Operator Algebras and Noncommutative Geometric Aspects in Conformal Field Theory

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Based on a joint papers with S. Carpi, Y. Kawahigashi and R. Hillier
Things to discuss

- Getting inspired by black hole entropy
- Symmetry and supersymmetry
- Local conformal nets
- Modularity and asymptotic formulae
- Fermi and superconformal nets
- Neveu-Schwarz and Ramond representations
- Fredholm index and Jones index
- Noncommutative geometrization
- Model analysis (in progress)
Prelude. Black hole entropy

Bekenstein: The entropy $S$ of a black hole is proportional to the area $A$ of its horizon

\[ S = A/4 \]

- $S$ is geometric
- $S$ is proportional to the area, not to the volume as a naive microscopic interpretation of entropy would suggest (logarithmic counting of possible states).
- This dimensional reduction has led to the *holographic principle* by t’Hooft, Susskind, …
- The horizon is not a physical boundary, but a submanifold where coordinates pick critical values $\rightarrow$ conformal symmetries
- The proportionality factor $1/4$ is fixed by Hawking temperature (*quantum* effect).
Black hole entropy

*Discretization* of the horizon (Bekenstein): horizon is made of cells or area \( \ell^2 \) and \( k \) degrees of freedom (\( \ell = \) Planck length):

\[
A = n \ell^2, \\
\text{Degrees of freedom} = k^n, \\
S = C n \log k = C \frac{A}{\ell^2} \log k, \\
dS = C \log k
\]

**Conclusion.**

Black hole entropy

\[\downarrow\]

Two-dimensional conformal quantum field theory with a “fuzzy” point of view

Legenda: Fuzzy = *noncommutative geometrical*
Symmetries in Physics

Spacetime symmetries
Lorentz, Poincaré, ...

Internal symmetries
Gauge, ...

SUSY

Bose-Fermi

SUSY: $H = Q^2$, $Q$ odd operator, $[\cdot, Q]$ graded super-derivation interchanging Boson and Fermions

Among consequences: Cancellation of some Higgs boson divergence
Conformal and superconformal

- Low dimension, conformal $\rightarrow$ infinite dim. symmetry
- Low dimension, conformal + SUSY $\rightarrow$ Superconformal symmetry (very stringent)
About three approaches to CFT

Vertex Algebras
\textit{(algebraic)}

\begin{itemize}
  \item Carpi—Weiner
  \item Fredenhagen—Jorss
\end{itemize}

Operator Algebras
\textit{(algebraic & analytic)}

\begin{itemize}
  \item Kac
\end{itemize}

Wightman fields
\textit{(analytic)}

\textit{partial relations known}
von Neumann algebras

$\mathcal{H}$ Hilbert space, $\mathcal{B}(\mathcal{H})$ *algebra of all bounded linear operators on $\mathcal{H}$.

**Def.** A *von Neumann algebra* $M$ is a weakly closed non-degenerate *-subalgebra of $\mathcal{B}(\mathcal{H})$.

- von Neumann density thm. $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ non-degenerate *-subalgebra

\[ \mathcal{A}' = \mathcal{A}'' \]

where $'$ denotes the commutant

\[ \mathcal{A}' = \{ T \in \mathcal{B}(\mathcal{H}) : TA = AT \ \forall A \in \mathcal{A} \} \]

*Double aspect, analytical and algebraic*

$M$ is a **factor** if its center $M \cap M' = \mathbb{C}$. 
The tensor category $\text{End}(M)$

$M$ an infinite factor $\rightarrow \text{End}(M)$ is a tensor $C^*$-category:

- **Objects**: $\text{End}(M)$
- **Arrows**: $\text{Hom}(\rho, \rho') \equiv \{ t \in M : t\rho(x) = \rho'(x)t \ \forall x \in M \}$
- **Tensor product of objects**: $\rho \otimes \rho' = \rho\rho'$
- **Tensor product of arrows**: $\sigma, \sigma' \in \text{End}(M)$, $t \in \text{Hom}(\rho, \rho')$, $s \in \text{Hom}(\sigma, \sigma')$,

\[
t \otimes s \equiv t\rho(s) = \rho'(s)t \in \text{Hom}(\rho \otimes \sigma, \rho' \otimes \sigma')
\]

