The Structure and Classification of Conformal Nets

Möbius covariants nets on S^1. A (local) *Möbius covariant net* $\mathcal{A}$ on S^1 is a map

$$I \in \mathcal{I} \to \mathcal{A}(I) \subset B(\mathcal{H})$$

$\mathcal{I} \equiv$ family of proper intervals of S^1, that satisfies:

A. *Isotony*. $I_1 \subset I_2 \implies \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$

B. *Locality*. $I_1 \cap I_2 = \varnothing \implies [\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$

C. *Möbius covariance*. $\exists$ unitary rep. U of the Möbius group Möb on $\mathcal{H}$ such that

$$U(g)\mathcal{A}(I)U(g)^* \; = \; \mathcal{A}(gI), \quad g \in \text{Möb}, \; I \in \mathcal{I}.$$

D. *Positivity of the energy*. Generator L_0 of rotation subgroup of U (conformal Hamiltonian) is positive.

E. *Existence of the vacuum.* $\exists!$ U-invariant vector $\Omega \in \mathcal{H}$ (vacuum vector), and Ω is cyclic for $\bigvee_{I\in\mathcal{I}} \mathcal{A}(I)$ and unique U-invariant.

First consequences

• *Irreducibility*: $\bigvee_{I\in\mathcal{I}} \mathcal{A}(I) = B(\mathcal{H})$.

• *Reeh-Schlieder theorem*: Ω is cyclic and separating for each $\mathcal{A}(I)$.

Proof. $H \geq 0$ Hamiltonian. $\xi \perp \mathcal{A}(I)\Omega \implies (\xi, e^{itH}X\Omega) = 0$, X localized in $I_0 \subset\subset I$, $|t| < \varepsilon$.

$(\xi, e^{izH}X\Omega)$ analytic in $\Im z > 0 \implies (\xi, e^{itH}X\Omega) = 0$ $\forall t$...

• *Bisognano-Wichmann property*: Tomita-Takesaki modular operator Δ_I and conjugation J_I of $(\mathcal{A}(I), \Omega)$, are

$$\boxed{\begin{array}{lr} U(\Lambda_I(2\pi t)) = \Delta_I^{it},\ t \in \mathbb{R}, & \text{dilations} \\ U(r_I) = J_I & \text{reflection} \end{array}}$$

(Guido-L., Frölich-Gabbiani)

$\Lambda_I(t) : x \mapsto e^{-t}x$ on $\mathbb{R}^+ \sim S^+$ (stereograph. map)

Proof based on the following:

Borchers thm. M vN algebra, Ω cyclic separating vector, $T(t)$ one-parameter unitary group, $T(t)\Omega = \Omega$

$$T(t)\mathcal{M}T(-t) \subset \mathcal{M}, \quad t \geq 0$$
$$T(t) = \exp(iHt), \quad H \geq 0$$

then

$$\boxed{\Delta^{is}T(t)\Delta^{-is} = T(e^{-2\pi s}t), \quad JT(t)J = T(-t)}$$

- *Haag duality*:

$$\boxed{\mathcal{A}(I)' = \mathcal{A}(I')}$$

Proof. $\mathcal{A}(I)' = J_I\mathcal{A}(I)J_I = \mathcal{A}(I')$.

- *Factoriality*: $\mathcal{A}(I)$ is III_1-factor (or $\mathcal{A}(I) = \mathbb{C}$).

Proof. Modular group is ergodic.

• *Additivity*: $I \subset \cup_i I_i \implies \mathcal{A}(I) \subset \vee_i \mathcal{A}(I_i)$ (Fredenhagen, Jorss).

Wiesbrock characterization (variant)

Thm. (Guido, Wiesbrock, L.)

$\boxed{\mathcal{A} \text{ local Möb covariant net} \Leftrightarrow (\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, \Omega)}$

$\mathcal{M}_k$ commuting vN algebras, Ω cycl.sep. vector, $\Delta_k^{is}\mathcal{M}_{k+1}\Delta_k^{-is} \subset \mathcal{M}_{k+1}$, $s \geq 0$, $k \in \mathbb{Z}_3$.

Split property and Buchholz nuclearity. $\mathcal{A}$ satisfies the *split* property if the von Neumann algebra

$$\mathcal{A}(I_1) \vee \mathcal{A}(I_2) \simeq \mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$$

(natural isomorphism) if $\bar{I}_1 \cap \bar{I}_2 = \varnothing$.

$$\boxed{\mathrm{Tr}(e^{-tL_0}) < \infty,\ \forall t > 0 \implies \text{split}}$$

(*nuclearity*)

Representations. A *representation* π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a map $I \in \mathcal{I} \mapsto \pi_I$, with π_I a normal representation of $\mathcal{A}(I)$ on $B(\mathcal{H})$ such that

$$\pi_{\tilde{I}}\restriction\mathcal{A}(I) = \pi_I, \quad I \subset \tilde{I}, \quad I, \tilde{I} \subset \mathcal{I} \ .$$

π is Möbius *covariant* if there is a projective unitary representation U_π of Möb on $\mathcal{H}$ such that

$$\pi_{gI}(U(g)xU(g)^*) = U_\pi(g)\pi_I(x)U_\pi(g)^*$$

for all $I \in \mathcal{I}$, $x \in \mathcal{A}(I)$ and $g \in$ Möb.

