Statistical Mechanics of the Universe

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"Il libro dell'Universo è scritto in lingua matematica"
Galileo Galilei

Credits and Contents

Credits

Work with *A. Alekseev, W. Aschbacher, O. Boyarsky, R. Brandenberger, A. Brandenburg, V. Cheianov, E. Lenzmann, B. Pedrini, I. Rogachevskii, O. Ruchayskiy, I. M. Sigal, T.-P. Tsai, Ph. Werner, H.-T. Yau*, and others, between 1997 and 2016. - Useful discussions with *R. Durrer, G. Isidori* and, most especially, with *N. Straumann* and *D. Wyler*, over many years.

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1. A List of Puzzles in Cosmology
2. Setting the Stage – The Geometry of the Universe
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7. Conclusions?

Disclaimer

This is an informal talk on matters I don't really know much about!

My interests in various *special problems of cosmology* arose accidentally: Since 1989/90 I was working on the theory of the quantum Hall effect (QHE). In 1998, I got interested in the seemingly purely academic question whether there are higher-dimensional cousins of the QHE $\rightarrow$ work with Alekseev and Cheianov. Ruth Durrer drew my attention to possible applications of our results to the problem of the origin of *cosmic magnetic fields*. In 1998, I also became interested in the *Mean Field Limit* of bosonic many-body systems, originally studied by Klaus Hepp (1974), and in solitary wave solutions of the limiting (Mean-Field) non-linear dynamics $\rightarrow$ *gravitational instabilities of boson- and neutron stars; possible models of axionic Dark Matter*.
In 2015, I got interested in the problem of *Dark Energy* (and possible interplays of DE and DM) and of the *Matter-Antimatter Asymmetry* in the Universe.

It is not clear whether my proposals are realistic.

1. A List of Puzzles in Cosmology

$\geq$ 7 basic puzzles in cosmology:

1. Formation of "classical" structure from initial quantum state of Universe: What are *"(Cosmological) Events"*, quantum-mech.? What is *"Dark Matter"*, how does it produce "classical" structure?
2. Role of *Inflation* – what does it explain, is it real, natural?
3. Why is expansion of Universe *accelerated* – what is *"Dark Energy"*?
4. Origin of *matter-antimatter asymmetry* in the Universe?
5. Why are there comparable amounts of *Visible Matter*, *DM* and *DE* in the Universe? Was this and will this always be the case?
6. Origin of *cosmic magnetic fields* ext. over intergalactic distances?
7. Cosmological hints at *Physics beyond the Standard Model* ? Neutrino masses, new degrees of freedom, such as *WIMP's, axions, new scalar fields, new gauge fields*, etc. ?

Comments on Puzzles 1 through 7

1. "ETH approach to QM" – instead of "Many-Worlds Int. of QM"! – Emergence of classical behaviour in Mean-Field Regime – tiny density, tiny gravitational coupling constant – of models of Visible Matter and (Fuzzy) Dark Matter; (Sect. 6).
2. Standard wisdom on inflation: would explain homogeneity, isotropy and spatial flatness ($\Omega_0 = 1$) of the Universe. *Observational indications of inflation:* CMB, nearly scale-inv. (red-shifted) fluctuations, acoustic peaks.
3. Proposal of a model of Dark Energy in Sect. 5.
4. New scalar field as a chemical potential for matter-antimatter asymmetry and a candidate of Dark Energy (?), Sect. 5.
5. This remains rather mysterious – vague ideas concerning "tracking DE"; see Sect. 5.
6. A QED axion as a source of cosmic magnetic fields; Sect. 4.
7. Clearly related to items 2 through 6! – Extra dimension(s)?

2. Setting the Stage – The Geometry of the Universe

From CMB: Up to an age of some $100'000$ years (before large-scale structures formed) Universe was remarkably *homogenous* and *isotropic*. Possible explanation: *Inflation!* Throughout, I will treat it as homogeneous and isotropic on very large distance scales; (e.g., dists. $\geq 10^7$ pc $\ll$ optical radius of Universe $< 10^{10}$ pc). *Consequence:* Universe foliated in space-like hypersurfaces, $\{\Sigma_t\}_{t\in\mathbb{R}}$, orthogonal to a time-like geodesic velocity field U, on which induced metrics are all proportional to one another $\Rightarrow$

$$d\tau^2 = dt^2 - a^2(t)ds^2, \tag{1}$$

where t is cosmological time, $a(t)$ is a scale factor, and ds^2 is the metric of 3D Riemannian manifold, Σ, of *constant curvature*,

$$k = \frac{\varepsilon}{R^2}\,, \qquad \varepsilon = 0, \pm 1.$$

Geometry of Universe – ctd.

