Inner functions, real Hilbert subspaces and new Boundary QFT nets of von Neumann algebras

Roberto Longo

University of Rome "Tor Vergata"

Beijing, July 2010

Mainly based on papers with K.H. Rehren
and and a joint work with E. Witten

Things to discuss

- Inner functions and Beurling-Lax theorem
- Real Hilbert subspaces
- Algebraic Boundary Conformal Field Theory (K.H. Rehren, R.L.)
- Models of Boundary QFT (E. Witten, R.L.)

Inner functions

$\mathbb{D} \equiv \{z \in \mathbb{C} : |z| < 1\}$ unit disk, $\mathbb{H}^\infty(\mathbb{D})$ Hardy space.

$\varphi \in \mathbb{H}^\infty(\mathbb{D}) \Rightarrow \exists\, \varphi(e^{i\theta}) \equiv \lim_{r \to 1^-} \varphi(re^{i\theta})$ a.e. on $\partial\mathbb{D}$

$\varphi \in \mathbb{H}^\infty(\mathbb{D})$ is an *inner function* if $|\varphi(z)| = 1$ for almost all $z \in \partial\mathbb{D}$.

Examples:
$B_0(z) \equiv z$, or its Möbius transform:
$B_a(z) = \frac{|a|}{a} \frac{z-a}{1-\bar{a}z}$ (Blaschke factor),

$$B(z) \equiv \prod_{n=1}^{\infty} B_{a_n}(z) \quad \text{(Blaschke product)},$$

$a_n \in \mathbb{D}$, $\sum_{n=1}^{\infty}(1 - |a_n|) < \infty$.

$B(z)$ has zeros exactly at $\{a_n\}$, with multiplicity.
If an inner function φ has no zeros on $\mathbb{D}$, then φ is called a *singular* inner function.
φ is an inner function iff (uniquely)

$$\varphi(z) = \alpha B(z) \exp\left(-\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \mathrm{d}\mu(e^{i\theta})\right),$$

where μ is a positive, Lebesgue singular measure on $\partial\mathbb{D}$, $B(z)$ is a Blaschke product and α is a constant with $|\alpha| = 1$. All the zeros of φ come from B so φ is singular if and only if B is the identity.
Inner functions form a (multiplicative) *semigroup*, singular inner functions a sub-semigroup.

One-param. semigroup $\{\varphi_t\}$ of inner functions:

$$\varphi_t(z) = e^{it\lambda} \exp\left(-t \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \mathrm{d}\mu(e^{i\theta})\right)$$

Symmetric inner functions:

φ is symmetric $\bar{\varphi}(z) = \varphi(\bar{z})$.

Notions go $\mathbb{S}_\infty$ and $\mathbb{S}_\pi$: $h(z) \equiv i\frac{1+z}{1-z}$,

$$\mathbb{D} \xrightarrow{h} \mathbb{S}_\infty \xrightarrow{\log} \mathbb{S}_\pi$$

$\varphi \in \mathbb{H}^\infty(\mathbb{S}_\pi)$ inner: $|\varphi(q)| = |\varphi(q + i\pi)| = 1$
symmetric: $\varphi(q + i\pi) = \bar{\varphi}(q)$, $q \in \mathbb{R}$ a.e.

$\varphi \in \mathbb{H}^\infty(\mathbb{S}_\infty)$ inner: $|\varphi(q)| = 1$, $q > 0$
symmetric: $\varphi(-q) = \bar{\varphi}(q)$ a.e.

Scattering functions
A scattering function is a symmetric innere function f on $\mathbb{S}_\pi$ s.t. $\varphi(-p) = \varphi(p)$.

Inverse scattering program: construct QFT models from scattering function.

Beurling-Lax theorem (1949-1959)

S shift operator on $H^2(\mathbb{D})$:

$$Sf(z) = zf(z)$$

A closed S-invariant subspace K of $H^2(\mathbb{D})$ has the form

$$K = \varphi H^2(\mathbb{D}), \quad \varphi \text{ an inner function}$$

$$\Downarrow$$

$f \in H^2$ (or $f \in H^p, p \geq 1$) has a factorization

$$f(z) = \varphi(z)\psi(z)$$

φ is inner and ψ is outer $\psi(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log|f(e^{it})| \mathrm{d}t\right)$

Lax generalization to $H^2(\mathbb{S}_\infty)$, one-param. unitary translations.