Version of DHR argument: given I and π rep. of $\mathcal{A}$, $\exists$ an endomorphism $\rho \simeq \pi$ of $\mathcal{A}$ localized in I; i.e. $\rho_{I'} = \mathrm{id}\restriction_{\mathcal{A}(I')}$.

Proof. $\mathcal{A}(I)$ is a type III factor, thus only one normal rep.

– Fix I: choose $\rho \simeq \pi$, $\pi_{I'} = \mathrm{id}$.

– By Haag duality $\rho_I(\mathcal{A}(I)) \subset \mathcal{A}(I)$.

Fredenhagen universal algebra.

$$\begin{array}{ccc} \mathcal{A}(I) & \xrightarrow{\iota_I} & C^*(\mathcal{A}) \\ \pi_I \downarrow & & \downarrow \pi \\ B(\mathcal{H}) & = & B(\mathcal{H}) \end{array}$$

Reps of $\mathcal{A} \leftrightarrow$ Endom. of $C^*(\mathcal{A})$

$$\Downarrow$$

Fusion of representations

$$\downarrow$$

$\boxed{\text{End}(C^*(\mathcal{A})) \text{ is braided tensor category}}$

$$||$$

canonical intertwiners $\varepsilon(\rho, \sigma) : \rho\sigma \to \sigma\rho$

(Fredenhagen, Rehren, Schroer)

Example. Let $\mathcal{A}$ be the local conformal net on S^1 associated with the $U(1)$-current algebra. In the real line picture $\mathcal{A}$ is given by

$$\mathcal{A}(I) \equiv \{W(f) : f \in C^\infty_{\mathbb{R}}(\mathbb{R}),\ \text{supp} f \subset I\}''$$

where W is the representation of the Weyl commutation relations

$$W(f)W(g) = e^{-i\int fg'}W(f+g)$$

associated with the vacuum state ω

$$\omega(W(f)) \equiv e^{-||f||^2}, \quad ||f||^2 \equiv \int_0^\infty |\tilde{f}(p)|^2 p \mathrm{d}p$$

where $\tilde{f}$ is the Fourier transform of f.

Buchholz-Mack-Todorov sectors There is a one parameter family $\{\alpha_q, q \in \mathbb{R}\}$ of irreducible sectors and all have index 1.

$$\alpha_q(W(f)) \equiv e^{2i\int Ff}W(f), \quad F \in C^\infty, \quad \int F = q\ .$$

Index-statistics thm.

$$\boxed{\text{DHR dim. } d(\rho) = \sqrt{\text{Jones index Ind}(\rho)}}$$

$$\begin{array}{c} \text{tensor category} \\ \text{End. local. in } I \end{array} \xrightarrow[\text{restriction}]{\text{full functor}} \begin{array}{c} \text{tensor category} \\ \text{End. of } \mathcal{A}(I) \end{array}$$

$$\mathrm{Hom}(\rho, \sigma) = \mathrm{Hom}(\rho_I, \sigma_I)$$

Local intertwiners = global intertwiners (Guido,L.)

Conformal spin-statistics thm. (Guido, L.)
π rep. of $\mathcal{A}$, λ_ρ DHR statistics parameter

$$\boxed{\kappa_\rho \equiv \mathrm{ph}(\lambda_\rho) = e^{2\pi i h_\rho}}$$

$h_\rho = =$ *spin*, i.e. lowest eigenvalue of L_ρ.

Proof. (some argument) $I_1 =$ upper half-circle, $I_2 =$ right half-circle ρ automorphism localized in $I_1 \cap I_2$.

$\rho|_{\mathcal{A}(I_i)} \to$ Araki-Connes-Haagerup unitary standard implementation V_i

V_1 and V_2 commute up to a phase

$$V_1 V_2 = \mu V_2 V_1.$$

μ algebraic invariant & geometric invariant: compare the two aspects. . .

Diff(S^1) and the Virasoro algebra. Diff(S^1) = smooth oriented diffeomorphisms of S^1. The (complexification of) Lie algebra of Diff(S^1) is Vect(S^1) (Witt algebra)

$$[L_m, L_n] = (m-n)L_{m+n}, \quad L_n = ie^{int}\frac{\mathrm{d}}{\mathrm{d}t}$$

The *Virasoro algebra* is the unique, non-trivial one-dim. central extension of De Witt alg.

$$\boxed{[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}}$$

and $[L_n, c] = 0$. c is called *central charge*.