- **Conjugation**: $\exists$ isometries $\nu \in \text{Hom}(\iota, \rho\rho)$ and $\bar{\nu} \in \text{Hom}(\iota, \bar{\rho}\rho)$ such that

\[
(\bar{\nu}^* \otimes 1_{\rho}) \cdot (1_{\rho} \otimes \nu) \equiv \bar{\nu}^* \rho(\nu) = \frac{1}{d}
\]
\[
(\nu^* \otimes 1_{\rho}) \cdot (1_{\rho} \otimes \bar{\nu}) \equiv \nu^* \rho(\bar{\nu}) = \frac{1}{d}
\]

for some $d > 0$. 
Dimension

The \textit{minimal} $d$ is the \textit{dimension} $d(\rho)$

$$[M : \rho(M)] = d(\rho)^2$$

(tensor categorical definition of the \textbf{Jones index})

$$d(\rho_1 \oplus \rho_2) = d(\rho_1) + d(\rho_2)$$
$$d(\rho_1 \rho_2) = d(\rho_1)d(\rho_2)$$
$$d(\bar{\rho}) = d(\rho)$$

\textit{End}(M) is a “universal” \textit{tensor category} (cf. Popa, Yamagami)

(generalising the Doplicher-Haag-Roberts theory)
Local conformal nets

A local Möbius covariant net $\mathcal{A}$ on $S^1$ is a map

$$I \in \mathcal{I} \rightarrow A(I) \subset B(\mathcal{H})$$

$\mathcal{I}$ ≡ family of proper intervals of $S^1$, that satisfies:

- **A. Isotony.** $I_1 \subset I_2 \implies A(I_1) \subset A(I_2)$
- **B. Locality.** $I_1 \cap I_2 = \emptyset \implies [A(I_1), A(I_2)] = \{0\}$
- **C. Möbius covariance.** $\exists$ unitary rep. $U$ of the Möbius group Möb on $\mathcal{H}$ such that

  $$U(g) A(I) U(g)^* = A(gI), \quad g \in \text{Möb}, \; I \in \mathcal{I}. $$

- **D. Positivity of the energy.** Generator $L_0$ of rotation subgroup of $U$ (conformal Hamiltonian) is positive.

- **E. Existence of the vacuum.** $\exists!$ $U$-invariant vector $\Omega \in \mathcal{H}$ (vacuum vector), and $\Omega$ is cyclic for $\bigvee_{I \in \mathcal{I}} A(I)$. 

First consequences

- **Irreducibility**: \( \bigvee_{I \in \mathcal{I}} \mathcal{A}(I) = B(H) \).
- **Reeh-Schlieder theorem**: \( \Omega \) is cyclic and separating for each \( \mathcal{A}(I) \).
- **Bisognano-Wichmann property**: Tomita-Takesaki modular operator \( \Delta_I \) and conjugation \( J_I \) of \( (\mathcal{A}(I), \Omega) \), are

\[
U(\Lambda_I(2\pi t)) = \Delta_I^{it}, \quad t \in \mathbb{R}, \quad \text{dilations}
\]

\[
U(r_I) = J_I \quad \text{reflection}
\]

(Fröhlich-Gabbiani, Guido-L.)

- **Haag duality**: \( \mathcal{A}(I)' = \mathcal{A}(I') \)

- **Factoriality**: \( \mathcal{A}(I) \) is III\(_1\)-factor (in Connes classification)

- **Additivity**: \( I \subset \bigcup_i I_i \implies \mathcal{A}(I) \subset \bigvee_i \mathcal{A}(I_i) \) (Fredenhagen, Jorss).
Local conformal nets

\[ \text{Diff}(S^1) \equiv \text{group of orientation-preserving smooth diffeomorphisms of } S^1. \]

\[ \text{Diff}_I(S^1) \equiv \{ g \in \text{Diff}(S^1) : g(t) = t \ \forall \ t \in I' \}. \]

A local \textit{conformal net} \( \mathcal{A} \) is a M"obius covariant net s.t.