Standard real Hilbert subspaces

$\mathcal{H}$ complex Hilbert space and $H \subset \mathcal{H}$ a real linear subspace.

Symplectic complement:

$$H' \equiv \{\xi \in \mathcal{H} : \Im(\xi, \eta) = 0 \quad \forall \eta \in H\}.$$

$H' = (iH)^{\perp}$ (real orthogonal complement)

$$H_1 \subset H_2 \Rightarrow H_1' \supset H_2' \ .$$

A *standard* subspace H of $\mathcal{H}$ is a closed, real linear subspace of $\mathcal{H}$ which is both cyclic ($\overline{H + iH} = \mathcal{H}$) and separating ($H \cap iH = \{0\}$). H is standard iff H' is standard.

H standard subspace $\to$ anti-linear operator $S \equiv S_H : D(S) \subset \mathcal{H} \to \mathcal{H}$, where $D(S) \equiv H + iH$,

$$S : \xi + i\eta \mapsto \xi - i\eta \ , \quad \xi, \eta \in H \ .$$

$S^2 = 1\restriction_{D(S)}$. S is closed and densely defined.

Conversely, S densely defined, closed, anti-linear involution on $\mathcal{H}$ gives

$$H = \{\xi : S\xi = \xi\} \quad \text{is a standard subspace}$$

$$H \longleftrightarrow S \quad \text{bijection}$$

Modular theory. Set

$$S_H = J_H \Delta_H^{1/2}$$

polar decomposition of $S = S_H$. Then J_H is an anti-unitary involution $\Delta \equiv S^* S > 0$

$$\Delta_H^{-it} H = H, \quad J_H H = H'$$

Borchers theorem (real subspace version) H standard subspace, U a one-parameter group with positive generator

$$U(s)H \subset H \quad s \geqslant 0.$$

Then:

$$\begin{cases} \Delta_H^{it} U(s) \Delta_H^{-it} = U(e^{-2\pi t} s), \\ J_H U(s) J_H = U(-s), \qquad t, s \in \mathbb{R}. \end{cases}$$

Note: Setting $K \equiv U(1)H$ we have

$$\Delta_H^{-it}K = \Delta_H^{-it}U(1)H = U(e^{2\pi t})\Delta_H^{-it}H$$
$$= U(e^{2\pi t})H \subset K, \quad t \geq 0\,.$$

$K \subset H$ is a half-sided modular inclusion.

About the proof (adapted from Florig). With $\xi \in H, \xi' \in H'$

$$f_U(z) = (\Delta^{i\bar{z}}\xi', U(e^{2\pi z}s)\Delta^{-iz}\xi).$$

is analytic in $\mathbb{S}_{1/2} = \{z \in \mathbb{C} : 0 < \Im\ z < \frac{1}{2}\}$ (the generator of $U(t)$ is positive and $\Im e^{2\pi z}s \geqslant 0$ for $z \in \mathbb{S}_{1/2}$).

$V(t) = JU(-t)J$ satisfies the same assumptions then U because of $JH = H'$

$$\begin{aligned} f_U\left(t + \frac{i}{2}\right) &= (\Delta^{-1/2}\Delta^{-it}\xi', U(e^{2\pi t + i\pi}s)\Delta^{-it}\Delta^{1/2}\xi) \\ &= (\Delta^{-1/2}\Delta^{-it}\xi', JV(e^{2\pi t}s)\Delta^{-it}\xi) \\ &= (\Delta^{-it}\xi', (J\Delta^{1/2})V(e^{2\pi t}s)\Delta^{-it}\xi) \\ &= (\Delta^{-it}\xi', V(e^{2\pi t}s)\Delta^{-it}\xi) = f_V(t) \end{aligned}$$

(KMS and positivity of energy) analogously $V(t) = JU(-t)J$ satisfies the same assumptions then U because of $JH = H'$

$$f_V\left(t + \frac{i}{2}\right) = f_U(t)$$

f_U and f_V glue to an entire bounded function, thus constant.

