\Omega\rangle \propto \sum_j c_j |j\rangle \otimes |j\rangle, \quad c_j \to 0$$

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$$|\Omega\rangle \propto \sum_j |j\rangle \otimes |j\rangle$$

Example: ∞ dimensions:

$$|\Omega\rangle \propto \sum_j e^{-2\pi E_j/\kappa} |j\rangle \otimes |j\rangle \quad (\rightarrow \text{Killing horizons, Unruh effect})$$

How to distinguish entangled states?

ω **entangled** (across A and B) $\Rightarrow$ ω **correlated**: There is a and b from the subsystems such that

$$\omega(ab) \neq \omega(a)\omega(b).$$

But converse is **not** usually true: Intuitively: correlations can have entirely "classical" origin, i.e. no relation with entanglement! Better measure:

Bell correlation:

If $E_B(\omega) > 2 \Rightarrow \omega$ entangled. Here

$$E_B(\omega) := \max\{\omega(a_1(b_1 + b_2) + a_2(b_1 - b_2))\} \tag{1}$$

maximum over all self-adjoint elements a_i (system A), b_i (system B) such that

$$-1 \leq a_i \leq 1, \quad -1 \leq b_i \leq 1\,. \tag{2}$$

Idea: Classical correlations "cancel out" in E_B. [Bell 1964, Clauser, Horne, Shimony, Holt 1969, Tsirelson 1980]

What to do with entangled states?

Now and then:

Then: EPR say (1935) Entanglement = "spooky action-at-a-distance"
Now: Entanglement = resource for doing new things!

Example: Teleportation of a state $|\beta\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$ from A to B. [Bennett, Brassard, Crepeau, Jozsa, Perez, Wootters 1993].

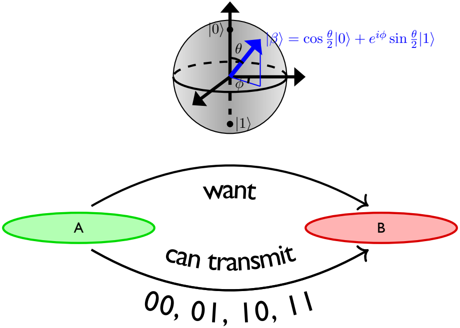


Figure: Teleportation of one q-bit.

Quantum teleportation

Lesson:

To teleport **one** "q-bit" $|\beta\rangle$ need **one** Bell-pair entangled across A and B!
$\Rightarrow$ For lots of q-bits need lots of entanglement.

Quantum teleportation

How it works: Choose “Bell-basis” of $\mathcal{H}_A \otimes \mathcal{H}_B$,

$$|\Psi_{00}\rangle \propto |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle, \quad |\Psi_{10}\rangle \propto |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle$$
$$|\Psi_{11}\rangle \propto |0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle, \quad |\Psi_{01}\rangle \propto |0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle$$

1. The state $|\beta\rangle_C \otimes |\Omega\rangle_{AB}$ for the combined system ABC is prepared.
2. **Local operation** (measurement) in AC: Some given observable of AC with four Bell-eigenstates is measured (by A). Afterwards, system is in one of the four states $U_i|\beta\rangle_B \otimes |\Psi_i\rangle_{AC}$ with $i \in \{00, 01, 10, 11\}$, and $U_i =$ unitaries from system B.
3. **Local operation** (unitary) $+$ **classical communication**: A communicates (classically) to B which of the four possibilities $i \in \{00, 01, 10, 11\}$ occurred ($=$ two classical bits of info), and, forgetting at this stage AC, B applies corresponding unitary U_i^* to extract $|\beta\rangle_B$!

When is a state more entangled than another?

More/less entanglement:

We quantify entanglement by listing the set of operations $\omega \mapsto \mathcal{F}^*\omega$ on states which (by definition!) do not increase it. $\to$ partial ordering of states.