Meaning of parameter ε:
$\varepsilon = -1$: open, expanding for ever/ $\varepsilon = 0$: flat, expanding /
$\varepsilon = 1$: closed, evt. collapsing; $R =$ "curvature radius" of Σ.

Plug ansatz (1) into *Einstein's Field Eqs.*, with energy-momentum tensor, $T = (T^{\mu}_{\nu})$, given by

$$T = \mathrm{Diag}(\rho, -p, -p, -p),$$

and appropriate *equations of state* relating ρ to p.

$\Rightarrow \sim$ *Friedmann Eqs.:*

$$\boxed{3H^2 + 3\frac{k}{a^2} = \kappa\rho + \Lambda,} \tag{2}$$

where $\kappa = 8\pi G_{Newton}$, $H(t) := \frac{\dot{a}(t)}{a(t)}$: Hubble "constant",
Λ: cosmological constant;

Geometry of Universe – ctd.

and

$$\boxed{2\dot{H} - 2\frac{k}{a^2} = -\kappa(\rho + p)} \qquad (3)$$

Inflation $\Rightarrow$ $k = 0$; we also set $\Lambda = 0$.

By (2),

$$\rho_{\text{crit.}} = \frac{3}{\kappa} H^2, \qquad \text{corresp. to } k = 0, \Lambda = 0.$$

Density parameter

$$\Omega_0 := \frac{\rho}{\rho_{\text{crit.}}}.$$

From data: $\Omega_0 \approx 1$, as would be explained by Inflation! This implies that, besides Visible Matter (VM, $\approx 5\%$), Dark Matter (DM, $\approx 27\%$), there must also exist Dark Energy (DE, $\Rightarrow \approx 68\%$), as confirmed by data from type IA supernovae (Perlmutter, Schmidt, Riess), CMB and Baryon oscillations (BAO – oscillations in power spectrum of matter).

Equations of State

(i) VM and DM: $p \approx 0$

(ii) Radiation: $T^{\mu}_{\mu} = 0 \quad \Rightarrow \quad p = \frac{\rho}{3}$ (conformal invariance)

(iii) DE (mimics Λ): $p \approx -\rho$

DE apparently dominates ($\approx 68\%$) $\Rightarrow$ Must solve *Friedmann Eqs.* with

$\rho + p = \delta\rho,\ 0 < \delta < 4/3$, (at present $\delta \approx 1/3$), yielding

$$\begin{aligned} a(t) &= a(t_0)\,(t/t_0)^{2/3\delta}\,, \\ H(t) &= (2/3\delta)t^{-1}\,, \\ \rho(t) &= (4/3\kappa\delta)t^{-2} = \text{const.}\, a(t)^{-3\delta}\,. \end{aligned} \tag{4}$$

For Rad.: $\delta = \frac{4}{3}$, $\rho(t) \propto a(t)^{-4} \propto t^{-2}$ (redshift!); for VM & DM: $\delta = 1$, $\rho(t) \propto a(t)^{-3} \propto t^{-2}$; for DE only: $\delta = 0, \rho(t) = const., H = const.$

Assuming Universe is in thermal equilibrium in radiation-dominated phase, before recombination, *Stefan-Boltzmann* implies that

$$\boxed{T(t) \propto \rho^{1/4} \propto \frac{1}{\sqrt{t}}\,, \qquad \text{with } T(t) = \text{const., for } \delta = 0!} \tag{5}$$

3. The State of the Universe Shortly After Inflation

Henceforth, U and Σ_t are always as above Eq. (1); $\mathrm{d}vol_t(x) =$ volume form of the metric $a(t)^2 ds^2$ on Σ_t. – Initially, all quantities encountered below are to be understood as *qm operators*.