Converse: Wiesbrock, Borchers, Araki-Zsido theorem (real subspace version)

Let H, K be standard subspaces. Assume half-sided modular inclusion:

$$\Delta_H^{-it} K \subset K, \qquad t \geq 0$$

Then $\{\Delta_K^{it}, \Delta_H^{is}\}$ generates a unitary represen-taion of the "$ax{+}b$" group with positive energy

$$\text{dilation group} = \Delta_H^{-is/2\pi}$$

$$\text{gen. of translations } P = \frac{1}{2\pi}(\log \Delta_K - \log \Delta_H)$$

Möbius covariant nets of real Hilbert subspaces

A *local Möbius covariant net* of standard subspaces $\mathcal{A}$ of real Hilbert subspaces on the intervals of S^1 is a map

$$I \to H(I)$$

with

1. Isotony : *If* I_1, I_2 *are intervals and* $I_1 \subset I_2$, *then*

$$H(I_1) \subset H(I_2) \ .$$

2. Möbius invariance**:** *There is a unitary representation* U *of* $\mathbf{Mob}$ *on* $\mathcal{H}$ *such that*

$$U(g)H(I) = H(gI) \ , \qquad g \in \mathbf{Mob}, \ I \in \mathcal{I}.$$

Here $\mathbf{Mob} \simeq PSL(2,\mathbb{R})$ acts on S^1 as usual.

3. Positivity of the energy : $L_0 \geq 0$

4. Cyclicity : *the* complex *linear span of all spaces* $H(I)$ *is dense in* $\mathcal{H}$.

5. Locality : *If* I_1 *and* I_2 *are disjoint intervals then*

$$H(I_1) \subset H(I_2)'$$

First consequences

Irreducibility: $\overline{\text{real lin.span}}_{I \in \mathcal{I}} \mathcal{H}(I) = H$.

Reeh-Schlieder theorem: $H(I)$ is a standard subspace for every I.

Bisognano-Wichmann property: Tomita-Takesaki modular operator Δ_I and conjugation J_I of

$H(I)$, are

$$U(\Lambda_I(2\pi t)) = \Delta_I^{-it},\ t \in \mathbb{R}, \qquad \text{dilations}$$
$$U(r_I) = J_I \qquad \text{reflection}$$

($\Lambda_{I_1}(t)x = e^t x, x \in \mathbb{R}$, $I_1 \simeq \mathbb{R}^+$ upper semi-circle)

Haag duality: $H(I)' = H(I')$ $\quad (I' \equiv S^1 \smallsetminus I)$.

Factoriality: $H(I) \cap H(I)' = 0$

Additivity: $I \subset \cup_i I_i \implies H(I) \subset \overline{\text{real lin.span}_i H(I_i)}$.

Modular theory and representations of $SL(2,\mathbb{R})$ (Brunetti, Guido, L.)

U a unitary, positive energy representation of $\mathbf{Mob}$ on $\mathcal{H}$ and J anti-unitary involution on $\mathcal{H}$ s.t.

$$JU(g)J = U(rgr), \quad g \in \mathbf{Mob}$$

where $r : z \mapsto \bar{z}$ reflection on S^1 w.r.t. the upper semicircle I_1. Then define

$$J_I \equiv U(g)JU(g)^*$$

where $g \in \mathbf{Mob}$ maps I_1 onto I.

$$\Delta_I^{it} \equiv U(\Lambda_I(-2\pi t)), \quad t \in \mathbb{R}$$

namely $-\frac{1}{2\pi}\log \Delta_I$ generator of dilations of I,

$$S_I \equiv J_I \Delta_I^{1/2}$$

is a densely defined, antilinear, closed involution on $\mathcal{H}$.

$H(I)$ standard subspace associated with S_I

$$\downarrow$$

Möbius covariant local net of real Hilbert spaces

A $\pm$*hsm factorization of real subspaces* is a triple K_0, K_1, K_2, where $\{K_i, i \in \mathbb{Z}_3\}$ is a set of standard subspaces s.t. $K_i \subset K'_{i+1}$ is a $\pm$hsm inclusion.

$$\text{Factorization}$$
$$\Updownarrow$$
$$\text{Local Möbius covariant net of real Hilbert spaces}$$
$$\Updownarrow$$
$$\text{Positive energy representation of } SL(2,\mathbb{R})/\{1,-1\}$$

Note: Irr. positive energy rep. of $SL(2,\mathbb{R})/\{1,-1\}$ are parametrized by $\mathbb{N}$

Endomorphisms of standard subspaces

A *standard pair* of $\mathcal{H}$ is a pair (H, T) such that

- H is a standard subspace,
- T is a one-par. unitary group, with positive generator P, s.t. $T(t)H \subset H$, $t \geq 0$.