What are these "operations"? **Single** system (channel):

- Time evolution/gate: unitary transformation: $\mathcal{F}(a) = UaU^*$
- Ancillae: n copies of system: $\mathcal{F}(a) = 1_{\mathbb{C}^n} \otimes a$
- v. Neumann measurement: $\mathcal{F}(a) = PaP$, where $P : \mathcal{H} \to \mathcal{H}'$ projection
- Arbitrary combinations $=$ completely positive maps [Stinespring 1955]

Bipartite system:

Separable operations ("$=$ channels $+$ classical communications"):

Normalized sum of product channels, $\sum \mathcal{F}_A \otimes \mathcal{F}_B$ acting on operator algebra $\mathfrak{A}_A \otimes \mathfrak{A}_B$

Example: Teleportation

Stated more abstractly in terms of channels, Teleportation is a combination of the following:

- Ancillae: $a \mapsto a \otimes 1_B \otimes 1_C$
- v. Neumann measurement: $a \otimes b \otimes c = P_i(a \otimes c)P_i \otimes b$
- Unitary gate: $a \otimes c \otimes b \mapsto a \otimes c \otimes U_i b U_i^*$
- v. Neumann measurement: $a \otimes c \otimes b \mapsto \langle \Omega | a \otimes c | \Omega \rangle \, b$ where $|\Omega\rangle =$ Bell state.

Teleportation

If $\mathcal{F}_i : A \to B$ is the channel defined by composing these separable operations, $i \in \{00, 01, 10, 11\}$, then the sum $\sum \mathcal{F}_i$ implements teleportation (in "Heisenberg picture").

Entanglement measures

Definition of entanglement measure is consistent with basic facts [Plenio, Vedral 1998]:

- No separable state can be mapped to entangled state by separable operation
- Every entangled state can be obtained from maximally entangled state (Bell state) by separable operation

An entanglement measure E on bipartite system should satisfy:

Minimum requirements for **any** entanglement measure:

- No increase "on average" under separable operations:

$$\sum_i p_i E(\tfrac{1}{p_i}\mathcal{F}_i^* \omega) \leq E(\omega)$$

  for all states ω (NB: $p_i = \mathcal{F}_i^* \omega(1) =$ probability that i-th separable operation is performed)
- E non-negative, $E(\omega) = 0 \Leftrightarrow \omega$ separable
- (Perhaps) various other requirements

Examples of entanglement measures

Example: Relative entanglement entropy [Lindblad 1972, Uhlmann 1977, Plenio, Vedral 1998,...]:

$$E_R(\rho) = \inf_{\sigma \text{ separable}} H(\rho, \sigma) \ .$$

Here, $H(\rho, \sigma) = \mathrm{Tr}(\rho \ln \rho - \rho \ln \sigma) =$ Umegaki's relative entropy [Araki 1970s]

Example: Distillable entanglement [Rains 2000]:

$$E_D(\rho) = \ln \Big(\text{max. number of Bell-pairs extractable via separable operations from } N \text{ copies of } \rho\Big) \Big/ \text{copy}$$

Example: Reduced v. Neumann entropy/mutual information [Schrödinger 1936?]:

$$E_{vN}(\rho) = -\mathrm{Tr}(\rho_A \ln \rho_A). \tag{3}$$

Reduced state $\rho_A = \mathrm{Tr}_{\mathcal{H}_B} \rho$ (restriction to A, or similarly B) or

$$E_I(\rho) = H_{\mathrm{vN}}(\rho_A) + H_{\mathrm{vN}}(\rho_B) - H_{\mathrm{vN}}(\rho_{AB}) \tag{4}$$

are not a reasonable entanglement measure except for pure states!

Examples of entanglement measures

Example: Bell correlations [Bell 196?, Tsirelson 1980,...]: (before)

Example: Logarithmic dominance [SH, Sanders 2017, ...]:

$$E_N(\rho) = \ln \left(\min\{\|\sigma\|_1 \mid \sigma \geq \rho\} \right)$$

Example: Modular nuclearity [SH & Sanders 2017]:

$$E_M(\rho) = \ln \nu_{A,B} \tag{5}$$

where ν is the nuclearity index ("trace") of the map $a \mapsto \Delta^{1/4} a|\Omega\rangle$ where $a \in \mathfrak{A}_A$, $|\Omega\rangle$ is the GNS-vector representing ρ and Δ is the modular operator for the commutant of $\mathfrak{A}_B$

Many other examples!

