Advances in Mathematics and Theoretical Physics

Roma September 21 2017

On the non linear Schrödinger equation on an n–dimensional torus

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Non linear PDE's

NON–LINEAR PDE's

Non linear PDE's the NLS

One of the most studied non–linear PDE's is the

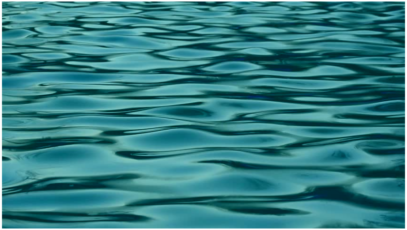
non linear Schrödinger equation, NLS

$$-iu_t + \Delta u = \kappa|u|^{2q}u, \quad q \geq 1 \in \mathbb{N}. \tag{1}$$

Here Δ is the Laplace operator. This is the completely resonant form of the NLS

The NLS equation and Waves

The NLS equation is used to model wave motion in water



The first thing to fix is the domain...
We are interested in bounded domains where we expect recurrent behavior to be typical.

Non linear PDE's the NLS

- I will discuss the periodic boundary conditions case, that is the equation on a torus $\mathbb{T}^n$.
- Thus $u := u(t, \varphi)$, $\varphi \in \mathbb{T}^n$ and Δ is the Laplace operator.
- The case $q = 1$ is of particular interest and is usually referred to as the *cubic NLS*.

There is an extensive literature in dimension $n = 1$

In dimension $n = q = 1$ the NLS has special good properties, it is

completely integrable

our treatment is for all n and q with special enfasis to $q = 1$.

Non linear PDE's the NLS

In Fourier representation

$$u(t,\varphi) := \sum_{k\in\mathbb{Z}^n} u_k(t) e^{\mathrm{i}(k,\varphi)} \,,$$

we have to study the evolution of the Fourier coefficients.

For the homogeneous linear equation we have:

$$\dot u(t,\phi) = -\mathrm{i}\Delta u(t,\phi) \quad \to \quad \dot u_k(t) = \mathrm{i}|k|^2 u_k$$

hence the formula of waves with integer frequencies:

$$u_k(t) = \sqrt{\xi_k} e^{i|k|^2 t} \,, \quad k \in \mathbb{Z}^n.$$

Non linear PDE's the NLS

We see that a solution which depends on finitely many frequencies

$$u(t) = \sum_{i=1}^{h} \sqrt{\xi_{k_i}} e^{i|k_i|^2 t}, \quad k_i \in \mathbb{Z}^n.$$

is necessarily periodic, this is a form of resonance for the linear NLS.

Notice also that $|k|^2$ is constant on large sets of frequencies.

Some interesting phenomena

1. Recurrent behavior

2. Energy transfer

Some interesting phenomena

1. Recurrent behavior

Start from an initial datum which is essentially localized on a finite number of Fourier modes...
the solution stays essentially localized on the same modes at all times.

2. Energy transfer

Some interesting phenomena

1. Recurrent behavior

2. Energy transfer

Start from an initial datum which is essentially localized on a finite number of Fourier modes...
the Fourier support of the solution spreads to higher modes.

Some interesting phenomena

1. Recurrent behavior

2. Energy transfer

One could also study

3. Shock waves.

Start with a smooth initial datum and after a finite time the solution is not smooth any more

There is an enormous literature and very active research on these topics!

The NLS in Hamiltonian formalism

The NLS can be described as an infinite dimensional Hamiltonian system where the Fourier coefficients are the symplectic coordinates and with Hamiltonian (for $q = 1$)

$$H = \sum_{k\in\mathbb{Z}^n} |k|^2 u_k \bar{u}_k + \sum_{k_i\in\mathbb{Z}^n : k_1+k_3=k_2+k_4} u_{k_1}\bar{u}_{k_2}u_{k_3}\bar{u}_{k_4},$$

$$\{u_h, u_k\} = \{\bar{u}_h, \bar{u}_k\} = 0, \; \{\bar{u}_h, u_k\} = \delta_k^h \mathrm{i}$$

