The Quantum Sine Gordon in pAQFT

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joint with Kasia Rejzner (CMP 2017)
and with Klaus Fredenhagen and Kasia Rejzner (arXiv:17xx.xxxxx)

Advances in Mathematics and Theoretical Physics
Accademia Nazionale dei Lincei, Roma
19 September 2017

Motivation

Perturbative Algebraic Quantum Field Theory (pAQFT) provides a framework for (perturbative) calculations – without Hilbert spaces. Idea: (Generalized) Haag-Kastler axioms for a perturbative setting (formal power series). Representations (on a Hilbert space) come at a later stage via a GNS construction for a suitable state on a suitable algebra.

This is important for QFT calculations in other settings than on flat (semi-)Riemannian vector spaces, e.g. a globally hyperbolic manifold M, where we have in general neither a distinguished vacuum state at hand, nor, by the way, a Wick rotation and related Euclidean theory.

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[Dütsch-Fredenhagen 00, Brunetti-Fredenhagen 00, Brunetti-Dütsch-Fredenhagen 09, . . . Hollands 08, Fredenhagen-Rejzner 11, . . .]. Based on [Radzikowski 96, Kay-Wald 91, Dimock 92 . . .]. Met with [Hollands-Wald 01, . . .]

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I will also comment on joint work with Klaus Fredenhagen and Kasia Rejzner, arXiv:17xx.xxxxx, where we can give up this restriction – and will hint at how to construct the Haag-Kastler net of local observables.

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Important not least because this makes a direct comparison of a Euclidean and a Minkowski theory feasible!

The ingredients

Main ingredient in pAQFT are certain distributions: the advanced and retarded fundamental solutions, E_A and E_R, respectively, of a linear (hyperbolic) PDE $Pf = 0$ on some spacetime M (the "free equation"), e.g. the wave operator $P = -\Box$ or the Klein-Gordon operator $P = -\Box + V$, $V \in C^\infty(M)$ smooth.

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On a globally hyperbolic manifold M, E_A and E_R exist as maps $C_c^\infty(M) \to C^\infty(M)$ and are uniquely characterized by their support properties [Duistermaat-Hörmander, ...],

$$P \circ E_{A/R} = E_{A/R} \circ P = \mathrm{id} \qquad \mathrm{supp} E_{A/R} f \subset J^{\mp}(\mathrm{supp} f)$$

E_A and E_R and a so-called 2-point function W (more below) are used to define star products on certain spaces of functionals $\mathcal{F}_{\dots}(M)$ on the theory's configuration space $\mathcal{E}(M)$.

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In our case (sine Gordon), the free theory is given by $P = -\Box$ on $M = \mathbb{R}^2$, the 2-dimensional Minkowski space ("massless scalar field"), and $\mathcal{E}(M) = C^\infty(M, \mathbb{R})$.

Star products – starting point

One of the arts in pAQFT is to determine suitable spaces $\mathcal{F}_{\dots}(M)$ of (i.g. nonlinear) functionals on $\mathcal{E}(M)$ on which certain star products are defined and then to possibly complete/extend to larger algebras containing physically relevant observables.

We start from functionals on $\mathcal{E}(M) = C^{\infty}(M, \mathbb{R})$ whose functional derivatives are compactly supported distributions, i.e. for any $\varphi \in \mathcal{E}(M)$, we have $F^{(k)}(\varphi) \in \mathcal{E}'(M^k)$ for any $k \in \mathbb{N}$.

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Quantization of the free theory as a deformation quantization (with formal paramter $\hbar$) of Peierl's bracket which for the free theory is a map $\mathcal{F}_{\mu c}(M) \times \mathcal{F}_{\mu c}(M) \to \mathcal{F}_{\mu c}(M)$,

$$\{F, G\} = \left\langle F^{(1)}, \Delta G^{(1)} \right\rangle ,$$

where $\Delta = E_R - E_A$ ("causal propagator"). The microcausal functionals $\mathcal{F}_{\mu c}(M)$ are characterized by a condition on their functional derivatives' wavefront set (microlocal methods).

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where $\Delta = E_R - E_A$ ("causal propagator"). The microcausal functionals $\mathcal{F}_{\mu c}(M)$ are characterized by a condition on their functional derivatives' wavefront set (microlocal methods). Note: WF characterizes the singularities of a distribution.

