Proof of the DOZZ Formula

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Liouville Model

2d field $\phi(z)$ with action functional

$$S(\phi) = \int (|\partial_z \phi(z)|^2 + \mu e^{\gamma\phi(z)}) d^2 z$$

Correlation functions of **vertex operators** $e^{\alpha\phi(z)}$, $\alpha \in \mathbb{C}$:

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle := \int \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} e^{-S(\phi)} D\phi$$

- **KPZ conjecture:** correlation functions of **2d quantum gravity** are given in terms of Liouville correlations
- **AGT conjecture:** Nekrasov partition functions of 4d SuSy Yang Mills are given in terms of Liouville correlations

We prove an **integrability** result for Liouville theory: the **DOZZ** formula for the 3-point functions

2d Gravity

Polyakov '81: string theory in terms of gravity on world sheet

(Euclidean) quantum gravity: **random** Riemannian metric g

In two dimensions

$$g = g(z)(dz \otimes d\bar{z} + d\bar{z} \otimes dz)$$

What is the probability law of $g(z)$?

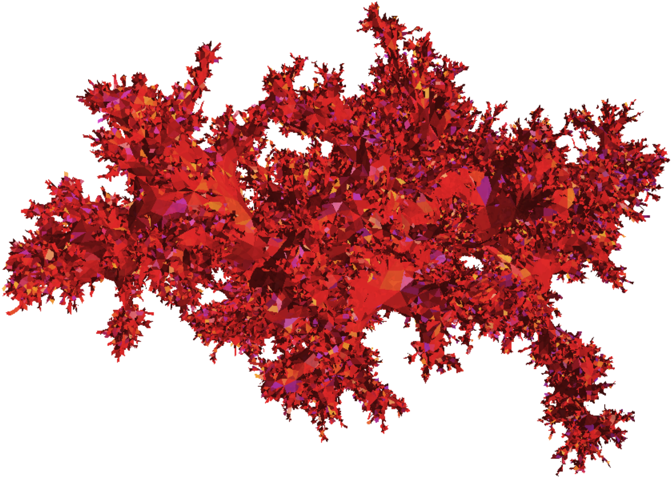
Knizhnik, Polyakov, and Zamolodchikov '88: Couple gravity to matter described by **conformal field theory**. Then

- $g = e^{\gamma\phi}$, ϕ the Liouville field
- μ is the cosmological constant
- γ depends on the central charge of the CFT:

$$c = 25 - 6Q^2, \quad Q = \frac{\gamma}{2} + \frac{2}{\gamma}$$

- $\gamma \in \mathbb{R} \implies Q^2 \geq 4 \implies c \leq 1$

$\gamma = \sqrt{2}$, ($c = -2$) Quantum Sphere



F. David

Gravitational dressing

Example: Ising model $c = \frac{1}{2} \implies \gamma = \sqrt{3}$. Let

- σ be scaling limit of Ising spin at T_c
- $\tilde{\sigma}$ be scaling limit of critical Ising spin on a random planar map uniformized to $\{\mathbb{S}^2, z_1.z_2.z_3\}$.

Then

$$\tilde{\sigma}(z) = e^{\alpha\phi(z)}\sigma(z)$$

with ϕ the $\gamma = \sqrt{3}$ Liouville field and $\alpha = \frac{5}{2\sqrt{3}}$.

More precisely

$$\langle \prod_{i=1}^{k} \tilde{\sigma}(x_i) \rangle = \frac{\langle e^{\gamma\phi(z_1)} e^{\gamma\phi(z_2)} e^{\gamma\phi(z_3)} \prod_{i=1}^{k} e^{\alpha\phi(x_i)} \rangle}{\langle e^{\gamma\phi(z_1)} e^{\gamma\phi(z_2)} e^{\gamma\phi(z_3)} \rangle} \langle \prod_{i=1}^{k} \sigma(x_i) \rangle$$

Gravitational dressing

Similar formuli for other Conformal Field Theories with $c \leq 1$:

Primary fields Ψ **dressed** to $e^{\alpha\phi}\Psi$

$$\Delta = \Delta_\alpha + \frac{\gamma^2}{4}\Delta_\alpha(\Delta_\alpha - 1), \quad \Delta_\alpha = \tfrac{\alpha}{2}(Q - \tfrac{\alpha}{2})$$

- Δ conformal weight of Ψ
- Δ_α conformal weight of $e^{\alpha\phi}$

Hence we need to understand correlations of **vertex operators**

$$V_\alpha(z) = e^{\alpha\phi(z)}$$

in Liouville theory.