Unitary irreducible representation of Virasoro alg. on Hilbert space $\mathcal{H}$ $\updownarrow$ Irr. family of operators L_n on $\mathcal{H}$ and $c \in \mathbb{R}$ with Virasoro relations and $L_n^* = L_{-n}$.

L_1, L_{-1}, L_0 = generators of $s\ell(2, \mathbb{R})$ (Lie algebra of Möbius group):

$$[L_1, L_0] = L_1,\ [L_{-1}, L_0] = -L_{-1},\ [L_1, L_{-1}] = 2L_0.$$

$L_0 \stackrel{\text{def}}{=}$ *conformal Hamiltonian* (= generator of rotations).

Positive energy unitary rep. U of Diff(S^1): $L_0 \geq 0$. Thus sp$U \subset \{h, h+1, h+2, \cdots\}$, $h \geq 0$. h is called *lowest weight*.

For every possible value of c and h $\exists!$ irr. pos. energy rep. $V_{c,h}$ of Diff(S^1). Possible values (Friedan, Qui, Shenker '86):

$$\boxed{c = 1 - \frac{6}{n(n+1)} \quad \text{or} \quad \mathrm{c} \geq 1}$$

$$h_{p,q} = \frac{((n+1)p - nq)^2 - 1}{4n(n+1)},$$

$1 \leq p \leq n-1$, $1 \leq q \leq n$, $p, q \in \mathbb{N}$, $(p.q) \sim (n-p, n+1-q)$. All values are taken (Goddard, Kent, Olive '86).

Reps. with the same c have *fusion* (internal tensor product).

Popa-Ocneanu clssification of subfactors (discrete series). $\mathcal{M}$ a finite amenable (inductive limit of finite-dim. *-algebras) factor. Subfactors $\mathcal{N} \subset \mathcal{M}$ with index < 4 are in Jones discrete series, i.e. $[\mathcal{M} : \mathcal{N}] = 4\cos^2\frac{\pi}{n}$, $n \geq 3$. Let $\iota : \mathcal{N} \to \mathcal{M}$ embedding

$$\mathrm{Hom}(\iota, \iota) \hookrightarrow \mathrm{Hom}(\iota\bar{\iota}, \iota\bar{\iota}) \hookrightarrow \mathrm{Hom}(\iota\bar{\iota}\iota, \iota\bar{\iota}\iota) \hookrightarrow \cdots$$

is a tower of multi-matrix algebras described by a Bratteli embedding graph. Moreover $\iota\bar{\iota}$ (canonical endomorphism) *shifts by 2* the tower. The remaining principal graph gives a complete

$$\boxed{A - D_{\text{even}} - E_{6,8} \text{ classification}}$$

$A - D$ case unique, E case two subfactors.

Long standing problem: is there a relation between Jones index discrete series and Virasoro central charge discrete series? We shall provide a connection below.

Conformal nets. A local conformal net $\mathcal{A}$ is a local Möbius covariant net s.t. $\exists$ proj. unitary rep. U of $\mathrm{Diff}(S^1)$, extendending the Möbius rep., s.t.

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI), \quad g \in \mathrm{Diff}(S^1),$$
$$U(g)xU(g)^* = x, \quad x \in \mathcal{A}(I),\ g \in \mathrm{Diff}(I'),$$

$\mathrm{Diff}(I) \stackrel{\text{def}}{=} \{g \in \mathrm{Diff}(S^1) : g(t) = t\ \forall t \in I'\}$.

U is unique (Weiner), hence canonical.

Virasoro nets Vir_c.

$$\mathrm{Vir}_c(I) \equiv V_c(\mathrm{Diff}(I))''$$

$V_c \equiv V_{c,h=0}$ (*vacuum representation*).

$\mathcal{A}$ (local) conformal net, Haag duality implies

$$U(\mathrm{Diff}(I)) \subset \mathcal{A}(I),$$

U is direct sum of reps $V_{c,h}$ with the same central charge c: the central charge of $\mathcal{A}$

$$\boxed{\begin{array}{c}\mathcal{A} \supset \mathrm{Vir}_c \\ \text{every local conformal net} \\ \text{is an extension of a Virasoro net}\end{array}}$$

On the other hand Vir_c is minimal, no nontrivial subnet (Carpi):

universal role of Vir_c

A (irred.) *representation* π of $\mathcal{A}$ on $\mathcal{H}$ is diffeomorphism *covariant* if $\exists$ projective unitary rep. U_π of $\mathrm{Diff}(S^1)$ extending the rep. U_π of Möb s.t.

$$\pi_{gI}(U(g)xU(g)^*) = U_\pi(g)\pi_I(x)U_\pi(g)^*$$

Automatic diff. covariance: D'Antoni, Fredenhagen, Koester, Weiner.