Quantum State of early Universe in radiation-dom. phase (before matter decouples): Local thermal equilibrium (LTE) at a temperature $T \approx$ (5). In order to identify this state, must know which quantities are (approx.) conserved in the hot, early Universe. Let's imagine these quantities correspond to approximately conserved currents, $J_a^\mu, a = 1, 2, \ldots$

$$T_{00} \propto T(U, U), \qquad j_a := \text{3-form dual to } J_a$$

Then LTE at time t is described by the (ill-def.) *"density matrix"*

$$P_{LTE} \propto exp\left(-\int_{\Sigma_t} \beta(x)\big[T_{00}(x)\,\mathrm{d}vol_t(x) - \sum_a \mu_a(x) j_a(x)\big]\right), \qquad (6)$$

where $\beta(x)$ is a (space-) time-dep. inverse temperature; "fields" $\mu_a(x)$ are local (space-time dep.) chemical potentials conjugate to (approx.) conserved currents J_a; normalisation factor multiplying R.S. of (6), chosen such that trace of P_{LTE} is $= 1$, is called inverse partition function.

Conserved and Anomalous Currents

From now on, adopt *thermodynamical interpretation* of $\beta(x), \mu_a(x)$ as TD state parameters/ "moduli".

A current, J, is said to be *conserved* iff

$$\nabla_\mu J^{\mu\cdots} = 0 \quad \Leftrightarrow \quad \mathrm{d}j^{\cdots} = 0 \quad (\text{d: exterior derivative})$$

Examples: (i) Electric current (density) J; (ii) $J_B - J_L \leftrightarrow$ matter-antimatter asymmetry;...

In the presence of gauge fields, or for massive matter fields, *axial currents*, J_5^μ, are usually *anomalous*, i.e., not (strictly) conserved:

$$\nabla_\mu J_5^\mu = \frac{\alpha}{4\pi} \varepsilon^{\mu\nu\sigma\rho} \mathrm{tr}(F_{\mu\nu} F_{\sigma\rho}) + \text{terms} \propto \text{masses.} \tag{7}$$

Here F is the field tensor of a gauge field, A, and $\alpha =$ analogue of fine structure constant.

An example of an anomalous current is the leptonic axial current, $J_{L,5}^\mu$, sensitive to the asymmetry between left-chiral and right-chiral leptons.

Conservation Laws Assoc. With Anomalous Currents

Let j_5 be the 3-form dual to an anomalous current J_5^μ. Let $\mathrm{tr}(A \wedge F)$ be the Chern-Simons 3-form of the gauge field A; (components given by $\mathrm{tr}(A_{[\mu} F_{\nu\rho]})$). If *masses of matter fields are negligible* then Eq. (7) $\Rightarrow$

$$\mathrm{d}\big(j_5 - \frac{\alpha}{2\pi}\mathrm{tr}(A \wedge F)\big) = 0,$$

i.e., the axial current dual to $j_5 - \frac{\alpha}{2\pi}\mathrm{tr}(A \wedge F)$ is *conserved*, though *not* gauge-invariant. However,

$$Q_5 := \int_{\Sigma_t} \big(j_5 - \frac{\alpha}{2\pi}\mathrm{tr}(A \wedge F)\big) \qquad (8)$$

is a gauge-invariant, conserved charge.
In order for the state (6) to be gauge-invariant, we then must require

$$\mathrm{d}(\beta\mu_5) \wedge F|_{\Sigma_t} = 0, \qquad (e.g.,\ \beta\mu_5 \text{ only dep. on time } , t), \qquad (9)$$

where μ_5 is the chemical potential conjugate to $j_5 - \frac{\alpha}{2\pi}\mathrm{tr}(A \wedge F)$.

Conserved Currents & Conjugate Chemical Potentials

Remark: Henceforth, fields and currents will be treated as classical, (i.e., as expectations of qm operators in the state of the Universe). – Quantum cosmology remains to be developed!

Conserved currents and conjugate chem. potentials:

I. Electric vector current density:

$J^{\nu} \leftrightarrow \mu_{el} = 0$ (local electric neutrality!)

II. $J^{\nu}_{B} - J^{\nu}_{L} \leftrightarrow \mu_{B-L}$ (tunes matter-antimatter asymmetry; μ_{B-L} related to a scalar field, σ, connected to DE (?))