Formal power series II

S-matrix encodes "the physics" of a model with interaction, written usually in terms of Dyson's series – formalized in pAQFT. Building block are time ordered products.

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Def: Time-ordering operator $\mathcal{T}$

$$(\mathcal{T}F)(\varphi) \doteq \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(2n)}(\varphi), (i\Delta_D)^{\otimes n} \right\rangle$$

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Def: Introduce a second formal parameter λ and define formal S-matrix (for an interaction F)

$$\mathcal{S}(\lambda F) \doteq \mathcal{T}\left(e^{i\mathcal{T}^{-1}(\lambda F)/\hbar}\right) = \sum_{n=0}^{\infty} \left(\frac{i\lambda}{\hbar}\right)^n \frac{1}{n!} F^{\cdot_{\mathcal{T}} n}$$

The Feynman propagator and normal ordering

Physics: Where's the Feynman propagator? Answer: Interaction is something that needs normal ordering (to make sense). This leads to a Feynman propagator (another fundamental solution) E_F entering into the formal S-matrix. Contrary to E_R and E_A, it is not uniquely determined, there is no canonical choice in a generic spacetime.

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$$E_F = \frac{i}{2}(E_R + E_A) + H$$

where H is some symmetric bisolution of P such that the "2-point function" (distribution)

$$W = \frac{i}{2}(E_R - E_A) + H$$

satisfies certain properties (positivity, wavefront set condition, ...).

Existence of left hand side W: by abstract arguments [Fulling - Narcowich - Wald 81, ...] related to the fact that actually, W is the 2-point function of a Hadamard state (on the CCR algebra built from $\Delta = E_R - E_A$).

Remarks

Feynman propagator

$$E_F = \frac{i}{2}(E_R + E_A) + H$$

The difference of two different choices of H is smooth.

H is symmetric, so the $\star$ product modfied by adding H, is still a quantization of Peierl's bracket. Different choices of H correspond to different choices of normal ordering, and the corresponding star products are equivalent (in the sense of formal power series) with an explicit formal map

$$\alpha_{H-H'} : \mathcal{F}_{\mu c}[[\hbar]] \to \mathcal{F}_{\mu c}[[\hbar]], \quad \alpha_{H-H'}(F) = \sum \frac{\hbar^n}{2^n \; n!} \left\langle (H - H')^{\otimes n}, F^{(2n)} \right\rangle$$

Formal S-matrix

The formula for the formal S-matrix then is (for a regular interaction $V \in \mathcal{F}_{reg}$, normally ordered w.r.t. H, $:V:_H = \alpha_H^{-1}(V)$)

$$\mathcal{S}(\lambda :V:_H) = \alpha_H^{-1} \sum_{n=0}^{\infty} \tfrac{1}{n!}(\tfrac{i\lambda}{\hbar})^n \ \mathcal{T}_n^H(V^{\otimes n})$$

where

$$\mathcal{T}_n^H \doteq \alpha_H \circ \mathcal{T}_n \circ (\alpha_H^{-1})^{\otimes n} = e^{\hbar \sum_{1 \le i < j \le n} \mathcal{D}_{\mathrm{F}}^{ij}} ,$$

where $e^{\hbar \cdots}$ is understood as a formal power series, where $\mathcal{T}_n$ abbreviates taking the n-fold time ordered product, where for a (regular) functional G,

$$\mathcal{D}_F(G) \doteq \langle \Delta_F, G^{(2)} \rangle$$

with "the" Feynman propagator Δ_F, and where the superscript i, j denotes on which of the n copies of V the functional derivatives act.

Expectation values

From the general setup of pAQFT [e.g. Fredenhagen-Rejzner 15]:

A Gaussian state with covariance H on a certain algebra is defined by evaluation of $\alpha_H(A)$ in a configuration $\varphi \in \mathcal{E}$,

$$\omega_{\varphi,H}(A) \doteq \alpha_H(A)(\varphi)$$

The choice $\varphi = 0$ is distinguished by the fact (which actually motivates the above definition) that $\omega_{0,H}$ is exactly the expectation value in the state whose 2-point function is given by $W = \frac{i}{2}\Delta + H$.