Non-uniqueness entanglement measures

In fact, for pure states one has basic fact [Donald, Horodecki 2002]:

Uniqueness

For pure states, basically all entanglement measures agree with v. Neumann entropy of reduced state.

For mixed states, uniqueness is lost. In QFT, we are always in this situation!

Relationships

Measure	Properties	Relationships	$E(\omega_n^+)$
E_B	OK		$\sqrt{2}$
E_D	OK	$E_D \leq E_R, E_N, E_M, E_I$	$\ln n$
E_R	OK	$E_D \leq E_R \leq E_N, E_M, E_I$	$\ln n$
E_N	OK	$E_D, E_R \leq E_N \leq E_M$	$\ln n$
E_M	mostly OK	$E_D, E_R, E_N \leq E_M$	$\frac{3}{2} \ln n$
E_I	not OK for mixed	$E_D, E_R \leq E_I$	$2 \ln n$

Entanglement measures in QFT

In QFT, systems are tied to spacetime location, e.g. system A

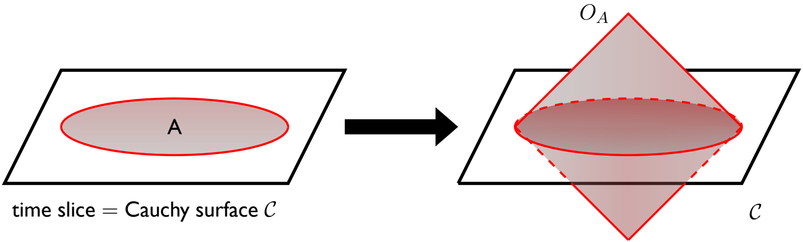


Figure: Causal diamond O_A associated with A.

Set of observables measurable within O_A is an algebra $\mathfrak{A}_A =$ "quantum fields localized at points in O_A". If A and B are regions on time slice (Einstein causality) [Haag, Kastler 1964]

$$[\mathfrak{A}_A, \mathfrak{A}_B] = \{0\} \ .$$

The algebra of all observables in A and B is called $\mathfrak{A}_A \vee \mathfrak{A}_B =$ v. Neumann algebra generated by $\mathfrak{A}_A$ and $\mathfrak{A}_B$.

Entanglement measures in QFT

Unfortunately [Buchholz, Wichmann 1986, Buchholz, D'Antoni, Longo 1987, Doplicher, Longo 1984, ... :

$$[\mathfrak{A}_A, \mathfrak{A}_B] = \{0\} \quad \text{does not always imply} \quad \mathfrak{A}_A \vee \mathfrak{A}_B \cong \mathfrak{A}_A \otimes \mathfrak{A}_B \ .$$

This will happen due to boundary effects if A and B touch each other (algebras are of type III_1 in Connes classification):

Basic conclusion

a) If A and B touch, then there are no (normal) product states, so no separable states, and no basis for discussing entanglement!

b) If A and B do not touch, then there are no pure states (without firewalls)!

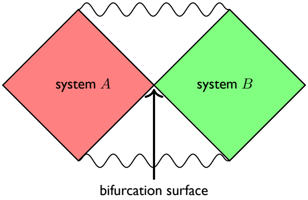
Therefore, if we want to discuss entanglement, we **must** leave a safety corridor between A and B, and we **must** accept b).

$$\Longrightarrow \textbf{no unique entanglement measure!}$$

In the rest of talk, I explain results obtained for relative entanglement entropy E_R for various concrete states/QFTs [Hollands, Sanders 2017, 104pp]

