Matematica e Fisica al crocevia

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The cycle Phys-Math-Math-Phys

"Nessuna humana investigazione si può dimandara vera scienzia s'essa non passa per le matematiche dimonstrazioni".
Leonardo da Vinci, Trattato della Pittura, 1500 circa

$$\begin{tikzcd}
\text{Physics} \arrow[r, "\text{experiment}", "\text{theory}"'] & \text{Physics} \arrow[d] \\
\text{Mathematics} \arrow[u] & \text{Mathematics} \arrow[l, "\text{art}"', "\text{formalism}"]
\end{tikzcd}$$

"*The Unreasonable Effectiveness of Mathematics in the Natural Sciences*."
Eugene Wigner, Comm. Pure Appl. Math. 1960

The unity of Mathematics

Euler's formula:

$$e^{i\pi} + 1 = 0$$

e: *Real Analisys*
i: *Complex Analysis*
π: *Geometry*
1: *Algebra*

Cartesio: Geometry to Algebra:

$$\bigcirc \longrightarrow x^2 + y^2 = r^2$$

Now x, y, r can be generalised!

The unity of Physics

Einstein:

$$E = mc^2$$

Hawking effect: a quantum black hole has a surface temperature

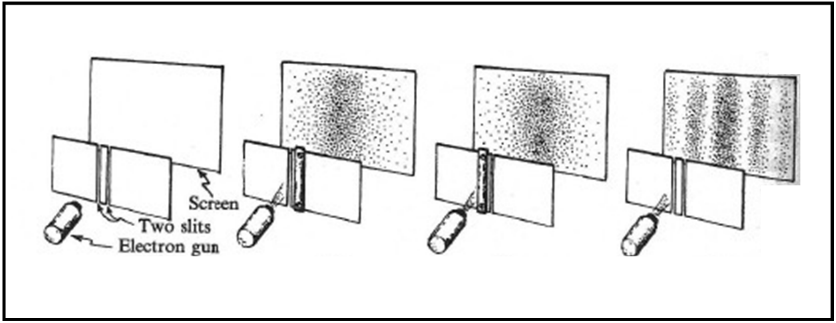
$$T = \frac{c^3 \hbar}{8\pi k M G}$$

c: speed of light
G: gravitational constant
k: Boltzmann constant
$\hbar$: Planck constant

Grand Unification and the dream of Quantum Gravity,
see part II by Massimo Bianchi

From classical to quantum physics

Until the XIX century, Mathematics and Physics had a natural and continuous interplay: Archimede, Galileo, Newton, Gauss, etc. At the beginning of the past century, Quantum Physics makes a discontinuity and opens a completely new frame.



Quantum Mechanics

$$\begin{tikzcd}
& \textit{Planck} \arrow[dl] \arrow[dr] & \\
\textit{Schroedinger} \arrow[dr] & & \textit{Heisenberg} \arrow[dl] \\
& \textit{von Neumann} &
\end{tikzcd}$$

- Schrödinger:

$$i\hbar \frac{\partial}{\partial t} \psi(x,t) = H\psi(x,t)$$

Differential equations

- Heisenberg:

$$PQ - QP = i\hbar I$$

Linear operators on a Hilbert space, <u>noncommutativity is essential</u>!

- von Neumann: The two QM are equivalent, uniqueness of CCR

Operator Algebras:

Classical Commutative	Quantum Noncommutative
Manifold X $C^\infty(X)$	$*$-algebra A
Topological space X $C(X)$	C^*-algebra $\mathfrak{A}$
Measure space X $L^\infty(X,\mu)$	von Neumann algebra $\mathcal{A}$

Thermal equilibrium states

Thermodynamics concerns heat and temperature and their relation to energy and work. A primary role is played by the equilibrium distribution.