Conformal Bootstrap

Belavin, Polyakov, Zamolodchikov '84: Conformal Field Theory is determined by

- **Spectrum**: the set of primary fields Ψ_i, $i \in I$
- **Three point functions** $\langle \Psi_i(z_1)\Psi_j(z_2)\Psi_k(z_3)\rangle$

By Möbius invariance suffices to find **structure constants**

$$C(i,j,k) = \langle \Psi_i(0)\Psi_j(1)\Psi_k(\infty)\rangle$$

BPZ found $C(i,j,k)$ for **minimal models** (e.g. Ising) but **failed** to find them for Liouville

CFT is an "unsuccesful attempt to solve the Liouville model" (Polyakov)

Bootstrap for Liouville

Spectrum of Liouville is conjectured to be given by (Braaten, Curtright, Thorn, Gervais, Neveu):

$$V_\alpha = e^{\alpha\phi}, \quad \alpha \in Q + i\mathbb{R}_+$$

Structure constants $C(\alpha_1, \alpha_2, \alpha_3) = \langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty) \rangle$

Bootstrap then gives e.g. for 4-point function

$$\langle V_{\alpha_1}(z) V_{\alpha_2}(0) V_{\alpha_3}(1) V_{\alpha_4}(\infty) \rangle = \int_{Q+i\mathbb{R}_+} |\mathcal{F}_{\alpha,\{\alpha_i\}}(z)|^2$$
$$\times\, C(\alpha_1, \alpha_2, \alpha) C(2Q - \alpha, \alpha_3, \alpha_4) d\alpha$$

where $\mathcal{F}_{\alpha,\{\alpha_i\}}(z)$ are explicit (universal conformal blocks)

Remains to find the structure constants.

In the '90s Dorn, Otto and Zamolodchikov, Zamolodchikov conjectured an explicit formula for $C(\alpha_1, \alpha_2, \alpha_3)$, the **DOZZ formula**.

DOZZ formula

$$C(\alpha_1,\alpha_2,\alpha_3) = \left(\pi\mu l(\tfrac{\gamma^2}{4})(\tfrac{\gamma}{2})^{\frac{4-\gamma^2}{2}}\right)^{-s} \frac{\Upsilon'(0)\Upsilon(\alpha_1)\Upsilon(\alpha_2)\Upsilon(\alpha_3)}{\Upsilon(\frac{\bar\alpha-2Q}{2})\Upsilon(\frac{\bar\alpha-\alpha_1}{2})\Upsilon(\frac{\bar\alpha-\alpha_2}{2})\Upsilon(\frac{\bar\alpha-\alpha_3}{2})}$$

$l(x) = \Gamma(x)/\Gamma(1-x)$, $\bar\alpha = \alpha_1+\alpha_2+\alpha_3$, $s = (\bar\alpha - 2Q)/\gamma$.

Υ is an entire function on $\mathbb{C}$ with simple zeros defined by

$$\log\Upsilon(\alpha) = \int_0^\infty \left((\tfrac{Q}{2}-\alpha)^2 e^{-t} - \frac{\sinh^2((\frac{Q}{2}-\alpha)\frac{t}{2})}{\sinh(\frac{t\gamma}{4})\sinh(\frac{t}{\gamma})}\right)\frac{dt}{t}$$

Hence $C(\alpha_1,\alpha_2,\alpha_3)$ is meromorphic in $\alpha_i \in \mathbb{C}$.