\textbf{F. Conformal covariance.} \exists a projective unitary representation \( U \) of \( \text{Diff}(S^1) \) on \( \mathcal{H} \) extending the unitary representation of M"ob s.t.

\[
U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI), \quad g \in \text{Diff}(S^1),
\]

\[
U(g)xU(g)^* = x, \quad x \in \mathcal{A}(I), \quad g \in \text{Diff}_I(S^1),
\]

\[ \longrightarrow \text{unitary representation of the Virasoro algebra} \]

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}
\]

\[ [L_n, c] = 0, \quad L_n^* = L_{-n}. \]
Representations

A representation $\pi$ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a map

$$I \in \mathcal{I} \mapsto \pi_I, \text{ normal rep. of } \mathcal{A}(I) \text{ on } B(\mathcal{H})$$

$$\pi_{\tilde{I}}|_{\mathcal{A}(I)} = \pi_I, \quad I \subset \tilde{I}$$

$\pi$ is automatically diffeomorphism covariant: $\exists$ a projective, pos. energy, unitary rep. $U_\pi$ of $\text{Diff}^{(\infty)}(S^1)$ s.t.

$$\pi_{gI}(U(g)xU(g)^*) = U_\pi(g)\pi_I(x)U_\pi(g)^*$$

for all $I \in \mathcal{I}$, $x \in \mathcal{A}(I)$, $g \in \text{Diff}^{(\infty)}(S^1)$ (Carpi & Weiner)

**DHR argument:** given $I$, there is an endomorphism of $\mathcal{A}$ localized in $I$ equivalent to $\pi$; namely $\rho$ is a representation of $\mathcal{A}$ on the vacuum Hilbert space $\mathcal{H}$, unitarily equivalent to $\pi$, such that $\rho_{I'} = \text{id} \upharpoonright \mathcal{A}(I')$.

- $\text{Rep}(\mathcal{A})$ is a braided tensor category (DHR, FRS, L.)
Index-statistics theorem

DHR dimension $d(\rho) = \sqrt{\text{Jones index } \text{Ind}(\rho)}$

tensor category $\text{Rep}_I(A)$ $\xrightarrow{\text{full functor}}$ $\xrightarrow{\text{restriction}}$ tensor category $\text{End}(A(I))$
Black hole incremental free energy

Define the *incremental free energy* \( F(\varphi_\sigma|\varphi_\rho) \) between the thermal states \( \varphi_\sigma \) and \( \varphi_\rho \) in reps \( \rho, \sigma \) localized in \( I \) (\( \beta^{-1} = \) Hawking temperature)

\[
F(\varphi_\sigma|\varphi_\rho) = \varphi_\rho(H_\rho) - \beta^{-1}S(\varphi_\sigma|\varphi_\rho)
\]

\( S(\varphi_\sigma|\varphi_\rho) = -(\log \Delta_{\xi_\sigma,\xi_\rho\xi_\rho}) \) is Araki relative entropy

Then

\[
F(\varphi_\sigma|\varphi_\rho) = \frac{1}{2} \beta^{-1}(d(\sigma) - d(\rho))
\]

\[
= \frac{1}{2} \beta^{-1}(\log m - \log n)
\]

If the charges \( \rho, \sigma \) come from a spacetime of dimension \( d \geq 2 + 1 \) then \( n, m \) integers by DHR restriction on the values \( d(\rho), d(\sigma) \).
Complete rationality

I₁, I₂ intervals \(\overline{I₁} \cap \overline{I₂} = \emptyset\), \(E \equiv I₁ \cup I₂\).

\(\mu\)-index: \(\muₘ \equiv [A(E')' : A(E)]\)

(Jones index). \(A\) conformal:

\(A\) completely rational \(\overset{\text{def}}{=} A\) split & \(\muₘ < \infty\)

**Thm.** (Y. Kawahigashi, M. Müger, R.L.) \(A\) completely rational: then

\[
\muₘ = \sum_i d(\rho_i)^2
\]

sum over all irreducible sectors. (F. Xu in SU\((N)\) models);

- \(A(E) \subset A(E')' \sim \text{LR inclusion (quantum double)}\);
- Representations form a modular tensor category (i.e. non-degenerate braiding).
Weyl’s theorem

$M$ compact oriented Riemann manifold, $\Delta$ Laplace operator on $L^2(M)$.