Gibbs states

Finite quantum system: $\mathfrak{A}$ matrix algebra with Hamiltonian H and evolution $\tau_t = \mathrm{Ad} e^{itH}$. Equilibrium state φ at inverse temperature β is given by the Gibbs property

$$\varphi(X) = \frac{\mathrm{Tr}(e^{-\beta H} X)}{\mathrm{Tr}(e^{-\beta H})}$$

What are the equilibrium states at infinite volume where there is no trace, no inner Hamiltonian?

von Neumann algebras

$\mathcal{H}$ a Hilbert space. $B(\mathcal{H})$ algebra of all bounded linear operators on $\mathcal{H}$.
$\mathcal{M} \subset B(\mathcal{H})$ is a von Neumann algebra if it is a $*$-algebra and is weakly closed. Equivalently (von Neumann density theorem)

$$\mathcal{M} = \mathcal{M}''$$

with $\mathcal{M}' = \{T \in B(\mathcal{H}) : TA = AT \quad \forall A \in \mathcal{M}\}$ the commutant.

A C^*-algebra is only closed in norm.

Observables are elements A of $\mathcal{M}$, *states* are normalised positive linear functionals φ,

$$\varphi(A) = \text{expected value of the observable in the state}$$

$\mathcal{M}$ abelian $\Leftrightarrow \mathcal{M} = L^\infty(X, \mu)$.

KMS states (HHW, Baton Rouge conference 1967)

Infinite volume. $\mathfrak{A}$ a C^*-algebra, τ a one-par. automorphism group of $\mathfrak{A}$. A state φ of $\mathfrak{A}$ is KMS at inverse temperature $\beta > 0$ if for $X, Y \in \mathfrak{A}$

$$\varphi\big(X\tau_{t+i\beta}(Y)\big) = \varphi\big(\tau_t(Y)X\big)$$

$$\varphi\big(\tau_t(Y)X\big)$$

$$\beta$$

$$\varphi\big(X\tau_t(Y)\big)$$

KMS $=$ *thermodynamical equilibrium condition*

Modular theory and Connes cocycles

Let $\mathcal{M}$ be a von Neumann algebra and φ a normal faithful state on $\mathcal{M}$. The Tomita-Takesaki theorem gives a *canonical evolution*:

$$t \in \mathbb{R} \mapsto \sigma_t^\varphi \in \mathrm{Aut}(\mathcal{M})$$

Non commutative measure theory is dynamical!

By a remarkable historical coincidence, Tomita announced the theorem at the 1967 Baton Rouge conference. Soon later Takesaki charcterised the modular group by the KMS condition.

The Connes Radon-Nikodym cocycle relates the modular groups of different states

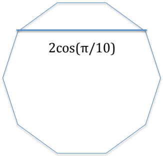
$$u_t = (D\psi : D\varphi)_t \in \mathcal{M} \ , \quad \sigma_t^\psi = u_t \sigma_t^\varphi(\cdot) u_t^*$$

Jones index

Factors (von Neumann algebras with trivial center) are "very infinite-dimensional" objects. For an inclusion of factors $\mathcal{N} \subset \mathcal{M}$ the Jones index $[\mathcal{M} : \mathcal{N}]$ measure the relative size of $\mathcal{N}$ in $\mathcal{M}$. Surprisingly, the index values are quantised:

$$[\mathcal{M} : \mathcal{N}] = 4\cos^2\left(\frac{\pi}{n}\right) \;, \quad n = 3, 4, \ldots \quad \text{or} \quad [\mathcal{M} : \mathcal{N}] \geq 4$$

Jones index appears in many places in math and in physics.



Quantum Field Theory

In QFT we have a quantum system with infinitely many degrees of freedom. The system is relativistic and there is particle creation and annihilation.

No mathematically rigorous QFT model with interaction still exists in 3+1 dimensions!

Haag local QFT:

$$O \text{ spacetime regions} \mapsto \text{von Neumann algebras } \mathcal{A}(O)$$

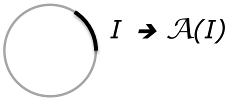
to each region one associates the "noncommuative functions" with support in O.

