INTRODUCTION TO LOEWNER THEORY IN ONE COMPLEX VARIABLE (PRELIMINARY PROGRAMME OF THE COURSE)

PAVEL GUMENYUK

Required courses:

- a standard university course of Complex Analysis (in one variable);
- Measure Theory and Lebesgue Integration (usually a part of a course in Real Analysis);
- advisable, but not necessary: Probability Theory and Stochastic Processes.

General characteristic. Topics of the course belong to Complex Analysis in one complex variable and are mainly related to problems in Conformal Mapping. At the same time, although it is not discussed in the course, the modern Loewner Theory has natural extension to several complex variables.

Loewner Theory combines deep results from Complex Analysis with fundamental ideas from Dynamics and Lie Group Theory. It includes, as a special case, the theory of one-parameter semigroups, which is classically known to have important applications to time-homogeneous Markov processes.

The course would be primarily useful for those students who are interested in Complex Analysis (one and/or several variables) or whose who wish to refresh and deepen his/hers knowledge in one complex variable. Furthermore, the course is advisable for students in Probabilities, especially for those who are interested in applications of Complex Analysis to Stochastic Processes. Finally, the course would be enjoyable for everybody who is curious to see how ideas and methods from different parts of Analysis can work together in the unit disk of the complex plane.

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PROGRAMME

I. PRELIMINARIES (approx. 8 hours)

(1) Holomorphic self-maps of the unit disk (half-plane):
Schwarz Lemma; Poincare's metric; boundary version's of the Schwarz Lemma for the unit disk and half-plane; angular limit and angular derivative. Boundary fixed and contact points.

(2) Normal families of holomorphic (or meromorphic) functions; Montel's criterion.

(3) Boundary behaviour in the unit disk: no-Koebe-arcs Theorem; Lindelöf's Theorem on angular limits; accessible boundary points.

(4) Elements of Potential Theory in the complex plane. Green function. Poisson integral and harmonic measure. The strong Markov property.

II. CLASSICAL LOEWNER THEORY (approx. 6 hours)

(1) Slit mappings and their properties. Loewner's parametric representation of slit mappings.

(2) Holomorphic functions with positive real part. Radial Loewner chains (in the sense of Pommerenke). Loewner – Kufarev PDE and ODE.

(3) Riemann Mapping Theorem and Carathéodory Convergence Theorem (without proof). Parametric representation of arbitrary conformal mappings of the unit disk.

(4) Remarks on chordal Loewner chains and "reversed time" setting.

III. GENERAL LOEWNER THEORY (approx. 6 hours)

(1) One-parameter semigroups of holomorphic self-maps and their infinitesimal generators.

(2) Generalized Loewner – Kufarev ODE and its evolution families. Intrinsic definition of evolution families.

(3) Generalized Loewner chains and their relationships to evolution families and Herglotz vector fields.

(4) Concluding remarks. Analogy with Lie Group Theory and the Embedding Problem. Evolution families in subclasses.

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