Conformal Field Theory, Operator Algebras and Vertex Operator Algebras

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Chiral conformal field theory and operator algebras

Relate vertex operator algebras and operator algebras directly for the first time and study their connections through abstract modular tensor categories.

Outline of the talk:

1. Quantum fields and chiral conformal field theory
2. Local conformal nets and representation theory
3. Vertex operator algebras (VOAs)
4. Realization problem of modular tensor categories
5. From VOAs to local conformal nets and back
6. Other types of conformal field theories

Quantum field theory (Wightman framework)

Mathematical/axiomatic ingredients: Spacetime, its symmetry group, quantum fields on the spacetime.

From a mathematical viewpoint, quantum fields are certain operator-valued distributions on the spacetime. Mathematical axiomatization of such operator-valued distributions on a Hilbert space is given by the Wightman axioms.

A pairing $\langle T, f\rangle$ for a quantum field T and a test function f supported in a spacetime region O gives an observable in O. For a fixed O, let $A(O)$ be the operator algebra generated by these observables. We have a family $\{A(O)\}$ of operator algebras of bounded linear operators parameterized with regions O, and it is called a net. We work on its mathematical axiomatization.

Chiral conformal field theory

Such an approach based on operator algebras is called algebraic quantum field theory and has been studied since 1960's. (Haag-Kastler, Doplicher-Haag-Roberts). One version of it in the setting of chiral conformal field theory has been studied since 1980's.

In the case of chiral conformal field theory, the spacetime is simply S^1 and its symmetry group is $\mathrm{Diff}(S^1)$, the orientation-preserving diffeomorphism group. (This circle is a one-point compactification of a light ray $\{x = \pm t\}$ of the 2-dimensional Minkowski space.) The spacetime regions are open connected intervals contained in S^1. One family $\{A(I)\}$ of operator algebras parameterized by $I \subset S^1$, satisfying a set of physically natural axioms, describes one chiral conformal field theory mathematically.

Local conformal nets

A chiral conformal field theory is described with a family $\{A(I)\}$ of operator algebras parameterized by an interval $I \subset S^1$ subject to certain axioms. Such a family is called a local conformal net.

Axioms for a local conformal net:

1. $I_1 \subset I_2 \Rightarrow A(I_1) \subset A(I_2)$.
2. $I_1 \cap I_2 = \varnothing \Rightarrow [A(I_1), A(I_2)] = 0$. (locality)
3. $\mathrm{Diff}(S^1)$-covariance (conformal covariance)
4. Positive energy, vacuum vector
5. Irreducibility

The locality axiom comes from the Einstein causality. (If two regions are spacelike separated, the observables living on them commute.)

Vertex operator algebras

A vertex operator algebra (VOA) gives another mathematical axiomatization of a chiral conformal field theory. It directly deals with Fourier expansions of operator-valued distributions, vertex operators, on S^1 in an algebraic manner.

It was initiated by Borcherds and Frenkel-Lepowsky-Meurman in 1980's in studies of the Moonshine conjecture describing mysterious relations between the Monster group and the j-function.

Both vertex operator algebras and local conformal nets give mathematical formulations of the same physical theory and many similarities between the both theories are apparent. We now directly connect them for the first time. This relation is somehow similar to the one between Lie algebras and Lie groups.

Examples of local conformal nets

Basic methods of constructions

1. Affine Lie algebras (Wassermann, Toledano Laredo)
2. Virasoro algebra (Loke, Xu)
3. Even lattices (K-Longo, Dong-Xu)

New constructions from given examples

1. Tensor product
2. Coset construction (Xu)
3. Orbifold construction (Xu — e.g. Moonshine net by K-Longo)
4. Simple current extension (Böckenhauer-Evans)
5. Extension by Frobenius algebras (Longo-Rehren, K-Longo, Xu)

Representation theory

Consider a local conformal net $\{A(I)\}$. Each $A(I)$ acts on the same Hilbert space (having the vacuum vector) from the beginning. We also consider a representation of these operator algebras on another Hilbert space (without a vacuum vector).

A classical Doplicher-Haag-Roberts theory of 1970's adapted to a local conformal net shows that one representation is given by one endomorphism of a single operator algebra, a factor, and the composition of two endomorphisms gives a proper notion of a tensor product. It also naturally has a notion of a dimension, using the Jones index. (The dimension takes a value in $[1, \infty]$.)

In this way, representation theory of a local conformal net gives a braided tensor category.

Rationality

We are often interested in a situation where we have only finitely many irreducible representations. Such a situation is usually called rational. (cf. Quantum groups at roots of unity.) K-Longo-Müger gave an operator algebraic characterization of complete rationality of a local conformal net, where we have only finitely many irreducible representations and each has a finite dimension, without using its representation theory.

This complete rationality is defined in terms of finiteness of some real number arising in the operator algebraic framework. If this finiteness holds, then the representation theory of the local conformal net gives a modular tensor category, which means the braiding is non-degenerate. (K-Longo-Müger)

Subfactors

An endomorphism λ of a factor M gives a subfactor $N = \lambda(M)$. Suppose the Jones index $[M : N]$ is finite and consider a bimodule ${}_N M_M$. Its relative tensor powers ${}_N M \otimes_M M \otimes_N M \cdots$ and their irreducible decompositions give 4 kinds of bimodules, N-N, N-M, M-N and M-M. If we have only finitely many irreducible M-M bimodules in this way, they make a fusion category, which is similar to a representation category of a finite group without braiding.

Finite or quantum groups produce many examples of such subfactors and fusion categories, but there are more examples which do not seem to arise from (quantum) groups in any known way. The most famous examples are the Haagerup subfactor, the Asaeda-Haagerup subfactor and the extended Haagerup subfactor.