H Poisson commutes with

$$\text{momentum} \quad M = \sum_{k\in\mathbb{Z}^n} k u_k \bar{u}_k\,, \quad \text{mass} \quad L = \sum_{k\in\mathbb{Z}^n} u_k \bar{u}_k\,.$$

This makes sense on Hilbert spaces of very regular functions with exponential decay on Fourier coefficients

The NLS in Hamiltonian formalism

The quadratic part $K := \sum_{k\in\mathbb{Z}^n} |k|^2 u_k \bar{u}_k$

describes the linear waves which behave as infinitely many independent oscillators! with integer frequencies

the non linear perturbation will deform these integer frequencies to possibly $\mathbb{Q}$–linearly independent frequencies or to cahotic behaviour.

Non linear PDE's the NLS

When we add the non linear term we expect that typical solutions are not periodic, but when we study *small solutions* we hope to be able to treat the problem with the methods of perturbation theory as in classical dynamical systems and hope to find special solutions:

quasi–periodic solutions.

The reason why has a long story.

WHY

QUASI PERIODIC ORBITS

Why quasi–periodic

As soon as we have at least two degrees of freedom (like rotation and revolution) each moving periodically the probability that the joint motion be periodic is clearly **zero**,

it means that the two frequencies have a rational ratio!!

- So we have to expect that n independent periodic motions describe a dense orbit in the n–dimensional torus (Kronecker).
- This is the notion of *quasi periodic orbit*.
- This is the usual picture, in action–angle variables, of a non degenerate completely integrable system.

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The ergodic hypothesis, a general system

The Fermi–Pasta–Ulam experiment

The ergodic hypothesis

For a long time, from qualitative considerations and from the ideas of statistical mechanics, it was believed that in a small perturbation of a completely integrable system almost all orbits should be *ergodic*.

A surprise 1955 (a recurrent behaviour)

As soon as the first computers were available there was a famous simulation by Fermi–Pasta–Ulam where they discovered, contrary to their intuition, that a small non–linear perturbation of a system of oscillators produced in long term, instead of an ergodic behaviour, complicated quasi–periodic behaviour.

The appearance of KAM theory

Small perturbation

The discovery of some (complicated) but large persistence of quasi–periodic solutions for a small perturbation of a non degenerate completely integrable system is the content of

KAM theory,

developed around 50 years ago by Kolmogorov, Arnold and Moser.

The theory is in a way constructive in the sense that the quasi–periodic orbits are built by an algorithm.

Birkhof normal form, a formal conjugacy Theorem

The KAM algorithm is a refinement of a purely algebraic procedure called **Birkhof normal form**.

A formal algebraic Theorem

1. Hamilton equations can be used to define a formal group of symplectic automorphisms of formal power series
2. Under this group a Hamiltonian of the form, $\sum_i \lambda_i I_i + P$ where P starts from degree 3 and the λ_i are *linearly independent over the rationals* is equivalent to a formal series

$$H \sim f(I_1, \dots, I_n)$$

in the Poisson commuting elements $I_i := (p_i^2 + q_i^2)/2$.

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Birkhof normal form, a formal conjugacy Theorem

A formal integrability

What this means is that formal perturbation theory brings the Hamiltonian in a *completely integrable form!*

But of course

usually this is completely divergent.

We cannot expect a perturbation to behave as a completely integrable systems!

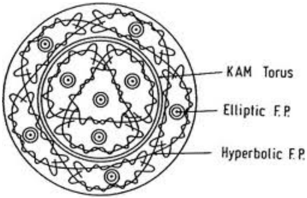
What should we expect?

An answer is given by KAM theory.

The answer of KAM theory

By introducing parameters for the orbits the previous algorithm can be made convergent on complicated Cantor subsets of the parameter space!

In a setting as before we should expect that, if P is small, lots of stable tori will persist, but in a very complex fractal way.



With infinitely many *holes* in which the behavior of the system may be quite complex and cahotic!