Local conformal nets

A local net $\mathcal{A}$ on the circle S^1 is a map

$$\text{interval } I \mapsto \text{von Neumann algebra } \mathcal{A}(I)$$

- *Isotony*. $I_1 \subset I_2 \implies \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$
- *Locality*. $I_1 \cap I_2 = \varnothing \implies [\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$
- *Diffeomorphism covariance with positive energy and vacuum vector*.



"Noncommutative chart"

Representations

A (DHR) *representation* ρ of local conformal net $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a map $I \in \mathcal{I} \mapsto \rho_I$, with ρ_I a normal rep. of $\mathcal{A}(I)$ on $\mathcal{H}$ s.t.

$$\rho_{\tilde{I}} \restriction \mathcal{A}(I) = \rho_I, \quad I \subset \tilde{I}, \quad I, \tilde{I} \subset \mathcal{I} \ .$$

Index-statistics theorem (R.L.):

$$d(\rho) = \Big[\rho_{I'}\big(\mathcal{A}(I')\big)' : \rho_I\big(\mathcal{A}(I)\big)\Big]^{\frac{1}{2}}$$

DHR dimension $= \sqrt{\text{Jones index}}$

Physical index

Anal. index

Classification of local conformal nets, $c = 1 - \frac{6}{m(m+1)}$

Local conformal nets with $c < 1$ are classified by pair of Dynkin diagrams $A - D_{2n} - E_{6,8}$ with difference of Coxeter numbers 1 (Kawahigashi, L. 2004)

m	Labels for Z
n	(A_{n-1}, A_n)
$4n+1$	(A_{4n}, D_{2n+2})
$4n+2$	(D_{2n+2}, A_{4n+2})
11	(A_{10}, E_6)
12	(E_6, A_{12})
29	(A_{28}, E_8)
30	(E_8, A_{30})

Four exceptional cases, one *new example* (A_{28}, E_8), probably not constructable as coset

Case $c = 1$ classified by Xu, Carpi (with a spectral condition)

Many new models by mirror symmetry, F. Xu.

Towards a QFT index theorem

D elliptic differential operator between vector bundles E and F over a compact manifold X.

Atiyah-Singer index theorem:

$$\text{Analytical index}(D) = \text{Topological index}(D)$$

The Index Theorem is one of the most influential theorem in Mathematics and in Physics of the past century.

Is there an index theorem with infinitely many degrees of freedom?

From classic to QFT

CLASSICAL	Classical variables Differential forms Chern classes	Variational calculus Infinite dimensional manifolds Functions spaces Wiener measure
QUANTUM	Quantum geometry Fredholm operators Index Cyclic cohomology	Subfactors Bimodules, Endomorphisms Jones index Supersymmetric QFT

An example: topological sectors (F. Xu, R.L.)

$\mathcal{A}$ local conformal net, $f : S^1 \to S^1$, $\deg(f) = n$,
$\mathcal{B} \equiv (\mathcal{A} \otimes \cdots \mathcal{A})^{\mathbb{Z}^n}$

τ_f topological sector of $\mathcal{B}$ associated with f. We have:

$$\text{Jones index}(\tau_f) = \mu_{\mathcal{A}}^{\deg(f)}$$

Anal. index *Topol. index*

Physical sector

Second example: incremental free energy in QFT

A QFT index theorem holds for a certain class of black holes (e.g. Rindler or Schwarschild)

$$\Delta(F_{\rho_1}|F_{\rho_2}) = \beta^{-1}\big(\log d(\rho_1) - \log d(\rho_2)\big)$$

- $\Delta(F_{\rho_1}|F_{\rho_2})$: incremental free energy adding the charge ρ_1 and removing the charge ρ_2
- β^{-1}: Hawking-Unruh temperature (Bisognano-Wichmann KMS modular property for a uniformly accelerated observer)
- $d(\rho_1), d(\rho_2)$: DHR statistical dimensions: integers! (Analog to the Fredholm index)

Thanks, now Part II