"It should be stressed that the arguments of this section have nothing to do with a derivation. These are rather some motivations and we consider the expression proposed as a guess which we try to support in the subsequent sections"

Heuristic attempts to derive of DOZZ formula

1. Perturbation theory in cosmological constant μ (DOZZ)
   - Order by order ∞, interpret terms as residues of poles in α, "analytically continue" from integers; cf. cite above
2. Assume the full machinery of CFT (Teschner '95)
   - Fusion rules of degenerate fields
   - Bootstrap of 4-point functions to 3-point functions
   - A mysterious **reflection relation** $V_\alpha = R(\alpha) V_{2Q-\alpha}$
3. Attempts for quantum integrability (Teschner '01)
4. Functional integral (Harlow, Maltz, Witten 2011)

Proof of DOZZ formula

Our proof of DOZZ:

- Probabilistic construction of Liouville functional integral DKRV2014
- Proof of the CFT machinery (Ward identities, BPZ equations) KRV2016
- Probabilistic derivation of reflection relation KRV2017

Probabilistic Liouville Theory

For DOZZ we need to work on the sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

- Liouville action

$$S(X) = \int_{\mathbb{C}} |\partial_z X|^2 d^2 z + \int_{\mathbb{C}} (QR_g X + \mu e^{\gamma X})g(z) d^2 z$$

- Smooth metric $g(z)$ on $\hat{\mathbb{C}}$, R_g scalar curvature
- "Background charge $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$ needed for conformal invariance

Regularize and renormalize the functional integral:

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(x_i)} \rangle = \lim_{\epsilon \to 0} \langle \prod_{i=1}^{n} e^{\alpha_i \phi(x_i)} \rangle_\epsilon \tag{1}$$

Superrenormalizable QFT if $\gamma \in [0, 2)$: simple Wick ordering of $e^{\gamma \phi}$ and V_α.

Existence of Liouville correlations

Theorem (DKRV 2014) *The limit* (1) *exists and is nontrivial if and only if:*

$$(A)\ \ \forall i : \alpha_i < Q \quad \text{and} \quad (B)\ \ \sum_i \alpha_i > 2Q$$

Remarks

- (A), (B) are called **Seiberg bounds**
- (A), (B) $\implies$ $n \geq 3$: **1- and 2-point functions are ∞**.
- $n \geq 3$ needed to fix Möbius invariance

0-mode

- Decompose $\phi(z) = c + \psi(z)$, $c \in \mathbb{R}$ zero mode
- Integrate over c: Gauss-Bonnet: $\int_{\mathbb{C}} R_g g d^2 z = 2$

$$\int e^{(\sum_i \alpha_i - 2Q)c - \mu e^{\gamma c} \int e^{\gamma \psi} g d^2 z} dc = \frac{\Gamma(s)}{\mu^s \gamma} \left(\int e^{\gamma \psi} g d^2 z \right)^{-s}$$

with $s = (\sum_i \alpha_i - 2Q)/\gamma$.

- Converges if and only if $s > 0$ i.e. (B) holds.

Upshot: let $\mathbb{E}$ be expectation in the gaussian free field ψ

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(x_i)} \rangle = \frac{\Gamma(s)}{\mu^s \gamma} \mathbb{E} \Big(\prod_i e^{\alpha_i \psi(x_i)} \big(\int e^{\gamma \psi} g d^2 z \big)^{-s} \Big)$$

μ **not** a perturbative parameter!

