# On the replica approach for statistical mechanics of random system

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- Spin glasses in the infinite range limit (the Sherrington Kirkpatrick model).
- The computation of the free energy in the SK model (mean field approximation).
- The heuristic approach base on replicas.
- A first step toward a different approach.
- Some interesting conjectures.

The simplest Ising spin glass (SK) has the following random Hamiltonian:

$$H_J[\vec{\sigma}] = \sum_{i,k=1,N} J_{i,k}\sigma(i)\sigma(k)$$

$\sigma_i = \pm 1$, $i = 1, N$.

The $J$ are random (e.g. Gaussian distributed with zero average).

$$E[J_{i,k}] \equiv \overline{J_{i,k}} = 0, \qquad E[J_{i,k}^2] \equiv \overline{J_{i,k}^2} = N^{-1}$$

Equivalently $H_J[\vec{\sigma}_1]$ is a random Gaussian function:

$$\overline{H_J[\vec{\sigma}_1]} = 0; \qquad \overline{H_J[\vec{\sigma}_1]H_J[\vec{\sigma}_2]} = N(\vec{\sigma}_1, \vec{\sigma}_2) \equiv N \sum_i \sigma_1(i)\sigma_2(i)$$

The probability distribution of the spectrum of $J$ is known: the Dyson semicircle law.

Partition function:

$$Z_J = \sum_{\vec{\sigma}} \exp(-\beta H_J[\vec{\sigma}]) \qquad 2^N \text{ terms} \qquad \left( H_J[\vec{\sigma}] = \sum_{i,k=1,N} J_{i,k}\sigma(i)\sigma(k) \right)$$

Free energy:

$$\beta N f_J^N(\beta) = -\log(Z_J)$$

We want to compute

$$\tilde{f}(\beta) = \lim_{N\to\infty} \overline{f_J^N(\beta)}$$

The overline represent the average over the $J$'s.

# Which is the result for the value of the free energy?

We start with a function $q(x)$ defined in the interval $0 \leq x \leq 1$.

The function $g_q(x,h)$ is defined in the strip $0 \leq x \leq 1$, $-\infty < h < \infty$.

Boundary condition: $g_q(1,h) = \log(\cosh(\beta h))$; the function $g_q(x,h)$ satisfies the following $q(x)$ dependent antiparabolic equation:

$$\frac{\partial g_q(x,h)}{\partial x} = -\frac{dq}{dx}\left(\frac{\partial^2 g_q(x,h)}{\partial h^2} + x\left(\frac{\partial g_q(x,h)}{\partial h}\right)^2\right)$$

$$F[q] = \frac{1}{2}\beta \int_0^1 dx \left(1 - q(x)^2\right) - g_q(0,0)$$

$$\tilde{f}(\beta) = \max_{q(x)} F[q]$$

This formula was found using replica approach. We need to compute

$$-\beta N f(\beta, N) \equiv \overline{\ln(Z)(\beta, N)}$$

while it is simple to compute for integer $n$:

$$f(n; \beta, N) = -\frac{\log\left(\overline{Z(\beta, N)^n}\right)}{\beta N n} \qquad \lim_{n \to 0} f(n; \beta, N) = f \ .$$

Nicola d'Oresme trick (1353). Starting form the definition of $A^n$,

$$A^{1/2} = \sqrt{A}, \qquad \ln(A) = \lim_{n \to 0} \frac{A^n - 1}{n}$$

After some algebra and Gaussian integrations we find the exact formula

$$\exp(-\beta n N f(n;\beta,N)) = \int dQ \exp(-\beta N n F[Q])$$

$$nF[Q] = -\frac{1}{2}\beta \mathrm{Tr} Q^2 + \beta^{-1}\log\left(\sum_{\{\sigma\}}\exp\left(\sum_{a,b} Q_{a,b}\sigma_a\sigma_b\right)\right)$$

The matrix $Q$ is symmetric, zero on the diagonal ($Q_{a.a} = 0$). We have $n$ variables $\sigma_a$ that takes the value $\pm 1$.

The integral is done on $\frac{n(n-1)}{2}$ variables.

Symmetries The symmetry group is $S_n$. If $\pi$ is a permutation

$$(Q^\pi)_{a,b} = Q_{\pi(a),\pi(b)} \qquad F[Q^\pi] = F[Q]$$

The proof is trivial:

$$nF[Q] = -\frac{1}{2}\beta \mathrm{Tr} Q^2 + \beta^{-1} \log\left(\sum_{\{\sigma\}} \exp\left(\sum_{a,b} Q_{a,b}\sigma_a\sigma_b\right)\right)$$

$$\sum_{\{\sigma\}} \exp\left(\sum_{a,b} Q^\pi_{a,b}\sigma_a\sigma_b\right) = \sum_{\{\sigma\}} \exp\left(\sum_{a,b} Q_{a,b}\sigma_{\pi(a)}\sigma_{\pi(b)}\right) = \sum_{\{\sigma\}} \exp\left(\sum_{a,b} Q_{a,b}\sigma\right.$$

Point of maximum for $N \to \infty$.

$$\exp(-\beta n N f(n,\beta,N)) = \int dQ \exp(-\beta N n F[Q])$$

$$f(n,\beta) \equiv \lim_{N\to\infty} f(n,\beta,N) = F[Q^*] = \min_Q F[Q]$$

We can compute $f(n)$ on the integers: the maximum is at $Q^*_{a,b} = q$, $\forall a, b$, i.e. the only matrix left invariant by the whole permutation group.