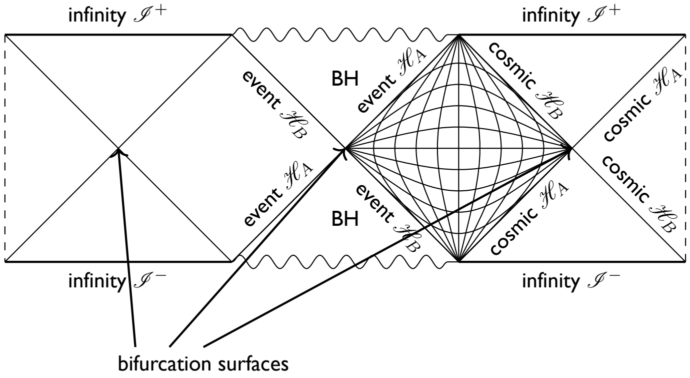
Entanglement in QFT

Natural application of entanglement ideas: Spacetimes with "bifurcate Killing horizons". Quantum state is strongly **entangled** (in a particular way!) between a "system A" and a "system B" across bifurcation surface:



Bifurcate Killing horizons

Such geometries are a generalization of familiar BH spacetimes such as the *extended* Schwarzschild(-deSitter) spacetime, containing as essential geometric feature one (or several) pairs of intersecting horizons:



Kay and Wald [Kay & Wald 1991] have shown

Hawking-Unruh effect

Any quantum state ω which is invariant under “boost” symmetry *and* “regular” across horizon necessarily has to be a thermal state at precisely the Hawking-temperature,

$$T_{\text{Hawking}} = \frac{\kappa}{2\pi} \tag{6}$$

The surface gravity, κ characterizes the geometry of the bifurcation surface (“horizon”). Related to [Bisognano, Wichmann 1972, Hawking 1975, Unruh 1976, Sewell 1982]

Consequences:

- A thermal state at a *different* temperature *necessarily* must have a singular behavior of the stress tensor $\omega(T_{ab}) \to \infty$ on the horizons $\mathscr{H}_A$ and $\mathscr{H}_B$, i.e. an observer made out of the quantum field (or coupled to it) will *burn when he/she crosses the horizon* (“firewall”).
- ω must be (infintely) *entangled across bifurcation surface*!

Overview

Results obtained in [Hollands, Sanders 2017]:

1. $1+1$-dimensional integrable models
2. $d+1$-dimensional CFTs
3. Area law
4. Free quantum fields
5. Charged states
6. General bounds for vacuum and thermal states

1) Integrable models

These models (i.e. their algebras $\mathfrak{A}_A$) are constructed using an "inverse scattering" method from their 2-body S-matrix, e.g.

$$S_2(\theta) = \prod_{k=1}^{2N+1} \frac{\sinh\theta - i\sin b_k}{\sinh\theta + i\sin b_k} \ ,$$

by [Schroer, Wiesbrock 2000, Buchholz,Lechner 2004, Lechner 2008, Allazawi,Lechner 2016, Cadamuro,Tanimoto 2016].
$b_i =$ parameters specifying model, e.g. sinh-Gordon model ($N = 0$).

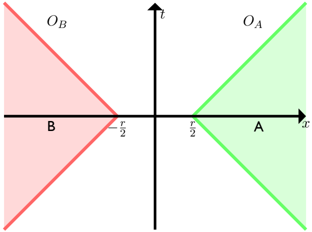


Figure: The regions A, B.

Results

For vacuum state $\rho_0 = |0\rangle\langle 0|$ and mass $m > 0$:

$$E_R(\rho_0) \lesssim C_\infty e^{-mr(1-k)} \ .$$

for $mr \gg 1$. The constant depends on the scattering matrix S_2, and $k > 0$.

The proof partly relies on estimates of [Lechner 2008, Allazawi,Lechner 2016]

Conjecturally (i.e. modulo one unproven estimate)

$$E_R(\rho_0) \lesssim C_0 \, |\ln(mr)|^\alpha \ ,$$

for $mr \ll 1$, with constants C_0, α depending on S_2.