SUMMARIZING

From an algorithmic viewpoint:

The appearance of Cantor sets

- in the KAM algorithm we have parameters ξ_i. In the algorithm we often divide by expressions in the ξ_i which may vanish, giving infinitely many poles which have to be avoided.
- These are *small divisors* and *resonances* so that the success of the algorithm depends on suitable *non degeneracy conditions* treated by a mixture of combinatorial and analytic methods.
-

A few years later similar approaches of KAM theory were developed successfully by several authors in order to study non linear PDE's treated as infinite dimensional dynamical systems.

The NLS in Hamiltonian formalism

The goal

We want to apply (infinite dimensional) KAM theory and prove the

existence of many quasi–periodic solutions.

The construction of quasi–periodic solutions is performed in four steps (corresponding to 4 papers with Michela, the second also with Nguyen Bich Van (her Ph.D. thesis)).

The plan

1. Construction of integrable normal forms, rectangle graph
2. Proof of non–degeneracy of the normal form, lots of algebra
3. $q = 1$, The quasi–Töpliz property and the KAM algorithm, hard analysis
4. General q.

Birkhof's algorithm and rectangles in the NLS for $q = 1$

$K := \sum_{k\in\mathbb{Z}^n} |k|^2 u_k \bar{u}_k$ is the *quadratic part* of H

In each step of Birkhof's algorithm one removes inductively the terms of the Hamiltonian which do not Poisson commute with K.

We see that each monomial $u_a\bar{u}_b u_c\bar{u}_d$ is an eigenvector of the operator $\{K, -\}$ (Poisson bracket) with eigenvalue

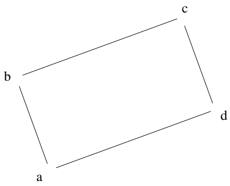
$$\mathrm{i}(|a|^2 - |b|^2 + |c|^2 - |d|^2)$$

Rectangles in the NLS

In particular the *resonant monomials* $u_a\bar{u}_b u_c \bar{u}_d$ which Poisson commute with L, M, K have

$$a + c = b + d, \quad |a|^2 + |c|^2 = |b|^2 + |d|^2.$$

That is the vectors a, b, c, d are **the vertices of a rectangle.**



Birkhof for the NLS

In each step of Birkhof's algorithm one removes all the terms of the Hamiltonian which do not commute with K.
So after one step (an analytic change of variables) the Hamiltonian

$$H = \sum_{k\in\mathbb{Z}^n} |k|^2 u_k \bar{u}_k + \sum_{k_i\in\mathbb{Z}^n : k_1+k_3=k_2+k_4} u_{k_1}\bar{u}_{k_2}u_{k_3}\bar{u}_{k_4},$$

becomes

$$K + \sum_{\substack{k_i\in\mathbb{Z}^n : k_1+k_3=k_2+k_4 \\ |k_1|^2+|k_3|^2=|k_2|^2+|k_4|^2}} u_{k_1}\bar{u}_{k_2}u_{k_3}\bar{u}_{k_4} + O(|u|^6).$$

Main remark

If we take the Hamiltonian in which we keep in the quartic part only the resonant monomials,

$$\text{resonant Hamiltonian} \quad K + \sum_{\substack{k_i \in \mathbb{Z}^n : k_1 + k_3 = k_2 + k_4 \\ |k_1|^2 + |k_3|^2 = |k_2|^2 + |k_4|^2}} u_{k_1} \bar{u}_{k_2} u_{k_3} \bar{u}_{k_4},$$

this has lots of easily described finite dimensional invariant subspaces where the Hamiltonian is completely integrable and non degenerate as follows:

Tangential sites

choose some generic frequencies $v_1, \ldots, v_m$ called *tangential sites*, set $u_j = 0, \ \forall j \notin \{v_1, \ldots, v_m\}$. On this subspace all solutions are quasi–periodic, they are ergodic on a family of tori.

The linearization of the problem

The idea is to

linearize the Hamiltonian vector field on the family of the normal bundles of these invariant tori.

This means in practice to choose polar coordinates on these tori and usual symplectic coordinates on the normal spaces.

Then we keep only the linear part of the vector field or the quadratic part of the Hamiltonian.