Complete rationality. Problem: characterize intrinsically a "rational" net (= finitely many irr. sectors, all with $d(\rho) < \infty$)

Def. $\mathcal{A}$ is *completely rational* if

- $\mathcal{A}$ is split, i.e. $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) \simeq \mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$

- The μ-*index* $\mu_{\mathcal{A}}$ is finite, i.e.

$$\mu_{\mathcal{A}} \equiv [\hat{\mathcal{A}}(E) : \mathcal{A}(E)] < \infty$$

$E = I_1 \cup I_2$, $I_1 \cap I_2 = \varnothing$, $\hat{\mathcal{A}}(E) = \mathcal{A}(E')'$ (failure of Haag duality for disconneted regions).

$\mu_{\mathcal{A}} < \infty$ for $SU(N)$ loop group models (F. Xu).

General theory (Kawahigashi, Müger, L.)

$\mathcal{A}$ completely rational $\Rightarrow$

- $$\boxed{\mu_{\mathcal{A}} = \sum_i d(\rho_i)^2}$$

sum of the indeces of all irreducible sectors

• $\mathcal{A}$ is *rational* and the representation tensor category is *modular* has non-degenerate braiding

• $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ is the quantum double inclusion of Rehren, L. (see below)

• All irreducible extensions of $\mathcal{A}$ have finite Jones index (by Izumi, Popa, L.)

• $\mathcal{A}$ is strongly additive (Xu, L.)

$$\mathcal{A}(I \setminus \{\text{point}\}) = \mathcal{A}(I)$$

Loop group and coset models. G compact Lie group,

LG loop group, i.e. $LG = \{g : t \in S^1 \to G\}$ (smooth maps with pointwise multiplication),

$U : LG \to B(\mathcal{H})$ pos. energy unitary rep. of LG, i.e. the action of Diff(S^1) on Aut(LG) is implementes by a pos. energy rep.

Vacuum irr. reps. (pos. energy) U of LG (0 eigenvalue of L_0) are labaled by a parameter, the *level* of U. Fix a level ℓ rep. U:

$$\mathcal{A}(I) \equiv \{U(g), g \in LG : \ g(t) = t, \ t \in I'\}''$$

is a conformal net.

$H \subset G$ closed subgroup

$$\mathcal{B}(I) \equiv \{U(g), g \in LH : \ g(t) = 1, \ t \in I'\}''$$

conformal subnet.

$C(I) = \mathcal{B}(I)' \cap \mathcal{A}(I)$ coset model of $H \subset G$.

$\boxed{\mathrm{Vir}_c = \text{coset } SU(2)_{m-1} \subset SU(2)_{m-1} \times SU(2)_1}$

$c = 1 - \frac{6}{m(m+1)}$ (GKO, Xu, Carpi, Kawahigashi, L.).

$\Rightarrow$ Vir_c is completely rational $c < 1$

$\Rightarrow$ All extensions of Vir_c have finite Jones index

$\Rightarrow$ Sectors of Vir_c have finite index (Loke)

The classification problem for the discrete series.

$$\boxed{\begin{array}{c}\text{Classify conformal nets with } c<1\\ \Updownarrow \\ \text{Classify all irreducible extensions of } \mathrm{Vir}_c\end{array}}$$

Verlinde-Rehren matrices. $\mathcal{A}$ rational, i.e. finitely many irr. sectors $\rho_o = \mathrm{id}, \rho_1, \dots \rho_n$

$$Y_{ij} \equiv d_i d_j \Phi_j(\varepsilon(\rho_j,\rho_i)^* \varepsilon(\rho_i,\rho_j)^*)$$

ε non degenerate $\Leftrightarrow$ $|\sigma|^2 = \sum d_i^2$ with $\sigma \equiv \sum \kappa_i^{-1} d_i^2$

$$S \equiv |\sigma|^{-1} Y, \qquad T \equiv \left(\frac{\sigma}{|\sigma|}\right)^{1/3} \mathrm{Diag}(\kappa_i)$$

$$\begin{aligned} SS^\dagger &= TT^\dagger = \mathrm{id},\\ STS &= T^{-1}ST^{-1},\\ S^2 &= C,\\ TC &= CT, \end{aligned}$$

where $C_{ij} = \delta_{i\bar{j}}$. In our case ($\mathrm{Vir}_c$) $C = \mathrm{id}$.

$\Rightarrow$ $\underline{T \text{ and } S \text{ generate unitary rep. of } SL(2,\mathbb{Z})}$.