Theorem (Weyl)

Heat kernel expansion as $t \to 0^+$:

$$\text{Tr}(e^{-t\Delta}) \sim \frac{1}{(4\pi t)^{n/2}} (a_0 + a_1 t + \cdots)$$

The spectral invariants $n$ and $a_0, a_1, \ldots$ encode geometric information and in particular

$$a_0 = \text{vol}(M), \quad a_1 = \frac{1}{6} \int_M \kappa(m) d\text{vol}(m),$$

$\kappa$ scalar curvature. $n = 2$: $a_1$ is proportional to the Euler characteristic $= \frac{1}{2\pi} \int_M \kappa(m) d\text{vol}(m)$ by Gauss-Bonnet theorem.
Modularity

With $\rho$ rep. of $\mathcal{A}$, set $L_{0,\rho}$ conf. Hamiltonian of $\rho$,

$$\chi_\rho(\tau) = \text{Tr} \left( e^{2\pi i \tau (L_{0,\rho} - c/24)} \right) \quad \text{Im} \tau > 0.$$  

specialized character, $c$ the central charge. $\mathcal{A}$ is modular if $\mu_\mathcal{A} < \infty$ and

$$\chi_\rho(-1/\tau) = \sum_\nu S_{\rho,\nu} \chi_\nu(\tau),$$

$$\chi_\rho(\tau + 1) = \sum_\nu T_{\rho,\nu} \chi_\nu(\tau).$$

with $S, T$ the (algebraically defined) Kac-Peterson, Verlinde Rehren matrices generating a representation of $SL(2, \mathbb{Z})$. One has:

- Modularity $\implies$ complete rationality
- Modularity holds in all computed rational case, e.g. $SU(N)_k$-models
- $\mathcal{A}$ modular, $\mathcal{B} \supset \mathcal{A}$ irreducible extension $\implies \mathcal{B}$ modular.
- All conformal nets with central charge $c < 1$ are modular.
A modular. The following asymptotic formula holds as $t \to 0^+$:

$$\log \text{Tr}(e^{-2\pi tL_0}) \sim \frac{\pi c}{12} \frac{1}{t} - \frac{1}{2} \log \mu_A - \frac{\pi c}{12} t$$

In any representation $\rho$, as $t \to 0^+$:

$$\log \text{Tr}(e^{-2\pi tL_{0,\rho}}) \sim \frac{\pi c}{12} \frac{1}{t} + \frac{1}{2} \log \frac{d(\rho)^2}{\mu_A} - \frac{\pi c}{12} t$$
Modular nets as NC manifolds (∞ degrees of freedom)

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<td>$\pi c/12$</td>
<td>NC area</td>
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<tr>
<td>$a_1$</td>
<td>$-\frac{1}{2} \log \mu_A$</td>
<td>NC Euler charact.</td>
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<tr>
<td>$a_2$</td>
<td>$-\pi c/12$</td>
<td>$2^{nd}$ spectral invariant</td>
<td>$2^{nd}$ order entr.</td>
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Rem. Physical literature: proposals for $2\pi c/12 = A/4$.

Question: What can we say for SUSY? (Dirac operator case)
Quantum calculus with infinitely many degrees of freedom

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McKean-Singer formula

Γ be a selfadjoint unitary on a Hilbert space \( \mathcal{H} \), thus \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \) is graded.

\( Q \) selfadjoint odd operator: \( \Gamma Q \Gamma^{-1} = -Q \) or

\[
Q = \begin{bmatrix} 0 & Q_- \\ Q_+ & 0 \end{bmatrix}
\]

\( \text{Tr}_s = \text{Tr}(\Gamma \cdot) \) the supertrace.

If \( e^{-tQ^2} \) is trace class then \( \text{Tr}_s(e^{-tQ^2}) \) is an integer independent of \( t \):

\[
\text{Tr}_s(e^{-tQ^2}) = \text{ind}(Q_+) \quad \forall t > 0
\]

\( \text{ind}(Q_+) \equiv \text{Dim ker}(Q_+) - \text{Dim ker}(Q_+^*) \) is the Fredholm index of \( Q_+ \).
Fermi conformal nets

\( \mathcal{A} \) is a Fermi net if locality is replaced by twisted locality:
\[ \exists \text{ self-adjoint unitary } \Gamma, \Gamma \Omega = \Omega, \Gamma \mathcal{A}(I) \Gamma = \mathcal{A}(I); \text{ if } I_1 \cap I_2 = \emptyset \]

\[ [x, y] = 0, \quad x \in \mathcal{A}(I_1), \ y \in \mathcal{A}(I_2). \]

\([x, y]\) is the graded commutator w.r.t. \( \gamma = \text{Ad} \Gamma \). Then

\[ \mathcal{A}(I') \subset Z \mathcal{A}(I)' Z^* \]

indeed \( \mathcal{A}(I') = Z \mathcal{A}(I)' Z^* \) twisted duality (where \( Z \equiv \frac{1-i\Gamma}{1-i} \))

The Bose subnet \( \mathcal{A}_b \equiv \mathcal{A}^\gamma \) of is local.