2) CFTs in $3+1$ dimensions

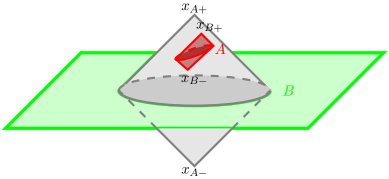


Figure: Nested causal diamonds.

Define conformally invariant cross-ratios u, v by

$$u = \frac{(x_{B+} - x_{B-})^2 (x_{A+} - x_{A-})^2}{(x_{A-} - x_{B-})^2 (x_{A+} - x_{B+})^2} > 0$$

(v similarly) and set

$$\theta = \cosh^{-1}\left(\frac{1}{\sqrt{v}} - \frac{1}{\sqrt{u}}\right), \quad \tau = \cosh^{-1}\left(\frac{1}{\sqrt{v}} + \frac{1}{\sqrt{u}}\right).$$

Results

For vacuum state $\rho_0 = |0\rangle\langle 0|$ in any $3+1$ dimensional CFT with local operators $\{\mathcal{O}\}$ of dimensions $d_{\mathcal{O}}$ and spins $s_{\mathcal{O}}, s'_{\mathcal{O}}$:

$$E_R(\rho_0) \leq \ln \sum_{\mathcal{O}} e^{-\tau d_{\mathcal{O}}} \frac{\sinh \frac{1}{2}(s_{\mathcal{O}}+1)\theta \ \sinh \frac{1}{2}(s'_{\mathcal{O}}+1)\theta}{\sinh^2(\frac{1}{2}\theta)} \ .$$

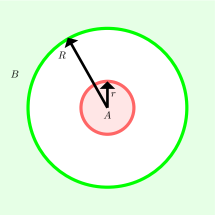


Figure: The regions A and B.

For concentric diamonds with radii $R \gg r$ this gives

$$E_R(\rho_0) \lesssim N_{\mathcal{O}} \left(\frac{r}{R}\right)^{d_{\mathcal{O}}} \ ,$$

where $\mathcal{O} =$ operator with the smallest dimension $d_{\mathcal{O}}$ and $N_{\mathcal{O}} =$ its multiplicity.

Tools: Hislop-Longo theorem [Brunetti, Guido, Longo 1994], Tomita-Takesaki theory

3) Area law in asymptotically free QFTs

A and B regions separated by a thin corridor of diameter $\varepsilon > 0$ in $d+1$ dimensional Minkowski spacetime, vacuum $\rho_0 = |0\rangle\langle 0|$.

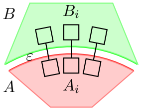


Figure: The the systems A, B

Result (“area law”)

Asymptotically, as $\varepsilon \to 0$

$$E_R(\rho_0) \gtrsim \begin{cases} D_2 \cdot |\partial A|/\varepsilon^{d-1} & d > 1, \\ D_2 \cdot \ln \frac{\min(|A|,|B|)}{\varepsilon} & d = 1, \end{cases}$$

where D_2 = distillable entropy E_D of an elementary “Cbit” pair

Tools: Strong super additivity of E_D, bounds [Donald, Horodecki 2002], also [Verch, Werner 2005, Wolf, Werner 2001,HHorodecki 1999]

4) Free massive QFTs

A and B regions in a static time slice in ultra-static spacetime, $\mathrm{d}s^2 = -\mathrm{d}t^2 + h(\text{space})$; lowest energy state: $\rho_0 = |0\rangle\langle 0|$.
Geodesic distance: r

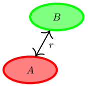


Figure: The the systems A, B

Results (decay + area law)

Dirac field: As $r \to 0$

$$E_R(\rho_0) \lesssim C_0 |\ln(mr)| \sum_{j \geq d-1} r^{-j} \int_{\partial A} a_j$$

where a_j curvature invariants of ∂A. Lowest order $\Longrightarrow$ area law.
Klein-Gordon field: As $r \to \infty$ decay

$$E_R(\rho_0) \lesssim C_\infty e^{-mr/2}$$

(Dirac: [Islam, to appear])