We compute everything for integer $n$ and we perform an analytic continuation at $n = 0$.

This gives a wrong result at high $\beta$. The function $f(n)$ must have a singularity at $0 < n < 1$: e.g.

$$f(n) = 0 \quad \text{for} \quad n > \frac{1}{2}; \qquad f(n) = \left(n - \frac{1}{2}\right)^5 \quad \text{for} \quad n < \frac{1}{2}$$

## Putting the finger on the origin of troubles.

If $F[Q]$ has minimum at $Q^*$ we must have

$$\frac{\partial F[Q]}{\partial Q_{a,b}} = 0 \qquad \mathcal{H} \geq 0 \qquad \mathcal{H}_{a,b;cd} \equiv \frac{\partial^2 F[Q]}{\partial Q_{a,b} \partial Q_{c,d}}$$

The non-negativity of the Hessian ($\mathcal{H} \geq 0$) is supposed to be equivalent (also for non-integer $n$) to:

The spectrum of $\mathcal{H}$ is non-negative.

At high $\beta$ the analytic continuation of the spectrum of $\mathcal{H}$ acquire a negative part for $0 < n < n^*(\beta)$: the de Almeida Touless instability.

A bold approach: we do the maximum point approximation at $n = 0$!

This lead to a strange mathematics: one introduces a is $n \times n$ matrix $Q$ and the symmetry group is $S_n$: eventually $n \to 0$.

If $\pi$ is a permutation

$$(Q^\pi)_{a,b} = Q_{\pi(a),\pi(b)} \qquad F[Q^\pi] = F[Q]$$

$F[Q]$ has a minimum at $Q^*$. We call $S^*$ the subgroup of $S_n$ that leaves $Q^*$ invariant i.e. the stabilizer subgroup (the little group).

$$S_n \supset S^*$$

If $S^* \neq S_n$ the (replica) simmetry group is "spontaneously broken".

## Explicit construction non-symmetric $Q*$

$Q^*_{a,b} = q_1 \quad$ if $I(a/m) = I(b/m) \qquad Q^*_{a,b} = q_0 \quad$ if $I(a/m) \neq I(b/m)$.

$$Q_{a,a} = 0.$$

i.e. $n$ obiects are divided into $n/m$ classes of $m$ elements. The little group corresponding to $Q^*$ is

$$S^* = S_{n/m} \otimes (S_m)^{n/m}.$$

When $n \to 0$, $S_{n/m} \to S_0$ so that $S_0 \supset S_0$.

$S_0$ is an infinite group!

The new estimate is (all functions depend also on $\beta$)

$$f_1^{RSB} = \max_{q_1,q_0,m} f(q_1, q_0, m)$$

The *minimum* is at $m^*$ with $0 < m^* < 1$. The old result $f_0^{RSB}$ is given by

$$f_0^{RSB} = \max_q f(q, q, m) \qquad\qquad f(q, q, m) \text{ does not depend on } m$$

In a recursive way ($S_0 \supset S_0$) we can define $f_k^{RSB}(\beta)$ in such a way that

$$f_0^{RSB} \leq f_1^{RSB} \leq f_2^{RSB} \ldots \leq f_k^{RSB} \ldots \leq f_\infty^{RSB} \equiv \lim_{k\to\infty} f_k^{RSB}$$

In 1979 it was conjectured that $f_\infty^{RSB} = f$. In 2002 Guerra proved that $f_k^{RSB} \leq f\ \forall k$. Less than one year later Talagrand twisted Guerra proof to prove that $f \leq \sup_k f_k^{RSB}$. Hence

$$f_\infty^{RSB} \leq f \leq f_\infty^{RSB} \qquad \to \qquad f_\infty^{RSB} = f$$

Can the replica derivation sligtly modified in such a way that it makes sense?

It should **not** involve sets whose cardinality is a non-integer real number !!!

Here I present a computation of $\tilde{f} \equiv \max_{q,m} f_1^{RSB}(q,0,m)$ that correspond in replicas to

$$Q^*_{a,b} = q \quad \text{if } I(a/m) = I(b/m) \qquad Q^*_{a,b} = 0 \quad \text{if } I(a/m) \neq I(b/m).$$

for non-integer $m$ (following Campellone, G.P. and Virasoro, inspired from Derrida).