Spin-statistics:

\[ U(2\pi) = \Gamma. \]

Therefore, in the Fermi case, \( U \) is representation of \( \text{Diff}^{(2)}(S^1) \).
Nets on a cover of $S^1$

A conformal net $\mathcal{A}$ on $S^1^{(n)}$ is a isotone map

$$I \in \mathcal{I}^{(n)} \mapsto \mathcal{A}(I) \subset B(\mathcal{H})$$

with a projective unitary, positive energy representation $U$ of $\text{Diff}^{(\infty)}(S^1)$ on $\mathcal{H}$ with

$$U(g)\mathcal{A}(I)U(g)^{-1} = \mathcal{A}(\dot{g}I), \quad I \in \mathcal{I}^{(n)}, \ g \in \text{Diff}^{(\infty)}(S^1)$$

conformal net $\mathcal{A}$ on $S^1$ \xrightarrow{\text{promotion}} conformal net $\mathcal{A}^{(n)}$ on $S^1^{(n)}$
Representations of a Fermi net

Let $\mathcal{A}$ be a Fermi net on $S^1$. A general representation $\lambda$ of $\mathcal{A}$ is a representation the cover net of $\mathcal{A}^{(\infty)}$ such that $\lambda_b \equiv \lambda|_{\mathcal{A}_b}$ is a DHR representation $\mathcal{A}_b$.

$\lambda$ is indeed a representation of $\mathcal{A}^{(2)}$. The following alternative holds:

(a) $\lambda$ is a DHR representation of $\mathcal{A}$. Equivalently $U_{\lambda_b}(2\pi)$ is not a scalar.

(b) $\lambda$ is the restriction of a representation of $\mathcal{A}^{(2)}$ and $\lambda$ is not a DHR representation of $\mathcal{A}$. Equivalently $U_{\lambda_b}(2\pi)$ is a scalar.

Case (a): Neveu-Schwarz representation
Case (b): Ramond representation
Super-Virasoro algebra

The super-Virasoro algebra governs the superconformal invariance:

local conformal $\leftrightarrow$ Virasoro
superconformal $\leftrightarrow$ super-Virasoro

Two super-Virasoro algebras: They are the super-Lie algebras generated by $L_n$, $n \in \mathbb{Z}$ (even), $G_r$ (odd), and $c$ (central):

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}
\]

\[
[L_m, G_r] = \left(\frac{m}{2} - r\right)G_{m+r}
\]

\[
[G_r, G_s] = 2L_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}
\]

Neveu-Schwarz case: $r \in \mathbb{Z} + 1/2$, Ramond case: $r \in \mathbb{Z}$.

Note: $G_0^2 = 2L_0 - c/12$ in Ramond sectors
FQS: admissible values for central charge $c$ and lowest weight $h$

Either $c \geq 3/2$, $h \geq 0$ ($h \geq c/24$ in the Ramond case) or

$$c = \frac{3}{2} \left(1 - \frac{8}{m(m+2)}\right), \quad m = 2, 3, \ldots$$

and

$$h = h_{p,q}(c) \equiv \frac{[(m+2)p - mq]^2 - 4}{8m(m+2)} + \frac{\varepsilon}{8}$$

where $p = 1, 2, \ldots, m-1$, $q = 1, 2, \ldots, m+1$ and $p - q$ is even or odd corresponding to the Neveu-Schwarz case ($\varepsilon = 0$) or Ramond case ($\varepsilon = 1/2$).