III. Leptonic axial current density, $J^{\nu}_{L,5}$, dual to $j_{L,5} - \frac{\alpha}{2\pi} A \wedge F$, where A is the electromagnetic vector potential, masses *neglected*:

$J^{\nu}_{L,5} \leftrightarrow \mu_5$ (tunes left-right asym., $\mu_5 \propto \dot{\theta}$, θ an "axion" field)

4. Generation of Primordial Magnetic Fields from Axion

Maxwell Equations in an Expanding Universe, with $\Sigma \simeq \mathbb{R}^3$ flat:

$$\vec{\nabla} \wedge \vec{E} + \dot{\vec{B}} + \frac{3}{2} H \vec{B} = 0, \quad \vec{\nabla} \cdot \vec{B} = 0 \tag{10}$$

$$\vec{\nabla} \wedge \vec{B} - \dot{\vec{E}} - \frac{3}{2} H \vec{E} + M^2 \vec{A}^{\perp} = \vec{J}, \quad \vec{\nabla} \cdot \vec{E} = \rho. \tag{11}$$

Here H is the Hubble "constant", $\vec{A}^{\perp}$ is the electromagnetic vector potential in the Coulomb gauge, $\vec{J}$ is the electric current density, ρ is the charge density, and $M \geq 0$ is a photon mass. It is reasonable to assume that in a hot plasma $\rho \equiv 0$. We have to find an expression for the current density $\vec{J}$. There is an Ohmic contribution to $\vec{J}$, but also one that mirrors a possible left-right asymmetry: chiral magnetic effect. For simplicity, we assume that the primordial plasma is $\sim$ at rest in the coordinates introduced in (1), above. Eq. (6) and the chiral anomaly imply ($\nearrow$ ACF)

$$\boxed{\vec{J} = \sigma \vec{E} + \frac{\alpha}{\pi} \mu_5 \vec{B}} \tag{12}$$

In (12), spatial dependence of μ_5 neglected; but μ_5 may depend on t!

An Instability

Plugging (12) into (11), taking the curl of the first eq. in (11), and using the first eq. in (10), one finds:

$$-\Delta\vec{B} + \ddot{\vec{B}} + (2h+\sigma)\dot{\vec{B}} + \dot{h}\vec{B} + h(h+\sigma)\vec{B} - \mu_5\vec{\nabla}\wedge\vec{B} + M^2\vec{B} = 0, \quad (13)$$

with $h := \frac{3}{2}H$. We solve (13) by Fourier transformation[1]:

$$\vec{B} = \vec{b}\, e^{i(kz-\omega t)}, \ \text{where } \vec{b} \perp \vec{e}_3 \quad (\vec{e}_3 = z - \text{ axis}) ,$$

using that $\vec{\nabla}\cdot\vec{B} = 0$. We then find that

$$\omega(k) = -i\left(h + \frac{\sigma}{2}\right) \pm \sqrt{-\left(h+\frac{\sigma}{2}\right)^2 + k^2 + M^2 + h(h+\sigma) + \dot{h} \pm \mu_5|k|} \quad (14)$$

We observe that the expansion of the Universe (i.e., $H > 0$) and Ohmic conductivity of the primordial plasma lead to power-law (actually, exp. if h =const.) damping of $\vec{B}$ in time, provided

$$\mu_5|k| < k^2 + M^2 + h(h+\sigma) + \dot{h} \quad (15)$$

[1] Time-dependence of μ_5 and of $h, \dot{h}$ assumed to be negligible

Instability – ctd

We will see that it is likely that $\mu_5 \searrow 0$, as $t \nearrow \infty$. Hence evolution of electromagnetic field is damped for large times if the mass M of the photon were positive. –*However*, for $M = 0$, one encounters a power-law (exponential, for $h = 0$) *instability* in the solutions of Eq. (13) for wave vectors k satisfying

$$\boxed{\frac{\mu_5 - \sqrt{\mu_5^2 - K}}{2} < |k| < \frac{\mu_5 + \sqrt{\mu_5^2 - K}}{2},} \tag{16}$$

where $K := 4\left[h(h+\sigma) + \dot{h}\right]$; growth rate $\propto \sqrt{\mu_5/(h + \frac{\sigma}{2})}$.
This is a *mechanism for the growth of very homogeneous primordial magnetic fields* from quantum fluctuations, (which, at late times, exhibit power-law decay dictated by H).

A systematic study of *Relativistic Magneto-Hydrodynamics* in the presence of the chiral magnetic effect (terms in the equations of motion proportional to $\mu_5 \ \propto$ time-derivative of axion field!) is presently carried out with Boyarsky, Brandenburg, Rogachevskii, Ruchayskiy, and others. We have identified various novel dynamos driven by chiral asymmetry.