Thm. Assume (H, T) to be irreducible and let $K \subset H$ be a real subspace. The following are equivalent:

(*i*) $T(t)K \subset K, t \geq 0$,

(*ii*) $K = VH$ where V is a unitary commuting with T,

(*iii*) $K = VH$ where $V = \psi(Q)$ with $Q \equiv \log P$ and $\psi \in L^\infty(\mathbb{R}, \mathrm{d}q)$ is the boundary value of an inner function in $H^\infty(\mathbb{S}_\pi)$ such that $\psi(q + i\pi) = \bar{\psi}(q)$, for almost all $q \in \mathbb{R}$.

The semigroup $\mathcal{E}(H)$ of endomorphisms of (H, T) is isomorphic to the semigroup of symmetric inner functions on the strip $0 < \Im z < \pi$.

2-dimensional CFT

$M = \mathbb{R}^2$ Minkowski plane.

$\begin{pmatrix} T_{00} & T_{10} \\ T_{01} & T_{11} \end{pmatrix}$ conserved and traceless stress-energy tensor.

As is well known, $T_L = \frac{1}{2}(T_{00} + T_{01})$ and $T_R = \frac{1}{2}(T_{00} - T_{01})$ are chiral fields,

$$T_L = T_L(t + x), \quad T_R = T_R(t - x).$$

Left and right movers.

Two-dimensional conformal fields and nets

Ψ_k family of conformal fields on M: T_{ij} + *relatively local fields*
$\mathcal{O} = I \times J$ double cone, I, J intervals of the chiral lines $t \pm x = 0$

$$\mathcal{A}(\mathcal{O}) = \{e^{i\Psi_k(f)}, \mathrm{supp} f \subset \mathcal{O}\}''$$

then by relative locality

$$\mathcal{A}(\mathcal{O}) \supset \mathcal{A}_L(I) \otimes \mathcal{A}_R(J)$$

$\mathcal{A}_L, \mathcal{A}_R$ chiral fields on $t \pm x = 0$ generated by T_L, T_R and other chiral fields

(completely) rational case: $\mathcal{A}_L(I) \otimes \mathcal{A}_R(J) \subset \mathcal{A}(\mathcal{O})$ finite Jones index

Local conformal nets

A local Möbius covariant net $\mathcal{A}$ on S^1 is a map

$$I \in \mathcal{I} \to \mathcal{A}(I) \subset B(\mathcal{H})$$

$\mathcal{I} \equiv$ family of proper intervals of S^1, that satisfies:

- **A.** *Isotony*. $I_1 \subset I_2 \implies \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$
- **B.** *Locality*. $I_1 \cap I_2 = \varnothing \implies [\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$
- **C.** *Möbius covariance*. $\exists$ unitary rep. U of the Möbius group Möb on $\mathcal{H}$ such that

$$U(g)\mathcal{A}(I)U(g)^* \;=\; \mathcal{A}(gI), \quad g \in \text{Möb},\; I \in \mathcal{I}.$$

- **D.** *Positivity of the energy*. Generator L_0 of rotation subgroup of U (conformal Hamiltonian) is positive.
- **E.** *Existence of the vacuum*. $\exists!$ U-invariant vector $\Omega \in \mathcal{H}$ (vacuum vector), and Ω is cyclic for $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I)$.

First consequences

- *Irreducibility*: $\bigvee_{I\in\mathcal{I}}\mathcal{A}(I) = B(H)$.
- *Reeh-Schlieder theorem*: Ω is cyclic and separating for each $\mathcal{A}(I)$.
- *Bisognano-Wichmann property*: Tomita-Takesaki modular operator Δ_I and conjugation J_I of $(\mathcal{A}(I),\Omega)$, are

$$U(\Lambda_I(2\pi t)) = \Delta_I^{it},\ t\in\mathbb{R}, \qquad \text{dilations}$$
$$U(r_I) = J_I \qquad \text{reflection}$$

  (Frölich-Gabbiani, Guido-L.)
- *Haag duality*: $\mathcal{A}(I)' = \mathcal{A}(I')$
- *Factoriality*: $\mathcal{A}(I)$ is III_1-factor (in Connes classification)
- *Additivity*: $I \subset \cup_i I_i \implies \mathcal{A}(I) \subset \vee_i \mathcal{A}(I_i)$ (Fredenhagen, Jorss).