I will start with writing some identities. I will exchange limits with integral, I will treat non convergent asymptotis series as convergent, but these are minor sins! The cardinality of sets will always be an integer number.

$$\overline{\ln Z_N} = \int_0^\infty \frac{dt}{t}\left(\exp(-t) - \overline{\exp(-tZ_N)}\right) \ .$$

Let us define

$$\exp(-\phi(t,N)) \equiv \overline{\exp(-tZ_N)} = \sum_{k=0,\infty} \frac{1}{k!}(-t)^k \overline{Z_N^k}.$$

$\overline{Z_N^k} = \int dQ \exp(-NF(k,Q))$, the integral is done over $k \times k$ matrices.

At the end of the game we have to evaluate $\exp(-\phi(t,N))$ for very large $t$.

We need a very good control of $\overline{Z_N^k}$:

$$\overline{Z_N^k} \approx \exp(-NF(k,Q^*)) \qquad F(k,Q^*) \equiv \min_Q F(k,Q)$$

gives the wrong replica symmetric result!

A different approximation could be to sum over all the critical points:

$$\overline{Z_N^k} \approx \sum_j \exp(-NF(k,Q_j^*)) \qquad \left.\frac{\partial F(k,Q)}{\partial Q_{a,b}}\right|_{Q_j^*} = 0$$

All critical poins are beyond my command. I will make an arbitrary selection:

We partition the set of $k$ elements into $l$ sets of size $m_i$, where $\sum_{i=1}^{l} m_i = k$. (Here $l$ is the total number of blocks of the matrix.) The off diagonal elements, i.e. $Q_{ab}$, have a constant value $q_i$ if $a$ and $b$ belong to the same set. $Q_{ab}$ is zero if $a$ and $b$ do not belong to the same set.

If we we take all the $m_i = m$ and $q_i = q$ we recover the replica computation for $q_1 = q$ and $q_0 = 0$.

Here we stick to integer $m_i$!

In this way each stationary point of this kind depends is characterized (apart from permutations) the size of the blocks $m_i$ and by the values of $q_i$ that are a function of the $m_i$. After some algebra we get

$$-\phi(t,N) = \sum_{m=1}^{\infty} \frac{(-t)^m}{m!} \exp(mNf(m)) \ .$$

When $N \to \infty$ we need to evaluate the previous formula when $t$ goes to to $\infty$ at constant $y = \ln t/N$: $t$ is very large, i.e. $O(\exp(yN))$. We can transform the previous sum into an integral in the complex plane

$$-\phi(t,N) = \frac{1}{2i} \int_{\mathcal{C}} dm \frac{\exp(N(my + mf(m)))}{\Gamma[1+m] \sin[\pi]}.$$

where $y = \ln t/N$ and $\mathcal{C}$ is an appropriate integration path in the complex plane and crosses the real line for $0 < x < 1$.

We deform the path $\mathcal{C}$ to a new path going from $-i\infty$ to $+i\infty$ crossing the real line for $0 < m < 1$.

We look for a saddle point in the complex plane. The equation for the saddle point (i.e. $m_{sp}$) is

$$f(m_{sp}(y)) + m_{sp}(y)f'(m_{sp}(y)) + y = 0.$$

Let us assume, for simplicity that $0 < m_{sp} < 1$. In this case we have at the leading order

$$\phi(t, N) \approx \exp\left(N m_{sp}(y)(y + f(m_{sp}(y))\right) \, .$$

where

$$f'(m_{sp}(y)) = 0 \, ,$$

After computing the integral on $t$ and after some simple estimates we recover the result of the replica approach.

## Crucial points!

$$-\phi(t,N)=\sum_{m=1}^{\infty}\frac{(-t)^m}{m!}\exp(mNf(m))=\frac{1}{2i}\int_{\mathcal{C}}dm\frac{\exp(N(my+mf(m)))}{\Gamma[1+m]\sin[\pi]}.$$

- We have a function $f(m)$ that we can write in an explicit way and we continue it from integer to non-integer values.
- We use this analytic continuation to write the result as a complex contour integral.
- We estimate the integral with the saddle point method.
- In a simple case, we can do the computation and we obtain the replica result.

Two conjectures:

- If $\overline{Z_N^k} \equiv \int d\mu_N(Z) Z^k = \int dQ \exp(-NF(k,Q))$, for $t = O(\exp(yN))$

$$\overline{\exp(-tZ_N)} = \sum_{k=0,\infty} \frac{1}{k!}(-t)^k \overline{Z_N^k} \approx \sum_{k=0,\infty} \sum_{j=1,C(k)} \frac{1}{k!}(-t)^k \exp(-NF(k,\tilde{Q}_j^k)).$$

The sum on $j$ runs over all the $(C(k))$ critical points $(\tilde{Q}_j^k)$ of $F(k,Q)$.

- If we use the previous formula together with

$$\overline{\ln Z_N} = \int_0^\infty \frac{dt}{t} \left( \exp(-t) - \overline{\exp(-tZ_N)} \right) \quad .$$

we get the replica broken result for $\lim_{N\to\infty} N^{-1} \overline{\ln Z_N}$ !