Neveu-Schwarz algebra has a vacuum representation, the Ramond algebra has no vacuum representation.
Super-Virasoro nets

c an admissible value, $h = 0$. Bose and Fermi stress-energy tensors:

$$T_B(z) = \sum_n z^{-n-2} L_n, \quad T_F(z) = \frac{1}{2} \sum_r z^{-r-3/2} G_r$$

in any NS/Ramond rep. same commutation relations ($w \equiv z_2/z_1$):

$$[T_F(z_1), T_F(z_2)] = \frac{1}{2} z_1^{-1} T_F(z_1) \delta(w) + z_1^{-3} w^{-3/2} \frac{c}{12} \left( w^2 \delta''(w) + \frac{3}{4} \delta(w) \right)$$

In the NS vacuum define the Super-Virasoro net of vN algebras:

$$SVir(I) \equiv \{ e^{iT_B(f_1)}, e^{iT_F(f_2)} : f_1, f_2 \in C^\infty(S^1) \text{ real}, \text{supp} f_1, \text{supp} f_2 \subset I \}''$$

Neveu-Schwarz rep. of SVir net $\leftrightarrow$ rep. of Neveu-Schwarz algebra
Ramond rep. of SVir net $\leftrightarrow$ rep. of Ramond algebra

- $SVir_b$ is modular (F. Xu)
- $SVir_b = (SU(2)_{N+2})' \cap (SU(2)_2 \otimes SU(2)_N)$ (GKO)
Supersymmetric representations

A general representation $\lambda$ of the Fermi conformal net $A$ is *supersymmetric* if $\lambda$ is graded

$$\lambda(\gamma(x)) = \Gamma_{\lambda} \lambda(x) \Gamma_{\lambda}^*$$

and the conformal Hamiltonian $H_{\lambda}$ satisfies

$$\tilde{H}_{\lambda} \equiv H_{\lambda} - c/24 = Q_{\lambda}^2$$

where $Q_{\lambda}$ is a selfadjoint odd w.r.t. $\Gamma_{\lambda}$.

Then

$$H_{\lambda} \geq c/24$$

McKean-Singer lemma:

$$\text{Tr}_s(e^{-t(H_{\lambda} - c/24)}) = \dim \ker(H_{\lambda} - c/24),$$

the multiplicity of the lowest eigenvalue $c/24$ of $H_{\lambda}$.

Super-Virasoro net:

$\lambda$ *supersymmetric* $\Rightarrow$ $\lambda$ *Ramond* (irr. iff $h = c/24$ i.e. minimal)
SUSY, Fredholm and Jones index

Assume $\mathcal{A}_b$ modular $\lambda|_{\mathcal{A}_b} = \rho \oplus \rho'$.

$$\text{Tr}_s(e^{-2\pi t\tilde{H}_\lambda}) = 2 \sum_{\nu \text{ Ramond}} S_{\rho,\nu} \text{Tr}(e^{-2\pi \tilde{L}_0,\nu/t}) .$$

If $\lambda$ is supersymmetric then

$$\text{ind}(Q_{\lambda+}) = 2 \sum_{\nu \text{ Ramond}} S_{\rho,\nu} \text{null}(\nu, c/24)$$

Therefore, writing Rehren definition of the $S$ matrix, we have

$$\text{ind}(Q_{\lambda+}) = \frac{d(\rho)}{\sqrt{\mu_A}} \sum_{\nu \text{ Ramond}} K(\rho, \nu)d(\nu)\text{null}(\nu, c/24)$$

The Fredholm index of the supercharge operator $Q_{\lambda+}$ and the Jones index both appear
Some consequences

- An identity for the $S$ matrix:

$$\sum_{\nu \text{ Ramond}} S_{\rho,\nu} d(\nu) = 0$$

- If $\text{ind}(Q_{\lambda^+}) \neq 0$ there exists a Ramond sector $\nu$ such that $c/24$ is an eigenvalue of $L_{0,\nu}$.

- Suppose that $\rho$ is the only Ramond sector with lowest eigenvalue $c/24$ modulo integers. Then

$$S_{\rho,\rho} = \frac{d(\rho)^2}{\sqrt{\mu A_b}} K(\rho, \rho) = \frac{1}{2}.$$
Complete list of superconformal nets, i.e. Fermi extensions of the super-Virasoro net, with \( c = \frac{3}{2} \left(1 - \frac{8}{m(m+2)}\right) \)

1. The super Virasoro net: \((A_{m-1}, A_{m+1})\).
2. Index 2 extensions of the above: \((A_{4m'}-1, D_{2m'+2})\), \(m = 4m'\) and \((D_{2m'+2}, A_{4m'+3})\), \(m = 4m' + 2\).
3. Six exceptionals: \((A_9, E_6)\), \((E_6, A_{13})\), \((A_{27}, E_8)\), \((E_8, A_{31})\), \((D_6, E_6)\), \((E_6, D_8)\).