Representations

A representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a map

$$I \in \mathcal{I} \mapsto \pi_I, \text{ normal rep. of } \mathcal{A}(I) \text{ on } B(\mathcal{H})$$

$$\pi_{\tilde{I}} \restriction \mathcal{A}(I) = \pi_I, \quad I \subset \tilde{I}$$

π is automatically diffeomorphism *covariant*: $\exists$ a projective, pos. energy, unitary rep. U_π of $\mathrm{Diff}^{(\infty)}(S^1)$ s.t.

$$\pi_{gI}(U(g)xU(g)^*) = U_\pi(g)\pi_I(x)U_\pi(g)^*$$

for all $I \in \mathcal{I}$, $x \in \mathcal{A}(I)$, $g \in \mathrm{Diff}^{(\infty)}(S^1)$ (Carpi & Weiner)

DHR argument: given I, there is an endomorphism of $\mathcal{A}$ localized in I equivalent to π; namely ρ is a representation of $\mathcal{A}$ on the vacuum Hilbert space $\mathcal{H}$, unitarily equivalent to π, such that $\rho_{I'} = \mathrm{id} \restriction_{\mathcal{A}(I')}$.

- $\mathrm{Rep}(\mathcal{A})$ is a *braided tensor category* (DHR, FRS, L.)

Split property

$\mathcal{A}$ satisfies the *split* property if the von Neumann algebra

$$\mathcal{A}(I_1) \vee \mathcal{A}(I_2) \simeq \mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$$

(natural isomorphism) if $\bar{I}_1 \cap \bar{I}_2 = \varnothing$.

$$\mathrm{Tr}(e^{-tL_0}) < \infty,\ \forall t > 0 \implies \text{ split }.$$

Complete rationality

I_1, I_2 intervals $\bar{I}_1 \cap \bar{I}_2 = \varnothing$, $E \equiv I_1 \cup I_2$.

$$\mu\textit{-index}: \quad \mu_{\mathcal{A}} \equiv [\mathcal{A}(E')' : \mathcal{A}(E)]$$

(Jones index). $\mathcal{A}$ conformal:

$$\mathcal{A} \textit{ completely rational} \stackrel{\text{def}}{=} \mathcal{A} \text{ split } \& \ \mu_{\mathcal{A}} < \infty$$

Thm. (Y. Kawahigashi, M. Müger, R.L.) $\mathcal{A}$ completely rational: then

$$\mu_{\mathcal{A}} = \sum_i d(\rho_i)^2$$

sum over all irreducible sectors. (F. Xu in $SU(N)$ models);

• $\mathcal{A}(E) \subset \mathcal{A}(E')' \sim$ LR inclusion (quantum double);

• Representations form a modular tensor category (i.e. non-degenerate braiding).

Boundary CFT

Stress-energy tensor left/right movers $T_L = \frac{1}{2}(T_{00} + T_{01})$ and $T_R = \frac{1}{2}(T_{00} - T_{01})$: $T_L = T_L(t+x)$, $T_R = T_R(t-x)$.

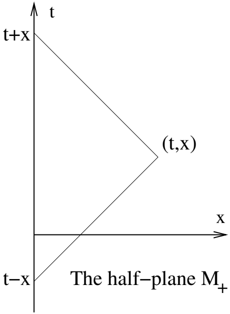
Boundary condition: no energy flow across the boundary:

$$T_{01}(t, x=0) = 0 \qquad \Leftrightarrow \qquad T_L = T_R \equiv T.$$

so $T_{10} = T_{01}$, $T_{11} = T_{00}$ are of the form

$$T_{00}(t,x) = T(t+x) + T(t-x), \qquad T_{01}(t,x) = T(t+x) - T(t-x),$$

i.e., *bi-local* expressions in terms of T

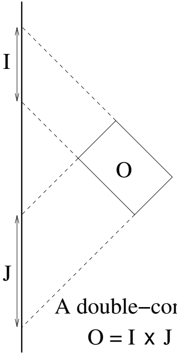


The chiral fields of a boundary CFT generate a net

$$O \mapsto A_+(O).$$

$A_+(O)$ is generated by chiral fields smeared in the variable $t + x$ over the interval I and in the variable $t - x$ over the interval J, where $O = I \times J$, $I > J$, is an open double-cone in M_+. The bi-localized structure translates into the form of the local algebras

$$A_+(O) = A(I) \vee A(J) \qquad (O = I \times J, \;\; I > J).$$



A double–cone
O = I x J

Definition of Boundary CFT

A *boundary CFT (BCFT)* associated with A is a local, isotonous net $O \mapsto B_+(O)$ over the double-cones within the half-space M_+, represented on a Hilbert space $\mathcal{H}_B$ such that

(i) there is a unitary representation $\mathcal{U}$ of the covering of the Möbius group $PSL(2, \mathbb{R})$ with positive generator for the subgroup of translations, such that

$$\mathcal{U}(g) B_+(O) \mathcal{U}(g)^* = B_+(gO)$$

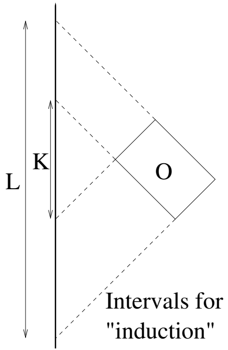
(ii) There is a representation π of A on $\mathcal{H}_B$ such that $B_+(O)$ contains $\pi(A_+(O))$ and π is U-covariant.

(iii) *"Joint irreducibility":* For each double-cone O, $B_+(O) \vee \pi(A_+)''$ is irreducible on $\mathcal{H}_B$ (almost automatic)

chiral extension $\longrightarrow$ boundary condition

If $I \mapsto B(I)$ is an irreducible chiral extension of $I \mapsto A(I)$ (possibly non-local, but relatively local with respect to A), then the *induced net* is defined by

$$O \mapsto B_+^{ind}(O) := B(L) \cap B(K)'.$$



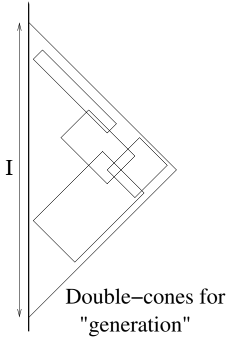
The observables of the induced BCFT localized in O belong to $B(L)$ and commute with $B(K)$.

BCFT $\rightarrow$ non-local chiral net

A boundary CFT $O \mapsto B_+(O)$ generates a chiral net $I \mapsto B^{gen}(I)$ (the associated *boundary net*) on $\mathcal{H}_B$, by

$$B^{gen}(I) := \bigvee_{O \subset W_L} B_+(O) \equiv B_+(W_L)$$

where W_L is the left wedge spanned by I



The observables of the associated chiral boundary net localized in I are generated by BCFT observables localized in double cones

(i) In the special case $B = A$, the induced net is the dual net A_+^{dual}:

$$B^{dual}(O) \equiv B(O')'$$

so $A_+(O) \subset A_+^{dual}(O)$ is the 2-interval inclusion.

(ii) If B is a chiral extension of A, then

$$(B_+^{ind})^{gen} = B$$

Conversely

$$(B_+^{gen})^{ind} = B_+^{dual}$$

(iii) Every induced net $B\,\mathrm{ind}_+$ is self-dual (Haag dual).

conclusion:

non-local chiral extensions of $A \leftrightarrow$ local extensions of A_+

Classification for $c < 1$: Kawahigashi, Pennig, Rehren, L.

Remarkable properties

Let B chiral extension of A, and $B\,\mathrm{ind}_+$ the induced BCFT net.

(i) The index of $\pi(A_+(O)) \subset B\,\mathrm{ind}_+(O)$ equals the μ-index μ_A of A. This index is thus the same for each chiral extension

(ii) When B_+ is Haag dual, then $\mu_{B_+} = 1$, and B_+ satisfies Haag duality also for the disconnected regions of the form $E = O_1 \cup O_2$

(iii) A Haag dual boundary CFT net B_+ has the no nontrivial DHR sectors.

