Stability of nearly-integrable (nearly) Hamiltonian systems

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CONTENT

1. Introduction

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Conservative case: KAM theory, Nekhoroshev's theorem, Greene's method, converse KAM, frequency analysis, FLI

Dissipative case: dissipative KAM theory, exponential stability estimates, approximating periodic orbits, converse KAM, basins of attractions

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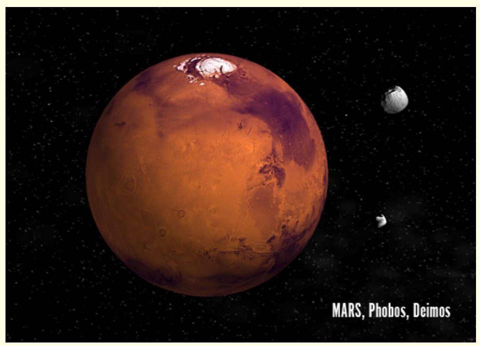
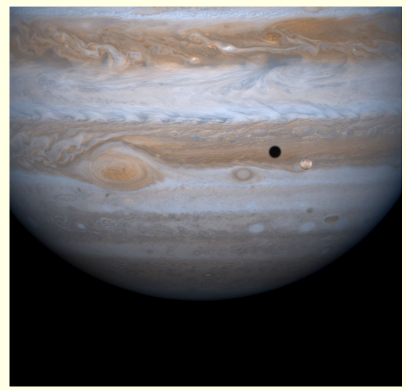
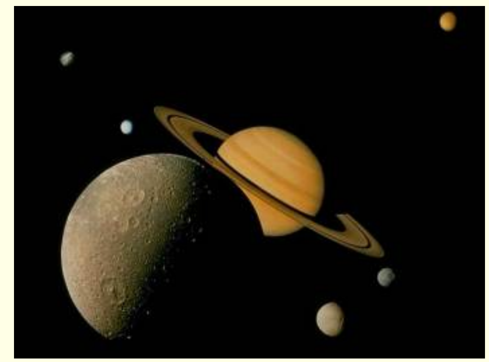
☞ MOTIVATION: rotational and orbital dynamics in Celestial Mechanics.

• **Moon,** and all evolved satellites: spin–orbit 1:1 resonance (1 rotation = 1 revolution). **Mercury:** 3:2 spin–orbit resonance (3 rotations = 2 revolutions). The analysis of the rotation provides information on the internal structure.

Does dissipation play a role in the selection of resonances?







• Effect of dissipation in astrodynamics **(Space Manifold Dynamics):** invariant manifolds, low–energy transfers, attitude–orbit dynamics.

Can dissipation be used to modify orbits?

☞ Examples of nearly–integrable systems with (weak) dissipation in the Solar System:

• **Tidal torque:** due to the non–rigidity of planets and satellites

• **Yarkovsky effect:** due to the joint action of solar lighting and rotation of the body; the rotation causes that the re-emission of the absorbed radiation occurs along a direction different from that of the Sun, thus provoking a variation of the angular momentum and therefore of the orbit

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- **Radiation pressure:** exerted on any surfaces subject to electromagnetic radiation

- **Solar wind:** caused by charged particles originating from the Sun

- **Stokes drag:** collision of particles with the molecules of the gas nebula during the formation of the planetary system

- **Poynting–Robertson effect:** due to the absorption and re-emission of the solar radiation, the velocity of a dust particle decreases

2. Conservative and dissipative systems

☞ Conservative Standard Map

$$
\begin{aligned}
y' &= y + \varepsilon\, g(x) \\
x' &= x + y' \ ,
\end{aligned}
\tag{1}
$$

where $y \in \mathbb{R}$, $x \in \mathbb{T} \equiv \mathbb{R}/(2\pi\mathbb{Z})$, $\varepsilon > 0$ *perturbing parameter*, $g = g(x)$ analytic, periodic function (typically $g(x) = \sin x$).

☞ Properties: integrable for $\varepsilon = 0$, non–integrable for $\varepsilon \neq 0$: periodic orbits are surrounded by *librational curves;* quasi–periodic invariant curves are slightly displaced and deformed (with w.r.t. the integrable case) and for a given value of ε they break–down, leaving place to *cantori,* which are still invariant sets, but they are graphs of a Cantor set.