Multiplicative Chaos

Shift the gaussian field ψ to dispose of $\prod_i e^{\alpha_i \psi}$:

Result: Liouville correlations are given by ($s = \frac{\sum_i \alpha_i - 2Q}{\gamma} > 0$)

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(x_i)} \rangle = \frac{C_g \Gamma(s)}{\mu^s} \prod_{i<j} |z_i - z_j|^{-\alpha_i \alpha_j}$$

$$\times \mathbb{E} \Big(\int \prod_i \frac{1}{|z - z_i|^{\gamma \alpha_i}} M(dz) \Big)^{-s}$$

where M is the multifractal **Multiplicative Chaos** measure

$$M(dz) =: e^{\gamma \psi(z)} : m(z) d^2 z$$

Seiberg bound (A): $\frac{1}{|z - z_i|^{\gamma \alpha_i}}$ is integrable almost surely **if and only if** $\alpha_i < Q$. In particular

$$e^{\alpha \phi} \equiv 0 \quad \alpha \geq Q$$

Conformal Field Theory

Theorem (DKRV 2014, KRV 2016)

(a) Liouville correlations are Möbius and Weyl covariant

(b)They satisfy ***Conformal Ward Identities*** *w.* $c = 1 + 6Q^2$

(c) Let $\chi = -\frac{\gamma}{2}$ *or* $-\frac{2}{\gamma}$ *and* $F(z) = \langle e^{\chi\phi(z)} \prod_i e^{\alpha_i \phi(x_i)} \rangle$. *Then*

$$\frac{1}{\chi^2}\partial_z^2 F + \sum_k \frac{\Delta_{\alpha_k}}{(z-z_k)^2} F + \sum_k \frac{1}{z-z_k}\partial_{z_k} F = 0.$$

*(**BPZ equation** for degenerate fields)*

(b) and (c) require delicate analysis of the regularity of correlation functions and proof of **operator product expansion** as insertions get together.

Structure constants

We obtain a probabilistic expression for structure constants

$$C(\alpha_1, \alpha_2, \alpha_3) = C_g \mu^{-s} \Gamma(s) \, \mathbb{E} \left(\int \frac{1}{|z|^{\gamma\alpha_1} |1 - z|^{\gamma\alpha_2}} M(dz) \right)^{-s}$$

DOZZ formula is an **integrability** result for multiplicative chaos.

Analogous to the Fyodorov-Bouchaud conjecture on REM on the circle.

Dilemma

The probabilistic expression is finite and nonzero **only if** α_i satisfy Seiberg bounds.

The DOZZ proposal is defined for **all** complex α_i and in particular $C_{DOZZ}(\alpha_1, \alpha_2, \alpha_3)$ is **nonzero** for $\alpha_i > Q$.

The probabilistic $C(\alpha_1, \alpha_2, \alpha_3)$ is **identically zero** in this region:

$$C(\alpha_1, \alpha_2, \alpha_3) \equiv 0 \quad \alpha_i \geq Q$$

What is going on? DOZZ is too beautiful to be wrong!

Remark. *One can renormalize $e^{\alpha\phi}$ for $\alpha \geq Q$ so that $C(\alpha_1, \alpha_2, \alpha_3) \neq 0$. However the result does not satisfy DOZZ.*

Analyticity

Theorem (KRV 2017) (A) *The Liouville correlation functions* $\langle \prod_{i=1}^{n} e^{\alpha_i \phi(x_i)} \rangle$ *and structure constants* $C(\alpha_1, \alpha_2, \alpha_3)$ *are analytic in* α_i *in a neighborhood of* $\alpha_i \in (0, Q)$.

(B) *The structure constants* $C(\alpha_1, \alpha_2, \alpha_3)$ *have a unique analytic continuation to meromorphic function* ***satisfying the DOZZ conjecture****.*

Note! Due to renormalization $|e^{i\beta\phi(x_i)}| = \infty$ so that even (A) is subtle!

Periodicity

Teschner in '95: DOZZ is the unique analytic solution of the **shift equations**

$$C_\gamma(\alpha_1+\chi,\alpha_2,\alpha_3)=D_\chi(\alpha_1,\alpha_2,\alpha_3)C_\gamma(\alpha_1-\chi,\alpha_2,\alpha_3)$$

with $\chi=\frac{\gamma}{2}$ **or** $\frac{2}{\gamma}$.