Some technical details on the linearization

- We put the variables corresponding to the frequencies v_i in polar coordinates and introduce parameters for the family of tori so we set

$$u_{v_i} := \sqrt{\xi_i + y_i}e^{\mathrm{i}x_i} = \sqrt{\xi_i}(1 + \frac{y_i}{2\xi_i} + \dots)e^{\mathrm{i}x_i}$$

- we call $u_k = z_k$ for the normal frequencies, give degree 2 to the y_i and 1 to z_k (and 0 to x_i)
- finally we $\boxed{\textit{collect the terms of degree 2}}$ in the resulting Hamiltonian, this is our *normal form* N.
- What is left one can show can be treated as *perturbation*.

We obtain thus the *normal form Hamiltonian N*

$$\boxed{N = K + \mathcal{Q}(x,z)}, \qquad S^c := \mathbb{Z}^n \setminus \{v_1, \dots, v_m\}$$

$$\text{where} \quad K := \sum_{1 \le i \le m} (|v_i|^2 - 2\xi_i) y_i + \sum_{k \in S^c} |k|^2 |z_k|^2 \tag{2}$$

$$\text{and} \quad \mathcal{Q}(x,z) = 4 \sum^{*}_{\substack{1 \le i \ne j \le m \\ h,k \in S^c}} \sqrt{\xi_i \xi_j} e^{\mathrm{i}(x_i - x_j)} z_h \bar{z}_k + \tag{3}$$

$$2 \sum^{**}_{\substack{1 \le i < j \le m \\ h,k \in S^c}} \sqrt{\xi_i \xi_j} e^{-\mathrm{i}(x_i + x_j)} z_h z_k + 2 \sum^{**}_{\substack{1 \le i < j \le m \\ h,k \in S^c}} \sqrt{\xi_i \xi_j} e^{\mathrm{i}(x_i + x_j)} \bar{z}_h \bar{z}_k .$$

This is a non-integrable potentially very complicated infinite dimensional quadratic Hamiltonian! Since the angle variables still interact with the normal ones.

The remaining terms form the *perturbation* which indeed is sufficiently small.

The symbols

$$\sum^{*} \quad \sum^{**}$$

mean that the sum is restricted to the quadruples v_i, v_j, h, k or v_i, h, v_j, k which are vertices of a rectangle.

The appearance of a graph

The combinatorics of these rectangles is given by a *RECTANGLE GRAPH*.
The connection with the NLS $N = K + \mathcal{Q}(x, z)$ is to single out the rectangles associated to the terms appearing in $\mathcal{Q}(x, z)$

The first MAIN GOAL

Theorem

For a generic S this Hamiltonian decomposes into an infinite sum of **non interacting** *finite blocks.*

One can make a symplectic change of variables which eliminates all the angles x_i in the formulas! (that is N becomes integrable!)

$$4 \sum^{*}_{\substack{1\leq i<j\leq m \\ h,k\in S^c}} \sqrt{\xi_i\xi_j} z_h \bar{z}_k + 2 \sum^{**}_{\substack{1\leq i<j\leq m \\ h,k\in S^c}} \sqrt{\xi_i\xi_j}(z_h z_k + \bar{z}_h \bar{z}_k).$$

This now is an integrable infinite dimensional quadratic Hamiltonian but decomposed into infinitely many non interacting finite dimensional blocks.

Linear algebra

At this point the study of this Hamiltonian is a very complicated problem of linear algebra, (canonical forms, eigenvalues).

How did I get involved?

The rectangle graph

A digression into geometry

My interest started when my daughter Michela proposed to me a strange problem of **elementary Euclidean Geometry.**

At that time I had no idea of its analytic implications.