Possible Origins of μ_5

Next, we search for origins of a non-trivial "chemical potential" μ_5. Let us look for a generally covariant form of Eq. (12), i.e., of $\vec{J} = \sigma \vec{E} + \frac{\alpha}{\pi}\mu_5 \vec{B}$, linear in the field tensor $F_{\nu\lambda}$:

$$J^\nu(x) = \frac{\alpha}{2\pi f}\varepsilon^{\nu\lambda\rho\tau}(\partial_\lambda\theta)(x)F_{\rho\tau}(x) - \sigma F^{\nu\lambda}(x)V_\lambda(x) + \rho(x)V^\nu(x). \quad (12')$$

Here f is the "axion decay constant", θ is a pseudo-scalar axion field, V^ν is the four-velocity field of the primordial plasma, and ρ is its charge density. Imposing local electric neutrality, i.e., $\rho \equiv 0$, we find that (in conformal time, τ)

$$\begin{aligned}\vec{J} &= \sigma\big(\vec{E} + \frac{1}{c}\vec{V}\times\vec{B}\big) + \frac{\alpha}{\pi f}\{[\dot{\theta} + \vec{V}\cdot\vec{\nabla}\theta]\vec{B} \\ &+ \vec{\nabla}\theta \times \big(\vec{E} + \frac{1}{c}\vec{V}\times\vec{B}\big)\},\end{aligned}$$

so that

$$\mu_5 = (\overline{\dot{\theta}}/f)\,. \quad (17)$$

Possible Origins of μ_5 – ctd.

We consider the special situation where $\sigma = \rho \equiv 0$. Then the Maxwell equations, with J^ν as in (12'), can be derived by varying the following action functional w. r. to the em vector potential A:

$$S(A,\theta) := \int \left[F_{\nu\lambda}(x)\, F^{\nu\lambda}(x)\, a(t)^3 d^4x - \frac{\alpha}{2\pi f}\theta(x) F(x) \wedge F(x)\right], \quad (18)$$

with $F \wedge F$ dual to $4\vec{E} \cdot \vec{B}$. Next, we search for an eq. of motion for θ. Let

$$q_5(t;\mu_5) := \text{ spatial average of } \rho_5(\vec{x},t) = \overline{\langle J_5^0(\vec{x},t)\rangle}_{\beta,\mu_5}$$

In the radiation phase ($a(t) \propto \sqrt{t}$, $T(t) \propto 1/\sqrt{t}$)

$$q_5(t;\mu_5) \approx \mu_5 \frac{\partial q_5}{\partial \mu_5}(t;\mu_5 = 0) \approx const.\, T^2 \mu_5 = a(t)^{-2}\mu_5\,,$$

with $q_5(t;\mu_5 = 0) = 0$.

For an incompressible plasma, this relation, Eq. (17) and the chiral anomaly imply that (in physical time, t)

$$\ddot{\bar{\theta}} + \text{ term } \propto H\dot{\bar{\theta}} = \frac{\alpha}{\pi}\,\overline{\vec{E}\cdot\vec{B}}. \quad (19)$$

Axion Equation of Motion

By (19), the field equation for θ must look like

$$\Box\theta + U'(\theta) = \frac{\alpha}{2\pi}\vec{E}\cdot\vec{B} \tag{20}$$

a non-linear wave equation; *or like*

$$\ddot{\theta} + 3H\dot{\theta} - D\Delta\dot{\theta} + U'(\theta) = \frac{\alpha}{2\pi}\vec{E}\cdot\vec{B}, \tag{20'}$$

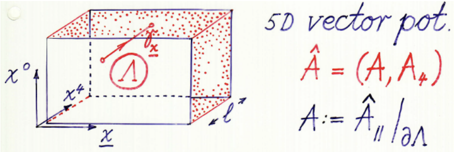
a non-linear diffusion eq., $D =$ diffusion constant, (with $\overline{U'(\theta)} = 0$). ...
Eq. (20) is reminiscent of the field equation for an *axion* $\rightarrow$ identify θ with a *pseudo-scalar axion field.*
The Maxwell equations, with J^ν as in (12') ($\sigma \equiv 0, \rho \equiv 0$), and Eq. (20) can be derived by varying the action functional

$$S_{\mathrm{tot}}(A,\theta) := S(A,\theta) + \int a(t)^3 d^4x \left[\partial_\nu\theta\partial^\nu\theta + U(\theta)\right]. \tag{21}$$