$D(\alpha_1,\alpha_2,\alpha_3)$ explicit

$$D_\chi(\alpha_1,\alpha_2,\alpha_3)=-\frac{1}{\pi\mu}\frac{\Gamma(-\chi^2)\Gamma(\chi\alpha_1)\Gamma(\chi\alpha_1-\chi^2)\Gamma(\frac{\chi}{2}(\bar\alpha-2\alpha_1-\chi))}{\Gamma(\frac{\chi}{2}(\bar\alpha-\chi-2Q))\Gamma(\frac{\chi}{2}(\bar\alpha-2\alpha_3-\chi))\Gamma(\frac{\chi}{2}(\bar\alpha-2\alpha_2-\chi))}$$
$$\times\frac{\Gamma(1-\frac{\chi}{2}(\bar\alpha-\chi-2Q))\Gamma(1-\frac{\chi}{2}(\bar\alpha-2\alpha_3-\chi))\Gamma(1-\frac{\chi}{2}(\bar\alpha-2\alpha_2-\chi))}{\Gamma(1+\chi^2)\Gamma(1-\chi\alpha_1)\Gamma(1-\chi\alpha_1+\chi^2)\Gamma(1-\frac{\chi}{2}(\bar\alpha-2\alpha_1-\chi))}$$

$\bar\alpha=\alpha_1+\alpha_2+\alpha_3$

Origin of shift equations

Shift equations follow if one assumes

- **Operator product** with degenerate fields ($\chi = \frac{\gamma}{2}$ or $\frac{2}{\gamma}$)

$$V_{-\chi} V_{\alpha} = V_{\alpha-\chi} + D(\alpha) V_{\alpha+\chi}$$

- **Reflection relation**: if $\alpha > Q$ then

$$V_{\alpha} = R(\alpha) V_{2Q-\alpha}$$

  with

$$R(\alpha) = -\left(\left(\frac{\gamma}{2}\right)^{\frac{\gamma^2}{2}-2} \tilde{\mu}\right)^{\frac{2(Q-\alpha)}{\gamma}} \frac{\Gamma(\frac{\gamma}{2}(\alpha-Q))\Gamma(\frac{2}{\gamma}(\alpha-Q))}{\Gamma(\frac{\gamma}{2}(Q-\alpha))\Gamma(\frac{2}{\gamma}(Q-\alpha))}.$$

  the **reflection coefficient**

- **Conformal bootstrap** for $\langle V_{\chi} V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} \rangle$

Structure of the proof

We prove OPE and the bootstrap relation in the form

- **If** $\alpha_1 + \chi < Q$

  $$\langle V_\chi(z) V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty)\rangle =$$
  $$C(\alpha_1 - \chi, \alpha_2, \alpha_3)|\mathcal{F}_-(z)|^2 + D(\alpha_1) C(\alpha_1 + \chi, \alpha_2, \alpha_3)|\mathcal{F}_+(z)|^2$$

  ($\mathcal{F}_\pm$ are hypergeometric functions).
- In the case $\alpha_1 + \chi > Q$ we get the same with the replacement

  $$C(\alpha_1 + \chi, \alpha_2, \alpha_3) \to R(\alpha + \chi) C(2Q - \alpha_1 - \chi, \alpha_2, \alpha_3)$$
- R is given by a **probabilistic** expression in terms of **tail behavior** of multiplicative chaos

Reflection coefficient

- Let $M(dz) =: e^{\gamma\phi(z)} : dz$ be Multiplicative Chaos of GFF ϕ
- Let $\alpha < Q$, $D \subset \mathbb{C}$ with $0 \in D$ and

$$Z := \int_D \frac{1}{|z|^\alpha} M(dz)$$

- Then

$$\mathbb{P}(Z > x) = R(\alpha)|x|^{-\frac{2(Q-\alpha)}{\gamma}}(1 + o(z))$$

$R(\alpha)$ has an explicit expression in terms of multiplicative chaos (**2-point Quantum Sphere** of Duplantier and Sheffield)

It is also (the nonexistent) Liouville two-point function

$$\lim_{\epsilon\to 0} \epsilon\langle e^{\epsilon\phi(0)} e^{\alpha\phi(1)} e^{\alpha\phi(\infty)}\rangle = 4R(\alpha)$$