☞ Dissipative Standard Map:

$$\begin{aligned} y' &= \lambda y + c + \varepsilon\, g(x) \qquad && y \in \mathbb{R}\ ,\ x \in \mathbb{T} \\ x' &= x + y'\ , && \lambda, c, \varepsilon \in \mathbb{R}_+\ , \end{aligned} \tag{2}$$

$g(x)(=\sin x)$ analytic periodic function, $0 < \lambda < 1$ dissipative parameter, $c =$ drift parameter ($\lambda = 1$, $c = 0$ conservative SM).

- Notice that for $\varepsilon = 0$ the trajectory $\{y = \omega \equiv \frac{c}{1-\lambda}\} \times \mathbb{T}$ is invariant:

$$y' = y = \lambda y + c\ , y = \omega \quad \rightarrow \quad \omega = \lambda\omega + c \quad \rightarrow \quad \omega \equiv \frac{c}{1-\lambda} \quad \rightarrow \quad c = (1-\lambda)\omega\ .$$

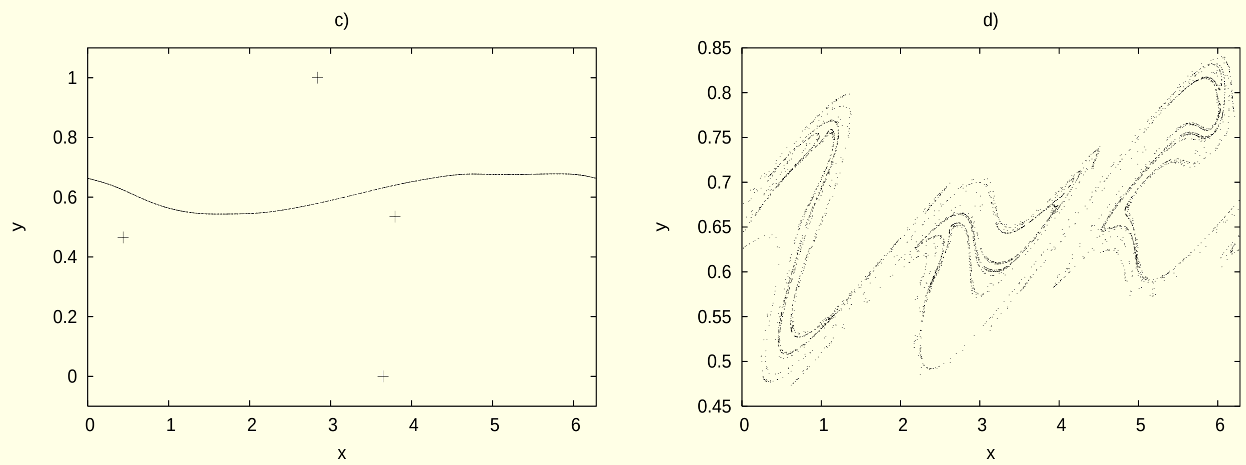

Figure 1: SMD attractors. Left: invariant attractor coexisting with 0/1, 1/2, 1/1 periodic orbits. Right: strange attractor.

• Basins of attraction for the coexisting case (500×500 random initial conditions with preliminary iterations).