Modular invariants. Given a unitary, finite-dim. rep. of $SL(2,\mathbb{Z})$, a *modular invariant* is a matrix $Z \in \mathrm{Mat}(\mathbb{Z}_+)$, $Z_{00} = 1$, s.t.

$$ZU = UZ$$

• Rational net with non-degenerate braiding $\rightarrow$ unitary rep. of $SL(2,\mathbb{Z}) \rightarrow$ modular invariants

• Thus (KLM): complete rational nets $\rightarrow$ modular invariants

• Capelli, Itzykson, Zuber '87: ADE classification of modular invariants for Vir_c, $c < 1$

• Böckenhaur, Evans, Kawahigashi 2000: $\mathcal{A} \subset \mathcal{B}$ conformal nets, $[\mathcal{B} : \mathcal{A}] < \infty$, then

$$\alpha - \text{induction} \longrightarrow \text{modular invariants}$$

$$Z_{\mu\nu} = \mathrm{dimHom}(\alpha_\mu^+, \alpha_\nu^-)$$

$\alpha_\mu^\pm =$ extension of DHR sector μ of $\mathcal{A}$ to right/left solitonic sector of $\mathcal{B}$ (Roberts, Rehren-L., Xu)

Q-systems. Recall: $\mathcal{M}$ factor, $\rho \in \mathrm{End}(\mathcal{M})$ then

$$\gamma_\rho = \rho\bar{\rho}$$

Converse problem: given $\gamma \in \mathrm{End}(M)$, when is γ canonical?

The problem is finding a "square root" ρ.

The conjugate equations give conditions:

γ canonical with finite index

$$\Downarrow$$

$\exists$ isometry $T \in \mathrm{Hom}(\iota, \gamma)$, and a co-isometry $S \in \mathrm{Hom}(\gamma^2, \gamma)$

$$\boxed{\begin{array}{c} SS = S\gamma(S) \\ S\gamma(T) \in \mathbb{C}\backslash\{0\}\ , \quad ST \in \mathbb{C}\backslash\{0\} \end{array}}$$

Def. A *Q-system* is a triple (γ, T, S) where $\gamma \in \mathrm{End}(M)$, $T \in \mathrm{Hom}(\iota, \gamma)$ is an isometry,

$S \in \mathrm{Hom}(\gamma^2, \gamma)$ is a co-isometry satisfying the above relations.

Thm. Q-system $(\gamma, T, S) \to$ finite-index subfactor $\mathcal{N} \subset \mathcal{M}$ with $\gamma : \mathcal{M} \to \mathcal{N}$ canonical endomorphism.

$\exists$ bijection

$$\boxed{\text{subfactors} \leftrightarrow \text{Q-systems}}$$

Proof. $\varepsilon \equiv S \cdot S^*$ is a positive map $\mathcal{M} \to \mathcal{M}$.

$\varepsilon^2 = \varepsilon$ (use $SS = S\gamma(S)$)

$\mathcal{N} \equiv \varepsilon(\mathcal{M})$ is a von Neumann subalgebra (again the relation) and ε is an expectation

Any $\rho \in \mathrm{End}(M)$, $\rho(\mathcal{M}) = \mathcal{N}$ is a "square root" with $\bar{\rho} = \rho^{-1}\gamma$ (last relations give the conjugate equations)

Application 1: Quantum double (Rehren, L.), see below.

Application 2: Duality for finite-dimensional complex semisimple Hopf algebras (L.).

An (irreducible) abstract *Q-system* is $(\mathcal{T}, \lambda, S)$ where λ an object of $\mathcal{T}$:

a): (ι, λ) is one–dimensional; namely there exists a unique element $T \in \text{Hom}(\iota, \lambda)$, up to a phase; T is proportional to an isometry.

b): there exists an arrow $S \in \text{Hom}(\lambda \otimes \lambda, \lambda)$ proportional to an coisometry ($SS^* = 1$) such that

$$\boxed{\begin{array}{c} b_1)\ S \circ 1_\lambda \otimes S = S \circ S \otimes 1_\lambda \\ b_2) \begin{cases} S \circ 1_\lambda \otimes T = 1_\lambda \\ S \circ T \otimes 1_\lambda = 1_\lambda \end{cases} \end{array}}$$

Thm. A finite-dimensional Hopf algebra is a Q-system s.t.

$$\lambda \otimes \lambda \simeq d\lambda$$

distiguished propery of regular representation.

Compare with Doplicher-Roberts duality for compact groups.

Two Q-systems (ρ, T_1, S_1) and (ρ, T_2, S_2) are *equivalent* if $\exists u \in \mathrm{Hom}(\rho, \rho)$ satisfying

$$T_2 = uT_1, \qquad uS_1 = S_2 u\rho(u).$$

Equivalence of Q-systems $\Leftrightarrow$ inner conjugacy of subfactors.

$$N \subset M \qquad \underset{\text{can. endomorphism}}{\overset{\text{Jones construction}}{\longleftrightarrow}} \qquad \tilde{M} \supset M$$

<u>Problem</u>: classify Q-systems up to equivalence when a system of endomorphisms is given and ρ is a direct sum of endomorphisms in the system.

Izumi-Kosaki cohomology for Q-systems: <u>finite</u> groups.

Classification of local extensions of the Virasoro nets (Kawahigashi, L.)