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For the S-matrix we find

$$\omega_{0,H}(\mathcal{S}(\lambda : V :_H)) = \alpha_H \left(\alpha_H^{-1} \sum_{n=0}^{\infty} \tfrac{1}{n!} (\tfrac{i\lambda}{\hbar})^n \, \mathcal{T}_n^H(V^{\otimes n}) \right) \Big|_{\varphi=0}$$

Observe: There is no vacuum state for massless scalar fields in 2D. This is often misinterpreted as the nonexistence of the field – but that is not correct: choose a Hadamard state – calculate.

Sine Gordon

Free theory: wave operator in 2-dimensional Minkowski space-time.
Interaction: $V = V^{a,g} = \frac{1}{2}\left(V_a(g) + V_{-a}(g)\right)$ with the (normally ordered) Vertex operators $V_a(g)$

$$: V_a(g) :_H = \alpha_H^{-1}\Big(\int \exp\left(ia\Phi_x\right) g(x) dx\Big)$$

where g is an arbitrary cutoff function, and Φ_x denotes the evaluation functional $\Phi_x(\varphi) = \varphi(x)$ for all $\varphi \in \mathcal{E}$.

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One of the simplest models – treated over the decades by numerous people and schools. Our approach resembles the calculations of vacuum expectation values as in [Fröhlich, Seiler, ... 1970ies]. But their calculations were done in a Euclidean setting. What's the difference? We do not have to worry about the singularities of the vacuum, but simply choose a Hadamard state and calculate the expectation value.

Convergence of the S-matrix

Theorem (DB, Rejzner CMP 2017)

There is a choice of the testfunction g (cutting of the interaction), such that the formal S-matrix $\alpha_H \circ \mathcal{S}(\lambda :V^{a,g}:_H)$ in the sine Gordon model with $\hbar a^2/4\pi < 1$ converges as a functional on the configuration space.

- Note that we actually prove (absolute) convergence of the series of the expectation value of the S-matrix in a state $\omega_{\varphi,H}$ with H given by a 2-point function $W = \frac{i}{2}\Delta + H$. By the pAQFT setup this implies the above. Existence of such states: by abstract reasoning, or cf. Schubert's Hamburg Diplom thesis 2013
- To be precise, we prove absolute convergence in a "state" given by the nonpositive Hadamard parametrix $\tilde{W} = \frac{i}{2}\Delta + \tilde{H}$. By general nonsense (shifting formal equivalences about), we can absorb the resulting equivalence $\alpha_{H-\tilde{H}}$ into the test function, otherwise our estimates are left undisturbed.

Main estimate

At n-th order perturbation theory, we have to estimate expressions of the form $\mathcal{T}_n^H(V_a^{\otimes k} \otimes V_{-a}^{\otimes n-k})$ with $0 \leq k \leq n$.

The key feature is that the time ordered products are (by definition) exponential sums of Feynman propagators, and that in our particular model, where $E_{R/A}(x,y) = -\frac{1}{2}\theta(\pm(x_0 - y_0) - |\mathbf{x} - \mathbf{y}|)$, and for our choice $H(x,y) = -\frac{1}{4\pi}\ln|(x-y)^2|$, the Feynman propgator is a logarithm.

Hence, everyting boils down to estimating expressions of the form

$$\prod_{1 \leq i < j \leq k} |\tau_{ij}^2 - \zeta_{ij}^2|^{\beta} \prod_{1 \leq i \leq k, k < j \leq n} |\tau_{ij}^2 - \zeta_{ij}^2|^{-\beta} \prod_{k < i < j \leq n} |\tau_{ij}^2 - \zeta_{ij}^2|^{\beta}.$$

with $\beta = \hbar a^2/4\pi > 0$, and with the time variable differences $\tau_{ij} = t_i - t_j$ and the space variable differences $\zeta_{ij} = \mathbf{x}_i - \mathbf{x}_j$.
Note that the assumption $\beta < 1$ ensures that we do not have to renormalize: all functions are locally integrable, the singularities at coinciding points (UV) are irrelevant (the model's finite regime).

Main trick

To estimate a term such as (for $0 \leq k \leq n$)

$$\prod_{1 \leq i < j \leq k} |\tau_{ij}^2 - \zeta_{ij}^2|^{\beta} \prod_{1 \leq i \leq k, k < j \leq n} |\tau_{ij}^2 - \zeta_{ij}^2|^{-\beta} \prod_{k < i < j \leq n} |\tau_{ij}^2 - \zeta_{ij}^2|^{\beta} .$$

we rewrite it as a Determinant of a Cauchy-Vandermonde Matrix. Using Laplace, we separate these different contributions.