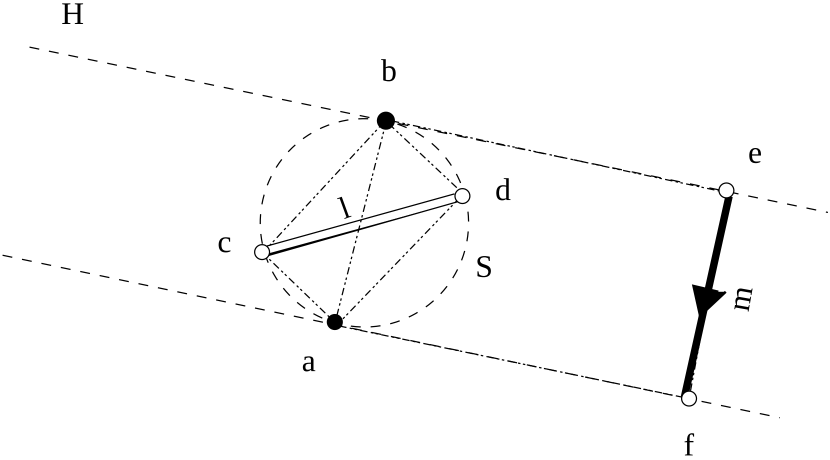
The culprit Hokkaido 2009

A strange problem: The rectangle graph Γ_S

In Euclidean space $\mathbb{R}^n$ choose m distinct points $S := \{v_1, \ldots, v_m\}$.

1. To this set S we associate a graph with vertices:
   all the points of $\mathbb{R}^n$.
2. Two points x, y are joined by an edge if we can complete them with two points $a = v_i, b = v_j \in S$ such that x, y, a, b are the vertices of a rectangle.

This can be done in 2 different ways:



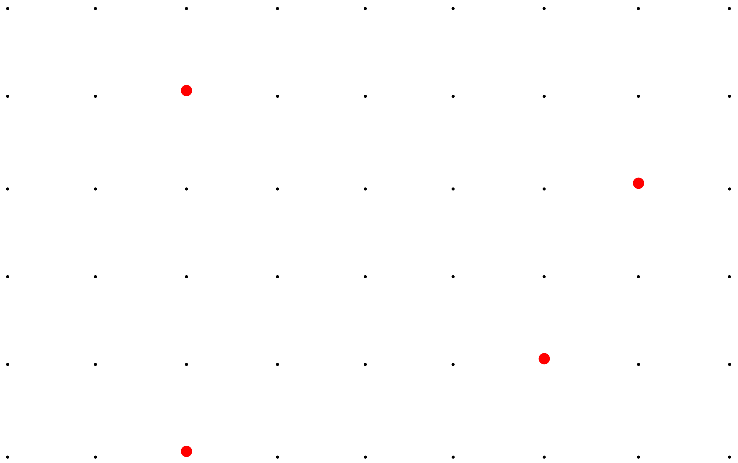
A colored and marked graph

In fact we have two different possibilities (two colors)

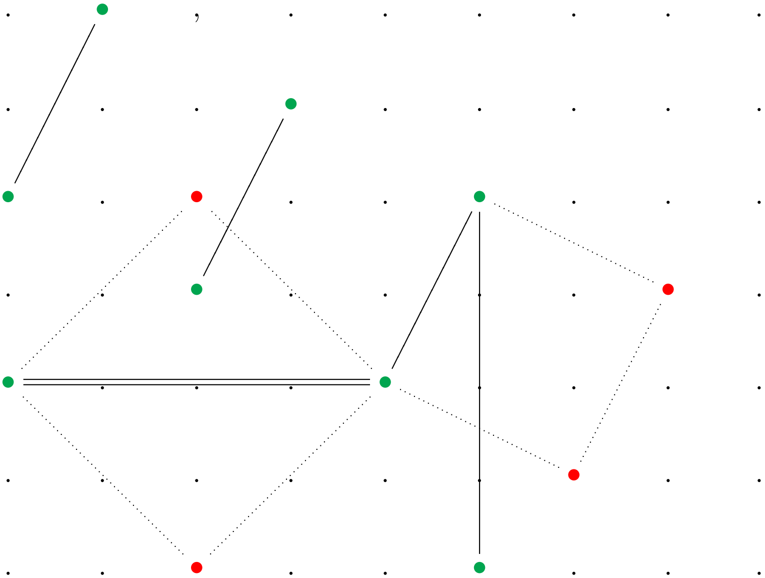
- A black edge $p \xrightarrow{e_i - e_j} q$ connects two points $p, \ q$ which are adjacent in the rectangle with vertices p, q, v_j, v_i.
- A red edge $p \overset{-e_i - e_j}{=\!=\!=} q$ connects two points p, q which are opposite in the rectangle with vertices p, v_j, q, v_i.