• Consider the Cappelli-Itzykson-Zuber classification of the modular invariants for the Virasoro nets with central charge $c = 1 - 6/m(m+1) < 1$, $m = 2, 3, 4, \ldots$.

• Show that each "type I" modular invariant is realized with α-induction for an extension $\mathrm{Vir}_c \subset \mathcal{M}$ as in Bockenhauer-Evans-Kawahigashi

• Use Q-system to detect the local extension of Vir_c, $c < 1$

$$\Downarrow$$

Classification of local conformal nets, $c = 1 - \frac{6}{m(m+1)}$

m	Labels for Z
n	(A_{n-1}, A_n)
$4n+1$	(A_{4n}, D_{2n+2})
$4n+2$	(D_{2n+2}, A_{4n+2})
11	(A_{10}, E_6)
12	(E_6, A_{12})
29	(A_{28}, E_8)
30	(E_8, A_{30})

Thm. (Kawahigashi,L.) Local conformal nets with $c < 1$ are classified by pair of Dynkin diagrams $A - D_{2n} - E_{6,8}$ s.t. the difference of Coxeter numbers is 1.

Simple current extensions. The simple current extensions of index 2

The four exceptional cases.

$(E_6, A_{12}), (E_8, A_{30})$ coset constructions (conjectuered by Böckenhauer-Evans

(A_{10}, E_6) coset construction (Köster)

One *new example* (A_{28}, E_8), most probably not constructable as coset.

Case $c = 1$ classified by Xu, Carpi (with a spectral condition, probably always true)

Subnet structure. Alternative labels for the classification.

Let $\mathcal{A}$ be an irreducible local conformal net with central charge $c < 1$. Let s be the number of finite-index conformal subnets, up to conjugacy (including $\mathcal{A}$ itself). Then $s \in \{1, 2, 3\}$. $\mathcal{A}$ is completely classified by the pair (m, s) where $c = 1 - 6/m(m+1)$. For any $m \in \mathbb{N}$ the possible values of s are:

- $s = 1$ for all $m \in \mathbb{N}$;

- $s = 2$ if $m = 1, 2$ mod 4, and if $m = 11, 12$;

- $s = 3$ if $m = 29, 30$.

Classification of 2-dimensional CFT.

Quantum double inclusion (Rehren, L., related to Popa and Ocneanu)

$\mathcal{T}$ a rational tensor subcategory of End$(\mathcal{M})$, objects $\{\rho_i\}$

$$\lambda \equiv \bigoplus_i \rho_i \otimes \rho_i^{\mathrm{opp}}$$

$\lambda \in$ End$(\mathcal{M} \otimes \mathcal{M}^{\mathrm{opp}})$

V_ℓ basis in Hom$(\rho_k, \rho_i\rho_j)$

$\rightarrow$ <u>canonical</u> element

$$\bigoplus_i V_i \otimes V_i^{\mathrm{opp}} \in \mathrm{Hom}(\rho_k \otimes \rho_k^{\mathrm{opp}}, \rho_i\rho_j \otimes \rho_i^{\mathrm{opp}}\rho_j^{\mathrm{opp}})$$

$\rightarrow$ canonical isometry $S \in$ Hom(λ, λ^2)

$\rightarrow$ canonical Q-system (λ, T, S^*)

$\rightarrow$ canonical inclusion

$$\mathcal{M} \otimes \mathcal{M}^{\mathsf{opp}} \subset \tilde{\mathcal{M}}$$

$[\tilde{\mathcal{M}} : \mathcal{M} \otimes \mathcal{M}^{\mathsf{opp}}] = \sum_i d(\rho_i)^2$

$\gamma : \tilde{\mathcal{M}} \rightarrow \mathcal{M} \otimes \mathcal{M}^{\mathsf{opp}}$ can. endomrphism

Tensor category generated by γ is braided, the quantum double of $\mathcal{T}$ in the sense of Drinfeld.

Sector structure studied by Izumi.

Canonical tensor product inclusions (Rehren)

The above generalizes to inclusion of the form $\mathcal{M} \otimes \mathcal{M}^{\mathsf{opp}} \subset \tilde{\mathcal{M}}$ with

$$\lambda = Z_{ij} \bigoplus_i \rho_i \otimes \rho_j^{\mathsf{opp}}$$

where $p \in \mathbb{S}_n$ and Z_{ij} is nonnegative integer matrix related to modular invariants.

Two-dimensional conformal nets. $\mathcal{A}$ net vN algebras on Minkowski space.

