KREIN’S TRACE THEOREM REVISITED

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Abstract. M. G. Krein’s celebrated Trace Theorem states that if \( A, B \) are self-adjoint operators in a separable Hilbert space such that \( A - B \) is a trace class operator, then, for any function \( f \) of a real variable, whose derivative \( f' \) in distributional sense has Fourier transform belonging to \( L^1(\mathbb{R}) \), the difference \( f(A) - f(B) \) is again a trace class operator and the formula

\[
\text{Trace}(f(A) - f(B)) = \int_{\mathbb{R}} f'(s) \xi(s) \, ds
\]

holds where the function \( \xi \in L^1(\mathbb{R}) \) depends only on \( A \) and \( B \) and is uniquely determined by the above formula. The function \( \xi \) is called the spectral shift function of the pair \( A, B \) and is an important ingredient in the perturbation theory of self-adjoint operators.

The original proof of Krein uses complex analysis and is quite involved. We supply a new proof which does not use complex analysis. Our proof works also for \( \sigma \)-finite von Neumann algebras \( M \) of type II and unbounded perturbations from the predual of \( M \). The exposition is based on joint work with D. Potapov and D. Zanin.

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