Reflection Relation

R provides **analytic continuation** of C to $\Re\alpha_1 > Q$:

$$C(\alpha_1, \alpha_2, \alpha_3) = R(\alpha_1) C_\gamma(2Q - \alpha_1, \alpha_2, \alpha_3)$$

To prove DOZZ need to prove **integrability** result for R:

$$R(\alpha) = -\left(\left(\tfrac{\gamma}{2}\right)^{\frac{\gamma^2}{2}-2} \tilde{\mu}\right)^{\frac{2(Q-\alpha)}{\gamma}} \frac{\Gamma(\frac{\gamma}{2}(\alpha - Q))\Gamma(\frac{2}{\gamma}(\alpha - Q))}{\Gamma(\frac{\gamma}{2}(Q - \alpha))\Gamma(\frac{2}{\gamma}(Q - \alpha))}.$$

This is done by proving analyticity of R in the probabilistic region $\alpha < Q$ and deriving shift identities for R.

OPE and Bootstrap

Let $\chi = \frac{\gamma}{2}$ or $\frac{2}{\gamma}$. BPZ equation for

$$F(z) = \langle e^{-\chi\phi(z)} \prod_i e^{\alpha_1\phi(0)} e^{\alpha_2\phi(1)} e^{\alpha_3\phi(\infty)} \rangle$$

becomes a hypergeometric equation.

- We prove its space of real solutions is **one dimensional**:

$$F(z) = A|\mathcal{F}_-(z)|^2 + B|\mathcal{F}_+(z)|^2$$

- $\mathcal{F}_\pm$ hypergeometric functions
- A/B **explicit** (ratio of 12 Γ-functions).
- A, B determined by studying fusion $z \to 0$
- Fusion determined by singularity at 0. This is controlled by the tail of the Chaos partition function $\implies$ R enters.

Summary

The mysteries of Liouville theory can be addressed with a rigorous analysis of the functional integral.

Future work:

- **Spectrum:** by Osterwalder-Schrader-reconstruction we get the physical Hilbert space and a unitary representation of the Virasoro algebra with $c = 1 + 6Q^2$. How does it reduce?
- **Conjecture**: highest weights $\frac{1}{4}(Q^2 + P^2)$ generated by vertex operators $e^{(Q+iP)\phi}$
- Prove conformal bootstrap using this.
- **Analytic continuation in γ**: e.g. for γ purely imaginary, $c < 1$, C is conjectured to give 3-point probabilities for FK Potts models.

Analytic continuation

Set $\alpha = \alpha_1 + \frac{\gamma}{2}$. We have obtained:

$$F(z) = \lambda_- |\mathcal{F}_-|^2 + \lambda_+ |\mathcal{F}_+|^2$$

with $\lambda_- = C(\alpha - \gamma, \alpha_2, \alpha_3)$ and

$$\lambda_+ = \begin{cases} B(\alpha - \frac{\gamma}{2}) C(\alpha, \alpha_2, \alpha_3) & \text{if } \alpha < Q \\ R(\alpha - \frac{\gamma}{2}) C(2Q - \alpha, \alpha_2, \alpha_3) & \text{if } \alpha > Q \end{cases}$$

and $\frac{\lambda_+}{\lambda_-} = D(\alpha_1, \alpha_2, \alpha_3)$ explicite, meromorphic. This implies

$$f(\alpha) := \begin{cases} C(\alpha, \alpha_2, \alpha_3) & \text{if } \alpha < Q \\ \frac{R(\alpha - \frac{\gamma}{2})}{B(\alpha - \frac{\gamma}{2})} C(2Q - \alpha, \alpha_2, \alpha_3) & \text{if } \alpha > Q \end{cases}$$

equals $\frac{D(\alpha_1, \alpha_2, \alpha_3)}{B(\alpha - \frac{\gamma}{2})} C(\alpha - \gamma, \alpha_2, \alpha_3)$ which is analytic around $\alpha = Q$!