The semigroup $\mathcal{E}(\mathcal{A})$

Let $\mathcal{A}$ be a local Möbius covariant net of von Neumann algebras on $\mathbb{R}$

$$I \subset \mathbb{R} \text{ interval} \to \mathcal{A}(I)$$

T one-parameter unitary translation group. Then $T(t)\mathcal{A}(I)T(-t) = \mathcal{A}(I+t)$, T has positive generator P and $T(t)\Omega = \Omega$ where Ω is the vacuum vactor.

Let V be a unitary on $\mathcal{H}$ commuting with T. The following are equivalent:

(*i*) $V\mathcal{A}(I_2)V^*$ commutes with $\mathcal{A}(I_1)$ for all intervals I_1, I_2 of $\mathbb{R}$ such that $I_2 > I_1$ (I_2 is contained in the future of I_1).

(*ii*) $V\mathcal{A}(a,\infty)V^* \subset \mathcal{A}(a,\infty)$ for every $a \in \mathbb{R}$.

(*iii*) $V\mathcal{A}(0,\infty)V^* \subset \mathcal{A}(0,\infty)$.

The semigroup $\mathcal{E}(\mathcal{A})$

$$\mathcal{E}(\mathcal{A}) \equiv \text{semigroup of unitaries } V \text{ as above}$$

$$\mathcal{A} \text{ conformal net \& } V \in \mathcal{E}(\mathcal{A}) \longrightarrow \text{Boundary QFT } \mathcal{A}_V$$

$$\mathcal{A}_V(\mathcal{O}) \equiv \mathcal{A}(I_1) \vee V\mathcal{A}(I_2)V^*$$

where I_1, I_2 are intervals of time-axis such that $I_2 > I_1$ and $\mathcal{O} = I_1 \times I_2$.

$\mathcal{A}$ with the split property, $V \in \mathcal{E}(\mathcal{A})$ then $\mathcal{A}_V$ is *locally isomorphic* to $\mathcal{A}_+ = \mathcal{A}_I$.

As an immediate consequence, if V_t is a one-parameter semigroup of unitaries in $\mathcal{E}(\mathcal{A})$, the family $\mathcal{A}_{V_t}$ gives a *deformation* of the conformal net $\mathcal{A}_+$ on M_+ with translation covariant nets on M_+ that are locally isomorphic to $\mathcal{A}_+$.

Constructing models

$\mathcal{A}$ free field on $\mathbb{R}$ acting on the Fock space $F(\mathcal{H})$.

H standard subspace of $\mathcal{H} \to$ von Neumann algebra on $F(\mathcal{H})$

$$\mathcal{A}(H) = \{W(h) : h \in H\}''$$

Take $H = H(0, \infty)$.

$$V \in \mathcal{E}(H) \to \Gamma(V) \in \mathcal{E}(\mathcal{A})$$

therefore

symmetric inner function $\to V \in \mathcal{E}(\mathcal{A}) \to$ Boundary QFT net $\mathcal{A}_V$ on M_+

In particular

φ scattering function $\to$ Boundary QFT

More general BQFT's

$\mathcal{A} = \mathcal{A}_N$ Buchholz-Mach-Tododorv extension of $U(1)$-current net:

symmetric inner function Hölder continuous at 0 & $V \in \mathcal{E}(\mathcal{A})$

$$\downarrow$$

Boundary QFT net $\mathcal{A}_V$ on M_+

Examples: $\mathcal{A}_1$ associated with level 1 $\widehat{su(2)}$-Kac-Moody algebra with $c = 1$, $\mathcal{A}_2$ Bose subnet of free complex Fermi field net, $\mathcal{A}_3$ appears in the $\mathbb{Z}_4$-parafermion current algebra analyzed by Zamolodchikov and Fateev, and in general $\mathcal{A}_N$ is a coset model $SO(4N)_1/SO(2N)_2$.

Outlook and problems

- Models on the full Minkowski plane
- Which BQFT's are associated with loop group models?
- Given a completely rational $\mathcal{A}$ CFT on the boundary, do all BQFT's $\mathcal{A}_V$ have the same *positive energy* representations?
- Construct BQFT on different spacetimes