In fact we are interested only when $S \subset \mathbb{Z}^n$ and on the part of the graph with vertices in $S_c = \mathbb{Z}^n \setminus S$!! Call this graph Γ_S.

EXAMPLE: S is given by 4 points marked



EXAMPLE: points connected by edges



As explained, the connection with $N = K + \mathcal{Q}(x,z)$ is to single out the rectangles associated to the terms appearing in $\mathcal{Q}(x,z)$

in order to decouple it into non interacting blocks.

In fact a black edge corresponds to a term

$\sqrt{\xi_i \xi_j} e^{\mathrm{i}(x_i - x_j)} z_h \bar{z}_k$ or $\sqrt{\xi_i \xi_j} e^{\mathrm{i}(x_j - x_i)} \bar{z}_h z_k$ of $\sum^*$

and a red edge corresponds to a term

$\sqrt{\xi_i \xi_j} e^{\mathrm{i}(x_i + x_j)} z_h z_k$ or $\sqrt{\xi_i \xi_j} e^{\mathrm{i}(x_i + x_j)} \bar{z}_h \bar{z}_k$ of $\sum^{**}$.

The Problem

The problem

The problem consists in the study of the connected components of this graph Γ_S for S *generic*.

Each connected component gives a block of $\mathcal{Q}(x, z)$, these blocks do not interact with each other.

It is not hard to see that, for generic values of S, the set S is itself a connected component which we call the *special component*.

In particular we are interested in the points of the graph with integer coordinates (assuming $S \subset \mathbb{Z}^n$).

The main combinatorial Theorem

Theorem

For generic choices of S the connected components of the graph Γ_S, different from the special component, are formed by

affinely independent points.

In particular each component has at most $n+1$ points.

The proof is quite complex

it requires some algebraic geometry and a very long and difficult combinatorial analysis.

The next algebraic step

- One can eliminate the angles x_i by suitable *rotations* of the variables z_k!. Then
- the new Hamiltonian is

$$4\sum_{\substack{1\le i<j\le m\\ h,k\in S^c}}^{*}\sqrt{\xi_i\xi_j}z_h\bar z_k+2\sum_{\substack{1\le i<j\le m\\ h,k\in S^c}}^{**}\sqrt{\xi_i\xi_j}(z_hz_k+\bar z_h\bar z_k),$$

each connected component of the graph

determines a matrix dependent on the variables ξ describing the Hamiltonian in that block.

- One wants to show that for generic values of ξ all eigenvalues of all these infinitely many matrices are all non zero and distinct! (This is the required non–degeneracy).
- These properties are called *Melnikov conditions*.

The matrices:

These infinitely many matrices are obtained from a

finite number of combinatorial matrices

by adding infinitely different scalar matrices, depending on the various geometric blocks with the same combinatorial structure, given by one of the possible combinatorial graphs.

The combinatorial matrices: example

combinatorial graph:

$$
\begin{tikzcd}
d \arrow[r, dash, "{4,3}"] \arrow[d, dash, "{4,1}"'] & c \arrow[d, dash, "{2,3}"] \\
a \arrow[r, dash, "{1,2}"] & b
\end{tikzcd}
$$

associated matrix:

$$
\begin{vmatrix}
0 & -2\sqrt{\xi_2\xi_1} & 0 & -2\sqrt{\xi_1\xi_4} \\
-2\sqrt{\xi_2\xi_1} & \xi_2 - \xi_1 & -2\sqrt{\xi_2\xi_3} & 0 \\
0 & -2\sqrt{\xi_2\xi_3} & -\xi_1 + \xi_3 & -2\sqrt{\xi_4\xi_3} \\
-2\sqrt{\xi_1\xi_4} & 0 & -2\sqrt{\xi_4\xi_3} & \xi_4 - \xi_1
\end{vmatrix}.
$$