Remark: For $U \equiv 0$, this action can be derived from Maxwell theory in 5D with a 5D Chern-Simons term by dimensional reduction, with

$$\theta(x) := \overline{\hat{A}_4(x,\cdot)} \;\; \Rightarrow \;\; \mu_5 = f^{-1}\overline{\hat{E}_4(x,\cdot)}, \;\; f^{-1} = \text{length scale of } 5^{\text{th}} \text{ dim.}$$

A 5D Cousin of the QHE



5D bulk, Λ, filled with heavy 4-component Dirac fermions coupled to $\hat{A}$
$\Rightarrow$ *PT*- breaking! $\rightarrow$ 5D analogue of anomalous Hall (bulk) current:

$$\boxed{j = \sigma_H \hat{F} \wedge \hat{F}, \ \sigma_H \sim \ \text{5D "Hall conductivity"} \ \Rightarrow \ \text{chiral surface currents.}}$$

Visible world located on $\partial\Lambda$, of 5D Univ.; light left-chiral- and right-chiral surface modes on different boundary branes; (masses gen. by tunneling)!

Instead of $-\frac{\alpha}{\pi f}\,\theta\,\vec{E}\cdot\vec{B}$ in the action (18), we may add ($\nearrow$ chir. anomaly)

$$f^{-1}\int \partial_\nu \theta(x) J^\nu_{L,5}(x)\, a(t)^3 d^4x = f^{-1}\int \mathrm{d}\theta \wedge j_{L,5} \ , \qquad (22)$$

which would introduce additional terms $\propto$ lepton masses. Formula (22) shows that $f^{-1}\dot{\theta}$ can be interpreted as a "chemical potential" for $j_{L,5}$.

5. Matter-Antimatter Asymmetry and Dark Energy

The current $J_{B-L} := J_B - J_L$ is conserved, and the corresponding charge, Q_{B-L}, is a measure of Matter-Antimatter Asymmetry. Assuming that this asymmetry originates in a phase when the Universe was in a state of local thermal equilibrium, i.e., before matter decoupled, it is natural to imagine that a chemical potential, μ_{B-L}, conjugate to Q_{B-L} tunes the Matter-Antimatter Asymmetry. Possible choices for μ_{B-L} might be

$$\mu_{B-L} = \dot{\sigma}, \text{ or } \mu_{B-L} = \sigma \cdot \dot{\phi}, \quad \text{where } \phi, \sigma \text{ are real scalar fields.} \tag{23}$$

In this section we introduce a scalar field σ suitable to tune the Matter-Antimatter Asymmetry and then argue that it gives rise to roughly the right amount of *Dark Energy*, as well as a promising amount of *Dark Matter*. Its action functional is chosen to be

$$S(\sigma; g) := \int \sqrt{-g}\, \mathrm{d}^4 x\, \mathcal{L}(\sigma, \partial_\mu \sigma; g)(x), \quad \text{where} \tag{24}$$

$$\mathcal{L}(\sigma, \partial_\mu \sigma; g)(x) := \tfrac{1}{2} \partial_\mu \sigma(x) g^{\mu\nu}(x) \partial_\nu \sigma(x) - \Lambda e^{-(\sigma(x)/f)},$$

$f \approx M_{\text{Planck}}$ is a constant, and $(g_{\mu\nu})$ is the metric on space-time.

Equation of Motion in a Homogeneous, Isotropic Space-Time

We make the ansatz that σ only depends on cosmological time t and that the metric of space-time satisfies the Friedmann eqs. Then

$$S(\sigma, g) \equiv S(\sigma) = \mathrm{vol}(\Sigma) \int \mathrm{d}t \underbrace{a(t)^3 \{ \frac{1}{2} \dot{\sigma}(t)^2 - \Lambda e^{-(\sigma(t)/f)} \}}_{\equiv L(\dot{\sigma}, \sigma, t)} \quad (25)$$

The Euler-Lagrange equation of motion reads

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\sigma}} - \frac{\partial L}{\partial \sigma} = 0$$

$$\Leftrightarrow \quad \boxed{\ddot{\sigma}(t) + 3H(t)\dot{\sigma}(t) - \frac{\Lambda}{f} e^{-(\sigma(t)/f)} = 0,} \quad (26)$$

with $3H(t) = \frac{2}{\delta} t^{-1}$; (w.l.o.g. $t + \tau \mapsto t$!) The parameter δ has been introduced in our discussion of equations of state:

$$\rho + p =: \delta \rho, \qquad \text{(with } \delta \approx \frac{1}{3}, \text{ at present)}$$