$\mathcal{A}$ is *conformal* if $r : \mathbf{x} \mapsto \mathbf{x}/||\mathbf{x}||^2$ is a symmetry ($\mathbf{x} = x^2 - t^2$)

Free field: $\Box\Psi = 0$ is preserved by r.
dim $= 2 \implies \Psi(\mathbf{x}) = \Psi_+(x+t) + \Psi_-(x-t)$

In general:

$$\boxed{\begin{array}{c}\mathcal{A} \text{ conformal net on } \mathbb{R}^2 \\ \text{restriction } \downarrow \text{ to } x \pm t = 0 \\ \text{two conf. net } \mathcal{A}_\pm \text{ on } \mathbb{R}\end{array}}$$

Thm (Rehren)

$$\mathcal{A}_+(I) \otimes \mathcal{A}_-(J) \subset \mathcal{A}_+^{\max}(I) \otimes \mathcal{A}_-^{\max}(J) \subset \mathcal{A}$$

$\mathcal{A}_+(I) \otimes \mathcal{A}_-(J) \subset \mathcal{A}$ c. t. p. subfactor

$\mathcal{A}_+^{\mathsf{max}}(I)\otimes\mathcal{A}_-^{\mathsf{max}}(J)\subset\mathcal{A}$ quantum double can. endom.

$$\lambda=\bigoplus_i \rho_i\otimes\rho_{p(i)}^{\mathsf{opp}}$$

The classification problem for two-dim CFT.

$c<1$ We have

$$\mathcal{A}_+(I)\otimes\mathcal{A}_-(J)\supset \mathsf{Vir}_c\otimes\mathsf{Vir}_c$$

and $\mathsf{Vir}_c\subset\mathcal{A}_\pm^{\mathsf{max}}$ is classified.

$c<1$, maximal nets.

$$\boxed{\begin{array}{c}\text{Classify irr. extensions of } \mathcal{A}_+^{\mathsf{max}}\otimes\mathcal{A}_-^{\mathsf{max}}\\ \text{with canonical endomorphism}\\ \bigoplus_i \rho_i\otimes\rho_{p(i)}^{\mathsf{opp}}\end{array}}$$

Classification problem

$$\updownarrow$$

<u>cohomological</u> problem for Q-systems

Vanishing of Izumi-Kosaki 2-cohomology for the tensor categories that appear.

Canoninal endom. $\lambda : \mathcal{A} \to \mathsf{Vir}_c \otimes \mathsf{Vir}_c$ is

$$\lambda = \bigoplus Z_{ij} \alpha_i \otimes \alpha_j^{\mathsf{opp}}$$

and Z is a modular invariant (Müger).

Modular invariants for the Virasoro tensor category Vir_c:

m	Labels for modular invariants	Type
n	(A_{n-1}, A_n)	I
$4n$	(D_{2n+1}, A_{4n})	II
$4n+1$	(A_{4n}, D_{2n+2})	I
$4n+2$	(D_{2n+2}, A_{4n+2})	I
$4n+3$	(A_{4n+2}, D_{2n+3})	II
11	(A_{10}, E_6)	I
12	(E_6, A_{12})	I
17	(A_{16}, E_7)	II
18	(E_7, A_{18})	II
29	(A_{28}, E_8)	I
30	(E_8, A_{30})	I

$\mathcal{A} \to Z$ is <u>bijection</u> two-dimensional max local conformal nets $\leftrightarrow$ modular invariants Z:

Thm. (Kawahigashi,L.) Two-dimensional maximal local conformal nets with $c < 1$ are classified by pair of Dynkin diagrams $A - D - E$ s.t. the difference of Coxeter numbers is 1.

Note: One-dimensional case D_{odd} and E_7 do not appaear.

Non-maximal are also classified.

Classification of non-local extensions. Kawahigashi, Rehren, L.

Classify all, relatively non-local, irreducible extensions $\mathcal{B}$ of $\mathcal{A} = \mathrm{Vir}_c$. $c < 1$.

For $c = 1 - 6/m(m+1)$, $m = 3, 4, 5, \ldots$

DHR sectors: $\sigma_{j,k}$, $j = 0, 1, \ldots, m-2$, $k = 0, 1, \ldots, m-1$, $\sigma_{j,k} = \sigma_{m-2-j, m-1-k}$ ($m(m-1)/2$ irreducible DHR sectors).

Consider the following sequence of commuting squares

$$\begin{array}{ccccccc} \mathrm{End}({}_{\mathcal{A}}\mathrm{id}_{\mathcal{A}}) & \subset & \mathrm{End}({}_{\mathcal{A}}\sigma_{1,0\,\mathcal{A}}) & \subset & \mathrm{End}({}_{\mathcal{A}}\sigma^2_{1,0\,\mathcal{A}}) & \cdots \\ \cap & & \cap & & \cap & \\ \mathrm{End}({}_{\mathcal{B}}\iota_{\mathcal{A}}) & \subset & \mathrm{End}({}_{\mathcal{B}}\iota\sigma_{1,0\,\mathcal{A}}) & \subset & \mathrm{End}({}_{\mathcal{B}}\iota\sigma^2_{1,0\,\mathcal{A}}) & \cdots \end{array}$$

Bratteli diagram of the second row $\to$ A-D-E Dynkin diagram with Coxeter number m, vertex v_1

$\sigma_{0,1}$ instead of $\sigma_{1,0}$ $\rightarrow$ graph G_2 Coxeter number m, vertex v_2.