With characteristic polynomial

$$-4\,\xi_1^3\,\xi_2 + 4\,\xi_1^2\,\xi_2\,\xi_3 - 4\,\xi_1^3\,\xi_4 + 8\,\xi_1^2\,\xi_2\,\xi_4 + 4\,\xi_1^2\,\xi_3\,\xi_4 - 8\,\xi_1\,\xi_2\,\xi_3\,\xi_4$$

$$+(\xi_1^3 - 9\,\xi_1^2\,\xi_2 - \xi_1^2\,\xi_3 + \xi_1\,\xi_2\,\xi_3 - 9\,\xi_1^2\,\xi_4 + 9\,\xi_1\,\xi_2\,\xi_4 + \xi_1\,\xi_3\,\xi_4 + 7\,\xi_2\,\xi_3\,\xi_4)\,t$$

$$+(3\,\xi_1^2 - 6\,\xi_1\,\xi_2 - 2\,\xi_1\,\xi_3 - 3\,\xi_2\,\xi_3 - 6\,\xi_1\,\xi_4 + \xi_2\,\xi_4 - 3\,\xi_3\,\xi_4)\,t^2$$

$$+(3\,\xi_1 - \xi_2 - \xi_3 - \xi_4)\,t^3 + t^4$$

The combinatorial matrices: example

$$a \overset{(1,2)}{=\!=\!=} b \overset{(i,j)}{\text{———}} c \overset{(h,k)}{\text{———}} d \ ,$$

$$\begin{vmatrix} 0 & 2\sqrt{\xi_2\xi_1} & 0 & 0 \\ -2\sqrt{\xi_2\xi_1} & \xi_2+\xi_1 & -2\sqrt{\xi_i\xi_j} & 0 \\ 0 & -2\sqrt{\xi_i\xi_j} & -\xi_i+\xi_j+\xi_2+\xi_1 & -2\sqrt{\xi_k\xi_h} \\ 0 & 0 & -2\sqrt{\xi_h\xi_h} & \xi_k-\xi_h-\xi_i+\xi_j+\xi_2+\xi_1 \end{vmatrix}$$

Discriminants and resultants

A priori to prove the condition that a polynomial has distinct roots is given by the non vanishing of the discriminant.

The condition that two polynomials have distinct roots is given by the non vanishing of their resultant.

For our matrices

this is impossible to verify directly. So we need a trick.

MAIN THEOREM of Step $\boxed{2}$

Main Theorem

The characteristic polynomials of the combinatorial matrices are all *irreducible* and *different from each other*.

Implications

1. Eigenvalue separation
2. Validity of the Melnikov conditions.
3. Symplectic coordinates which diagonalize the Hamiltonian.

The KAM algorithm

At this point one must show

Theorem

The KAM algorithm converges on some Cantor set of positive measure in the parameters ξ *producing the desired quasi–periodic solutions.*

This is based on the introduction of a norm on Hamiltonians, the quasi–Töpliz norm, (a rather non trivial development of ideas originated from Eliasson and Kuksin), then one has to show that this norm controls the algorithm.

An idea of the algebraic part of the proof

SOME ALGEBRA: the Cayley graphs

Given a group G and a set $S \subset G$ with $S = S^{-1}$

The Cayley graph has vertices the elements of G and edges the pairs a, b such that $ab^{-1} \in S$.

The graph Λ_S associated to $v_1, \dots, v_m$

1. We start from the group $G := \mathbb{Z}^m \ltimes \mathbb{Z}/(2) = \{(\sum_i n_i e_i, \pm 1)\}$, a semi–direct product.
2. In G consider the Cayley graph associated to the elements $(e_i - e_j, 1), (-e_i - e_j, -1)$.
3. One next defines an *energy function*: Given $a = \sum_i \nu_i e_i,\ \sigma = \pm 1$ set
$$K((a,\sigma)) := \frac{\sigma}{2}(|\sum_i a_i v_i|^2 + \sum_i a_i |v_i|^2). \tag{4}$$
4. Finally Λ_S is the subgraph of the Cayley graph in which each edge preserves the energy.