The Energy-Momentum Tensor of σ

Here ρ is the total energy density and p is the total pressure. These quantities are constrained by the Friedmann equations (2), (3). The energy-momentum tensor T is given by

$$T \equiv (T^{\mu}_{\ \nu}) = \mathrm{Diag}(\rho, -p, -p, -p).$$

The contribution of the field σ to T is calculated from the formula

$$T_{\mu\nu} = \frac{\delta S(\sigma, g)}{\delta g^{\mu\nu}} \ \Rightarrow \ T^{\mu}_{\ \nu}(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \sigma(x))} \cdot (\partial_\nu \sigma)(x) - \delta^{\mu}_{\ \nu} \mathcal{L}(\sigma, ...)(x).$$

For our special ansatz, $\sigma = \sigma(t)$ (indep. of $\vec{x}$), this yields

$$\rho_\sigma = \frac{1}{2}\dot{\sigma}^2 + \Lambda e^{-(\sigma/f)}, \quad p_\sigma = \frac{1}{2}\dot{\sigma}^2 - \Lambda e^{-(\sigma/f)}. \tag{27}$$

Setting $\rho = \rho_\sigma + \rho_M$, $p = p_\sigma + p_M$, ($\rho_M :=$ energy density of matter, $p_M \approx 0$, Radiation neglected), the Friedmann equations yield

$$-\frac{2}{\kappa}\dot{H} = \rho + p = \dot{\sigma}^2 + \rho_M = \delta\rho = \frac{3\delta}{\kappa} H^2. \tag{28}$$

A Special Solution of the Equation of Motion (26)

A special solution of Eq. (26) is given by

$$\sigma(t) \equiv \sigma^{(0)}(t) = \sigma_0 \, \ell n(\frac{t}{t_0}), \text{ with } \sigma_0 = 2f, \; t_0 = \sqrt{\frac{4-2\delta}{\delta \, \Lambda}} \, f \quad (29)$$

For this solution, we have that

$$\rho_\sigma(t) + p_\sigma(t) = \delta \rho_\sigma(t), \quad \forall \delta \, , \tag{30}$$

with

$$\rho_\sigma(t) = \frac{4}{\delta} f^2 t^{-2}, \quad p_\sigma(t) = (4 - \frac{4}{\delta}) f^2 t^{-2}.$$

Thus, *the Friedmann equations are solved, provided*

$$\boxed{\rho_M, p_M = 0, \text{ and } f^2 = (3\delta\kappa)^{-1}} \tag{31}$$

Tantalizingly, $f^2 = \kappa^{-1}$, for $\delta = \frac{1}{3}$!

Interpretation of results (29, (30) and (31)

Remarks:

I. Relation (31) suggests that the field σ is a *gravitational degree of freedom*. In previous work with *Chamseddine* and *Grandjean*, it has been argued that $exp[-(\sigma/f)]$ is related to the scale of an extra dimension (chosen to be discrete in CFG), and that $f \approx \kappa^{-(1/2)}$ is a consequence of deriving the action functional for σ from a higher-dimensional *Einstein-Hilbert action* by "dimensional reduction". This fits well with the idea that the QED axion introduced in the Section 4 can be interpreted as arising from electromagnetism (with Chern-Simons term) on a 5D space-time, with a continuous or discrete extra dimension.

II. Results (30) and (31) suggest that, as time $t \to \infty$ (when matter and radiation become negligible), the solution $\sigma^{(0)}$ is an *"attractor"* in solution-space. This expectation is supported by the following result.

Linear and Non-Linear Stability of $\sigma^{(0)}$

Theorem: General solutions, $\sigma(t)$, of (26) approach $\sigma^{(0)}(t)$, as $t \to \infty$.

Linear Stability:
Inserting the ansatz $\sigma(t) := \sigma^{(0)}(t) + \sigma^{(1)}(t)$, with $\sigma^{(1)}(t) \ll \sigma^{(0)}(t)$, for large t, into (26) and linearizing in $\sigma^{(1)}$, we find that

$$\sigma^{(1)}(t) \propto t^{-\alpha}, \quad \alpha = \beta \pm \sqrt{\beta^2 - 4\beta}, \quad \beta := \delta^{-1} - \frac{1}{2} > \frac{1}{4}.$$

Note that $\Re\alpha > 0$, $\forall \delta \leq \frac{4}{3}$, hence $\sigma^{(1)}(t) \searrow 0$, as $t \to \infty$.
If $\beta < 4$, i.e., $\delta > \frac{2}{9}$, then $\Im\alpha \neq 0 \Rightarrow \sigma^{(1)}$ describes *oscillations* (with a tiny time-dependent mass $\propto f(t_0/t)^2$) that die out like $t^{\frac{1}{2}-\delta^{-1}}$. These oscillations may contribute to Dark Matter.