Thm. The quadruple $(G_1, [v_1], G_2, [v_2])$ gives a complete invariant for irreducible extensions of nets Vir_c, and an arbitrary quadruple, subject to the conditions on the Coxeter numbers as above, arises as an invariant of some extension.

ABC: Algebraic Boundary CFT. H.K. Rehren, L.

$M_+ \equiv \{(t,x) \in \mathbb{R}^2 : x > 0\}$

Chiral net $\mathcal{A} \longrightarrow$ two local nets on double-cones of M_+

1) *trivial boundary CFT*

$$O \mapsto A_+(O) := A(I) \vee A(J)$$

2) its *dual*

$$O \mapsto A_+^d(O) := A(L) \cap A(K)' \ \Big(= A_+(O')' \Big)$$

A^d is local and $A_+(O) \subset A_+^d(O)$ is the "two-interval subfactor".

Def. A *boundary CFT (BCFT)* associated with A is a local, isotonous net

$$O \mapsto B_+(O)$$

on double-cones of M_+, of vN algebras on a fixed Hilbert space $\mathcal{H}_B$ s.t.

(i) $\exists$ unitary, pos. energy rep. U of the covering $G = \widetilde{\text{Möb}}$ s.t.

$$U(g)B_+(O)U(g)^* = B_+(gO)$$

"within M_+" .

(ii) $\exists$ rep. π of A on $\mathcal{H}_B$ such that $B_+(O)$ contains $\pi(A_+(O))$, and

$$U(g)\pi(A_+(O))U(g)^* = \pi(A_+(gO))$$

(iii) *"Joint irreducibility":* $\forall\, O$, $B_+(O) \vee \pi(A_+)''$ is irreducible on $\mathcal{H}_B$.

Dual net:

$$B_+^d(O) = B_+(O')' \equiv B_+(O_<)' \cap B_+(O_>)'.$$

$B_+(O_<)$, $B_+(O_>)$ vN algebras of wedges.

Prop. Wedge duality:

$$B_+^d(O) := B_+(O'_>) \cap B_+(O'_<).$$

$\Rightarrow B_+^d$ is local

Shall assume A is completely rational

The non-local chiral net associated with a BCFT.

A boundary CFT $O \mapsto B_+(O)$ generates a chiral net $I \mapsto B^{\mathrm{gen}}(I)$ (the associated *boundary net*) on $\mathcal{H}_B$, by

$$B^{\mathrm{gen}}(I) := \bigvee_{O \subset W_L} B_+(O) \equiv B_+(W_L)$$

where W_L is the left wedge spanned by I.

Prop. (i) The boundary net B^{gen} generated from B_+ is isotonous, and it is covariant:

$$U(g) B^{\mathrm{gen}}(I) U(g)^* = B^{\mathrm{gen}}(gI)$$

whenever $I \subset \mathbb{R}$, $gI \subset \mathbb{R}$

$$\pi(A(I)) \subset B^{\text{gen}}(I) \subset \pi(A(I'))'.$$

(ii) There is a consistent family of vacuum-preserving expectations $\mathcal{E}^I : B^{\text{gen}}(I) \to A(I)$.

(iii) The local subfactors $\pi(A(I)) \subset B^{\text{gen}}(I)$ are irreducible and have finite index. The index is independent of I.

In general, the boundary net B^{gen} is a *non-local*.

Prop. If B_+ is relatively local with respect to $\pi(A_+^d)$, then $B^{\text{gen}} = A$, and B_+ lies between A_+ and A_+^d.

By the definition of the boundary net B and locality of B_+, we obviously have $B_+(O) \subset$

$B^{\text{gen}}(L) \cap B^{\text{gen}}(K)'$. This suggests the following definition of a local boundary CFT *induced* by a given (possibly non-local) chiral net:

Def. If $I \mapsto B(I)$ is an irreducible chiral extension of $I \mapsto A(I)$ (possibly non-local, but relatively local with respect to A), then the *induced net* is defined by

$$O \mapsto B_+^{\text{ind}}(O) := B(L) \cap B(K)'.$$

Prop. B^{ind} is a boundary CFT associated with A (special case $B = A$: $B^{\text{ind}} = A_+^{\text{dual}}$)

$$\text{gen} \circ \text{ind} = \text{id}, \ \text{ind} \circ \text{gen} = \text{dual}$$

$$\Rightarrow \text{dual} \circ \text{ind} = \text{ind and dual} \circ \text{dual} = \text{dual}$$

Every induced net B_+^{ind} is self-dual (Haag dual).

$$\boxed{\begin{array}{c}\text{Classif. Haag dual ABC on } A,\ c<1 \\ \Updownarrow \\ \text{Classif. non-local chiral extension of } A,\ c<1\end{array}}$$