We expect our methods to yield similar results to hold generally on spacetimes with bifurcate Killing horizon, as studied by Kay and Wald in 1991 paper:

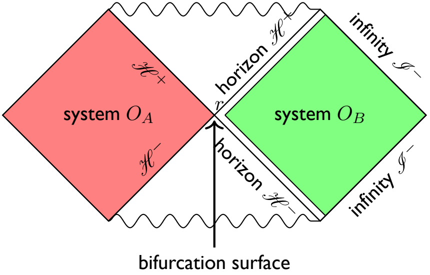


Figure: Spacetime with bifurcate Killing horizon.

5) Charged states

A and B regions, ω any normal state in a QFT in $d+1$ dim. $\chi^*\omega$ state obtained by adding "charges" χ in A or B.

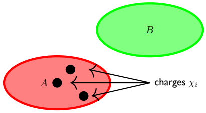


Figure: Adding charges to state in A

Result

$$0 \leq E_R(\omega) - E_R(\chi^*\omega) \leq \ln \prod_i \dim(\chi_i)^{2n_i} \ ,$$

n_i: # irreducible charges χ_i type i, and

$$\dim(\chi_i) = \text{quantum dimension} = \sqrt{\text{Jones index}}$$

Tools: Index-statistics theorem [Longo 1990], Jones subfactor theory, Doplicher-Haag-Roberts theory

Examples

Example: $d = 1$, Minimal model type $(p, p+1)$, χ irreducible charge of type (n, m)

$$0 \leq E_R(\omega) - E_R(\chi^*\omega) \leq \ln \frac{\sin\left(\frac{\pi(p+1)m}{p}\right) \sin\left(\frac{\pi p n}{p+1}\right)}{\sin\left(\frac{\pi(p+1)}{p}\right) \sin\left(\frac{\pi p}{p+1}\right)} \ .$$

Example: $d > 1$, general QFT, irreducible charge χ with Young tableaux statistics $\begin{array}{|c|c|c|c|c|c|} \hline 8 & 6 & 5 & 4 & 2 & 1 \\ \hline 5 & 3 & 2 & 1 \\ \cline{1-4} 1 \\ \cline{1-1} \end{array}$.

$$0 \leq E_R(\omega) - E_R(\chi^*\omega) \leq 2 \ln 5,945,940$$

6) Decay in general QFTs

A and B regions in a time slice of Minkowski. Distance: r. QFT satisfies nuclearity condition a la Buchholz-Wichmann

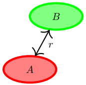


Figure: The the systems A, B

Results (Decay)

Vacuum state in massive theory:

$$E_R(\rho_0) \lesssim C_0 e^{-(mr)^k} \ ,$$

for any given $k < 1$ (our C_0 diverges when $k \to 1$)

Thermal state:

$$E_R(\rho_\beta) \lesssim C_\beta r^{-\alpha+1} \ ,$$

for $\alpha > 1$ a constant in nuclearity condition. Similar for massless theory.

In this talk, I have

- Explained what entanglement is, and how it can be used.
- Explained what an entanglement measure is, and given concrete examples
- Explained how entanglement arises in Quantum Field Theory, and why there always has to be a finite safety corridor between the systems.
- Evaluated (in the sense of upper and lower bounds) a particularly natural entanglement measure in several geometrical setups, quantum field theories and states of interest.
- Shown how the "area law" emerges.

I think that our entanglement measure deserves to be studied further, especially its relation with the considerable literature on v. Neumann entropy in the theoretical physics literature! Especially:

- 2d CFTs Calbrese, Cardy, Nozaki, Numasawa, Takayanagi,...
- 2d integrable models Cardy, Doyon,...
- Modular theory, c-theorems: Casini, Huerta,...
- Holographic methods Hubeny, Myers, Rangamani, Ryu, Takayanagi,...