Technical difficulty: In the estimate of the Vandermonde-Determinant, we have to control the support of the testfunction g in order to force a constant to be less than 1.

Note that we only have Vandermonde Matrix contributions if $k \neq n/2$, i.e. if there are more (or less) tensor powers of V_a than of V_{-a} in the contribution $\mathcal{T}_n^H(V_a^{\otimes k} \otimes V_{-a}^{\otimes n-k})$ at hand.

Relation to the Euclidean estimate

In [Fröhlich 1976], vacuum expectation values of the Euclidean sine Gordon model were investigated, where there is no choice of states (just the vacuum). In order to make the connection to this model, we also considered the expectation value with respect to the singular vacuum, but for the Minkowski version of the theory.

The trick here is to introduce an auxiliary mass $m > 0$, calculate the expectation values and study the limit $m \to 0$ [Wightman 67]. We showed that this gives a special case of our estimate, where all contributions with $k \neq n/2$ vanish (in the limit $m \to 0$).

In this case, we only have Cauchy determinants (very similar to the estimates in [Fröhlich 1976]) and we can get rid of the restriction on the support of g in order to show convergence of the expectation value.

The Derezinski-Meissner representation

In [DM 2006] the following representation space for the massless scalar field in 2D was given:

- Carrier space is $\mathcal{H} = \mathcal{H}_0 \otimes L^2(\mathbb{R})$ where $\mathcal{H}_0$ is the usual Fock space for derivatives of the field. and $L^2(\mathbb{R})$ an auxiliary Hilbert space.
- Choose a test density ψ with total integral 1. Then

$$\pi_\psi(\phi(g)) = \phi_c\left(g - \psi \int g\right) \otimes 1 + 1 \otimes \left(\int g\right) q - 1 \otimes \left(\int g \Delta\psi\right) p$$

  where ϕ_c is the free field on Fock space (well defined because it is evaluated in a test function of total integral 0) and where q and p are standard position and momentum operators in the Schrödinger representation.
- Observe: The 2-point function of the (vector) state $\Omega_0 \otimes \Omega \in \mathcal{H}$ is the 2-point function of Schubert.

The S matrix in the Derezinski-Meissner representation (with KF, KR, unpublished)

Take the expectation value of the S-matrix (of the sine Gordon model) with respect to a vector state given by $e^{i\phi_c(f)}\Omega_0 \otimes \xi \in \mathcal{H}$ where ξ has support $\subset [-\frac{b}{2}, \frac{b}{2}]$ and f has vanishing integral. Then

$$\left\langle e^{i\phi_c(f)}\Omega_0 \otimes \xi, \mathcal{T}_n(: V_{a_1}(g) :_0 \otimes \ldots \otimes : V_{a_n}(g) :_0) e^{i\phi_c(f)}\Omega_0 \otimes \xi \right\rangle = 0$$

if $\sum a_j = a \sum \mathrm{sign}(a_j) > b$.

It follows that in the estimates on the expectation values, we can get rid of the restriction on the support of g but still have weak convergence of the S-matrix.

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Observe that such states are dense in $\mathcal{H}$.

Outlook: Constructing the net of von Neumann algebras (with KF, KR, unpublished)

Bogoliubov's formula

$$\mathcal{R}_{\lambda V}(F) = -i\hbar \frac{d}{d\mu}|_{\mu=0}\big(\underbrace{\mathcal{S}(\lambda V)^{-1}\mathcal{S}(\lambda V + \mu F)}_{=\mathcal{S}_{\lambda V(g)}(\mu F)}\big)$$

for a functional F on $\mathcal{E}$.

Main input is the relative S-matrix. Its properties (as operators on $\mathcal{H}$) are currently under investigation (have: unitarity, covariance, causal factorization property) for the interacting field ($F(h) = \int \Phi_x h(x)dx$) and vertex operators.

From [Fredenhagen, Rejzner 2015], this suffices to construct the local net (there, given in terms of formal power series, which we can show to be summable here).

What else?

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Work in progress: Conserved currents... quantum integrability?!