Remark. Follows the classification of local conformal nets with \( c < 1 \) with the construction of new models (Kawahigashi, L., also F. Xu and K.H. Rehren)
Def. A ($\theta$-summable) graded spectral triple $(\mathcal{A}, \mathcal{H}, Q)$ consists of a graded Hilbert space $\mathcal{H}$, with selfadjoint grading unitary $\Gamma$, a unital $*$-algebra $\mathcal{A} \subset B(\mathcal{H})$ graded by $\gamma \equiv \text{Ad}(\Gamma)$, and an odd selfadjoint operator $Q$ on $\mathcal{H}$ as follows:

- $\mathcal{A}$ is contained in $D(\delta)$, the domain of the superderivation $\delta = [Q, \cdot]$;
- For every $\beta > 0$, $\text{Tr}(e^{-\beta Q^2}) < \infty$ ($\theta$-summability).

The operator $Q$ is called the supercharge operator, its square the Hamiltonian. $Q$ is also called Dirac operator and denoted by $D$.

A spectral triple is a fundamental object to define NCG.
Assume we have a quantum algebra (essentially a spectral triple). Then the JLO cocycle (Chern character) on the Bose algebra

$$\tau_n(a_0, a_1, \ldots, a_n) \equiv (-1)^{-\frac{n}{2}} \int_{0 \leq t_1 \leq \cdots \leq t_n \leq 1} \text{Tr}_\rho (e^{-H} a_0 \alpha_{it_1}(\delta a_1) \alpha_{it_2}(\delta a_2) \cdots \alpha_{it_n}(\delta a_n)) dt_1 dt_2 \ldots$$

($n$ even) is an entire cyclic cocycle, so it gives an element in Connes entire cyclic cohomology that pairs with K-theory.
Spectral triples in CFT

A supersymmetric representation $\rho$ of a Fermi net $\mathcal{A}$ gives rise to a $\theta$-summable spectral triple if the superderivation $\delta$

$$\delta(a) \equiv [a, Q_\rho]$$

has a *dense domain* in the representation $\rho$ ($\theta$-summability is essentially automatic).

Then the JLO cocycle (Chern character) on the Bose algebra

$$\tau^\rho_n(a_0, a_1, \ldots, a_n) \equiv (-1)^{-\frac{n}{2}} \int_{0 \leq t_1 \leq \cdots \leq t_n \leq 1} \text{Tr}_s \left( e^{-H_\rho a_0 \alpha_{it_1}(\delta a_1) \alpha_{it_2}(\delta a_2) \ldots \alpha_{it_n}(\delta a_n)} \right) dt_1 dt_2 \ldots$$

($n$ even) is entire cyclic cocycle.
Noncommutative geometrization

We want to associate to each supersymmetric sector the above Chern character

\[ \rho \rightarrow \tau^\rho \]

**Thm.** (Carpi, Hillier, Kawahigashi, R.L.)

The supersymmetric Ramond sectors of $SVir$ give rise to $\theta$-summable spectral triple ($\delta$ has a dense domain)

For the super-Virasoro net the index map

\[ \rho \rightarrow \sum \tau^\rho_n(1,1,\ldots,1) = \text{Tr}_s(e^{-tH_\rho}) \]

for Ramond sectors is given by

\[ \text{Index}(\rho_{h=c/24}) = 1, \quad \text{Index}(\rho_{h\neq c/24}) = 0 \]
Further model analysis (in progress)

- Free supersymmetric CFT on the circle: all Buchholz-Mack-Todorov sectors give the same JLO cocycle (deformation argument). Even JLO cocycle is trivial, odd JLO cocycle probably non-trivial.

- Extension of $U(1)^2 \otimes \text{(Fermions)}$ gives a non-trivial JLO cocycle (there is a unitary that does not distinguish sectors).

- Super-Virasoro net: Ramond lowest energy sector is non-trivial (see above), other Ramond sectors give trivial JLO cocycles (NS case is not interesting).

- Richer structure is expected in the $N = 2$ superconformal case.