The graph Γ_S can be studied through Λ_S.

Components of Λ_S

So components correspond to geometric–combinatorial graphs

Example (Combinatorial graph)

$$
\begin{tikzcd}
 & -e_2-e_1-\mu & -e_2-e_3-\mu \\
e_4-e_5+\mu & \mu \arrow[l, "e_4-e_5"'] \arrow[u, equal, "-e_2-e_1"] \arrow[ur, equal, "-e_2-e_3"'] \arrow[r, "e_4-e_2"'] & \mu+e_4-e_2
\end{tikzcd}
$$

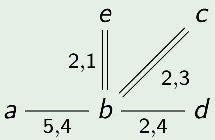
$\mu = \sum_i \mu_i e_i \in \mathbb{Z}^m$

In order to understand energy conservation we then pass to the geometric graph mapping $\mu \to \pi(\mu) = \sum_i \mu_i v_i := -b$ and have the corresponding

Components of Γ_S

geometric components satisfying a system of equations dependent on the parameters v_i:

Example (Geometric Avatar of previous graph)



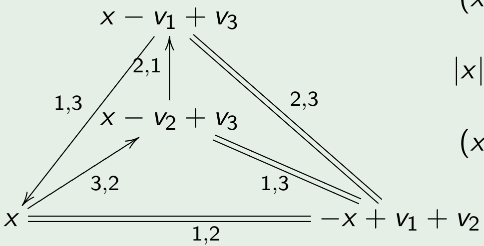
$$a - b = v_5 - v_4,\ \ e + b = v_1 + v_2,\ \ b + c = v_2 + v_3,\ \ b - d = v_2 - v_4$$

$$|a|^2 - |b|^2 = |v_5|^2 - |v_4|^2,\ \ |e|^2 + |b|^2 = |v_1|^2 + |v_2|^2, \dots$$

Another example, after eliminating the linear equations

Example

The equations that x has to satisfy are:



$$(x, v_2 - v_3) = |v_2|^2 - (v_2, v_3)$$

$$|x|^2 - (x, v_1 + v_2) = -(v_1, v_2)$$

$$(x, v_1 - v_3) = |v_1|^2 - (v_2, v_3)$$

All the vector variables except 1 (for instance b) can be eliminated by the linear equations.

Thus one needs to prove that certain *resonant graphs* which we want to exclude produce a system of equations *without real solutions* for generic values of the v_i.

The lists of non generic vectors which instead satisfy some of the equations give the resonant choices.

This turns out to be a very delicate combinatorial problem but it has a nice positive solution.

Every graph with affinely dependent vertices is resonant and does not appear for generic choices of v_i.

For the irreducibility of the characteristic polynomials one proceeds by induction.

The idea is this

assume that all the polynomials relative to graphs with less then k elements are irreducible.

For a given graph with $k+1$ elements

by setting one of the $\xi_i = 0$ one removes all edges where i appears in the marking.

The given polynomial specialyzes to a product of polynomials, irreducible by induction, for smaller graphs.
Doing this for different i one gets the result

This is a very long case analysis plus some basic inductive steps done by computer.

(Ph. D. thesis of Nguyen Bich Van)

Example

$$\gamma := \quad a \overset{(1,2)}{=\!=\!=} b \overset{(i,j)}{\text{———}} c \overset{(h,k)}{\text{———}} d\ ,$$

set $\xi_i = 0$ get

$$a \overset{(1,2)}{=\!=\!=} b \qquad c \overset{(h,k)}{\text{———}} d\ ,$$

set $\xi_1 = 0$ get

$$a \qquad b \overset{(i,j)}{\text{———}} c \overset{(h,k)}{\text{———}} d\ ,$$

so if the characteristic polynomial associated to γ is reducible, by the first it must split into 2 quadratic irreducibes by the second into a linear and a cubic. INCOMPATIBLE!