Non-Linear Stability:

$$\rho_\sigma = \frac{1}{2}\dot{\sigma}^2 + \Lambda e^{-(\sigma/f)}$$

is a *Lyapunov functional* that decreases in time on solutions of (26).
All solutions of (26) are bounded above by $\ell n(\frac{t}{t_*})$, for some t_*.

Matter-Antimatter Asymmetry

To the action $S(\sigma, g)$ one can add the "topological term"

$$G \int \mathrm{d}\sigma \wedge j_{B-L}, \qquad \text{where } G \text{ is a constant.} \tag{32}$$

This term does not appear in the equation of motion for σ, because it is a *pure surface term*. However, it *will* appear in our formula for the state, P_{LTE}, of the Universe describing local thermal equilibrium. In the formula for P_{LTE}, the time derivative of σ plays the role of a *time-dependent chemical potential* conjugate to the conserved charge Q_{B-L}. Before matter decouples, $\dot{\sigma}$ could be large and, hence, might trigger a substantial asymmetry between Matter and Antimatter.

Yet, there is another problem we have to tackle! Since the Friedmann equations are automatically satisfied for the solution $\sigma^{(0)}$ of (26) displayed in (29), provided Relation (31) between f, δ and κ holds – *without* introducing additional fields – one may worry that there won't be room for *Radiation* and *Visible -*, as well as *Dark Matter* in the Universe.

The fate of the Universe

However, one expects that, if ordinary matter is introduced into the model and contributes to ρ, then, *for small* t, $H(t)$ will deviate from $\frac{2}{\delta}$, and the true solution, $\sigma(t)$, of (26) will deviate from $\sigma^{(0)}(t)$! As a test example one may want to add to the degrees of freedom of the model a massive free scalar field, φ, which only couples directly to the metric tensor $g^{\mu\nu}$ (but not to σ). One then has to solve the resulting coupled equations for σ and φ, along with the Friedmann Equations.
If, for some reason, this does not yield satsifactory results one might have to impose a "slow-roll" condition on $\sigma(t)$, effective at *early times*. There are various possibilities to do this.
One has reasons to hope that a model of the kind suggested here will also trigger *inflation* at very early times!

The late-time fate of the Universe

To conclude, it appears safe to predict that the late-time fate of the Universe will be very boring: The solution $\sigma^{(0)}$ will dominate, as t tends to ∞; and all other degrees of freedom will become negligible!

7. Conclusions?

For an outsider like myself, theoretical cosmology – in contrast to observational, phenomenological and computational cosmology – does not look like a firmly established theoretical science, yet. My impression is that the following features tend to drive one into serious difficulties:

1. The equations describing the evolution of Visible and Dark Matter and Dark Energy are highly non-linear. They might exhibit instabilities, most obviously gravitational instabilities and gauge-field dynamos, which we do not know how to treat properly, yet.
2. In every serious analysis of the dynamical evolution of the Universe one faces the problem that all forms of matter and energy are quantum-mechanical, but all gravitational degrees of freedom (g and $\sigma, \dots$) are treated classically. While there may be various self-consistent ways (e.g., semi-classical approx.) of dealing with this basic problem, it is deeply disturbing – in just about any serious study of cosmology – that we still do not know how to combine Quantum Theory with a Relativistic Theory of Gravitation.
3. On the positive side, we have made a case for the existence of extra dimensions and of an additional gravitational degree of freedom, in the form of the field σ., accounting for Dark Energy.

“Survivre et Vivre” – 47 years later

… depuis fin juillet 1970 je consacre la plus grande partie de mon temps en militant pour le mouvement *Survivre*, fondé en juillet à Montréal. Son but est la lutte pour la survie de l’espèce humaine, et même de la vie tout court, menacée par le déséquilibre écologique croissant causé par une utilisation indiscriminée de la science et de la technologie et par des mécanismes sociaux suicidaires, et menacée également par des conflits militaires liés à la prolifération des appareils militaires et des industries d’armements. …

Alexandre Grothendieck