Exceptional modular tensor categories

These exceptional subfactors produce exceptional fusion categories and their Drinfeld centers give exceptional modular tensor categories. For the Haagerup subfactor, the Asaeda-Haagrup subfactor and the extended Haagerup subfactor, the S- and T-matrices of the Drinfeld centers have been explicitly computed. They do not look like anything arising from combinations of known other constructions such as coset/orbifold constructions for Wess-Zumino-Witten models.

These three exceptional subfactors have been found through a combinatorial search of subfactors for a very small range of the Jones index values just above 4. Then it is natural to believe there is a huge variety of exceptional subfactors and modular tensor categories beyond what are known today.

Realization problem

We consider a problem whether a given (unitary) modular tensor category is realized as the representation category of a local conformal net.

In the history of classification theory on operator algebras, we have had no nontrivial obstructions to realization of algebraic invariants as long as we have an analytic property called amenability. All the algebraic structures are always amenable for a completely rational local conformal net.

We then strongly believe any modular tensor category is realized as the representation category of a local conformal net. This would imply there would be a huge variety of new exotic chiral conformal field theories beyond what are known today.

α-induction

We recall a classical notion of induction of a representation of a group and introduce a similar construction for a local conformal net.

Let $\{A(I) \subset B(I)\}$ be an inclusion of local conformal nets. We can produce an almost representation of $\{B(I)\}$ from a representation of $\{A(I)\}$, using the braiding. ($\alpha^\pm$-induction: Longo-Rehren, Xu, Ocneanu, Böckenhauer-Evans-K)

Böckenhauer-Evans-K have shown that the matrix $(Z_{\lambda,\mu})$ defined by

$$Z_{\lambda,\mu} = \dim \mathrm{Hom}(\alpha^+_\lambda, \alpha^-_\mu)$$

is in the commutant of the representation of $SL(2,\mathbb{Z})$ arising from the braiding. Such a matrix Z is called a modular invariant.

Modular invariants are often quite limited for a given $\{A(I)\}$.

Classification theory

Based on this, K-Longo have obtained the following complete classification of local conformal nets with $c < 1$.

(1) Virasoro nets $\{\mathrm{Vir}_c(I)\}$ with $c < 1$.

(2) Their simple current extensions with index 2.

(3) Four exceptionals at $c = 21/22$, $25/26$, $144/145$, $154/155$.

The simple current extensions in (2) are easy to understand. Three of the four exceptionals in the above (3) are identified with coset constructions, but the remaining one $c = 144/145$ does not seem to be related to any other known constructions so far, and is given as an extension by a Frobenius algebra.

Note that this appearance of a modular invariant matrix is different from its original context in 2-dimensional conformal field theory.

Unitarity and energy bounds

Now we construct a local conformal net from a VOA V. We first need a Hilbert space, the completion of V with respect to an inner product, but a general VOA does not have a nice one. So we have to assume to have such an inner product. One having a *nice* inner product is called unitary. Many natural examples are unitary.

Let V be a unitary VOA. We say that $u \in V$ satisfies energy-bounds if we have positive integers s, k and a constant $M > 0$ such that we have

$$\|u_n v\| \leq M(|n|+1)^s \|(L_0+1)^k v\|,$$

for all $v \in V$ and $n \in \mathbb{Z}$. If every $u \in V$ satisfies energy-bounds, we say V is energy-bounded.

Strong locality

For every $u \in V$, we define the operator $Y_0(u, f)$ with domain V by $Y_0(u, f)v = \sum_{n\in\mathbb{Z}} \hat{f}_n u_n v$ for $v \in V$, where f is a C^∞ function on S^1 and $\hat{f}_n$ is its Fourier coefficient.

Let $Y(u, f)$ be the closure of $Y_0(u, f)$ on the completion of V. The (unbounded) operators $Y(u, f)$, where $u \in V$ and $\mathrm{supp}\, f \subset I$, generate an operator algebra $A(I)$. The family $\{A(I)\}$ satisfies all the axioms of a local conformal net except for locality. If we also have locality, we say the original VOA has strong locality. (Something more than the usual locality of a VOA.)

If a simple VOA V has a set of *nice* generators, we have strong locality. This sufficient condition applies to many known examples of VOA's. (Carpi-K-Longo-Weiner)

Going back to a VOA

We construct a local conformal net from a VOA and now would like to recover the original VOA.

Based on an idea of Fredenhagen-Jörss and with help of the Tomita-Takesaki theory, we can recover a smeared vertex operator $Y(u, f)$ for $u \in V$ and $f \in C^\infty(S^1)$, using one of the Virasoro generators, L_{-1}. Here the space V is first recovered as an algebraic direct sum of the eigenspaces of the Virasoro generator L_0. Hence we can also recover $u_n \in \mathrm{End}(V)$, which gives the entire structure of a VOA. (Carpi-K-Longo-Weiner)

If we start with a general local conformal net, we can define an analogue of $Y(u, f)$, but we do not know whether this gives a vertex operator in general or not.

Other types of conformal field theory

For a full conformal field theory, we study a net of operator algebras parameterized by rectangles in a 2-dimensional Minkowski space. Such a theory produces two local conformal nets, and if they are completely rational, we have a general classification machinery.

For a boundary conformal field theory, we study a net of operator algebras parameterized by rectangles in a 2-dimensional half Minkowski space $\{x > 0\}$. Such a theory lives on one local conformal net, and if it is completely rational, we have a general classification machinery.

For a superconformal field theory, we have a $\mathbb{Z}_2$-grading on the Hilbert space and on the operators. Then locality is replaced with graded locality. Super Virasoro algebras naturally appear.