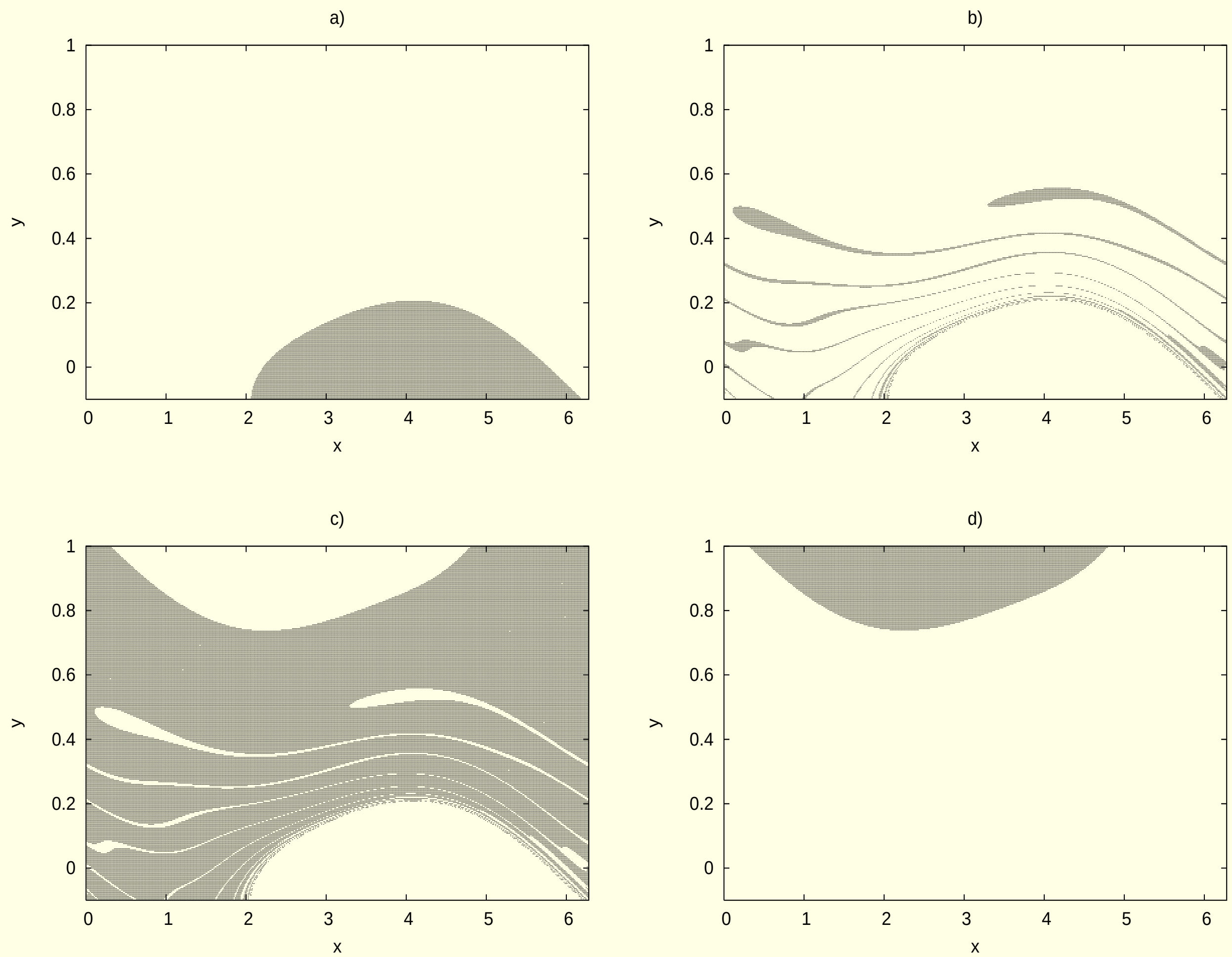

Figure 2: Basins of attraction of *a*) 0/1 periodic orbit; *b*) 1/2 periodic orbit; *c*) quasi–periodic attractor; *d*) 1/1 periodic orbit.

3. Stability: KAM and Nekhoroshev

• **KAM theorem:** persistence of invariant tori on which a quasi–periodic motion takes place.

2–dimensional systems: dim(phase space)=4, dim(constant energy level)=3, dim(invariant tori)=2 $\rightarrow$ confinement in phase space for ∞ times between bounding invariant tori

• No more valid for $n > 2$: the motion can diffuse through invariant tori, reaching arbitrarily far regions (Arnold's diffusion)

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• **Nekhoroshev theorem:** bound on the action variables $|y(t) - y(0)| \leq \rho$ for exponential times $t \leq T = T_0 \ e^{(\frac{\varepsilon_0}{\varepsilon})^a}$.

J.E. Littlewood: *"While not eternity, this is a considerable slice of it"*.

4. Conservative KAM results

Nearly–integrable Hamiltonian system:

$$\mathcal{H}(y,x,t) = h(y) + \varepsilon f(y,x,t) \ , \qquad y \in \mathbb{R}^n, \ (x,t) \in \mathbb{T}^{n+1} \ .$$

Assume that the frequency $\omega = \omega(y_0) \equiv h'(y_0)$ is Diophantine:

$$|\omega \cdot q + p| \geq \nu |q|^{-\tau} \ , \qquad \forall (q,p) \in \mathbb{Z}^{n+1} \backslash \{0\} \ , \ \ \nu > 0 \ , \ \tau > 0 \ . \tag{3}$$

KAM theorem: *Assume ω diophantine and h not–degenerate $\det(h'') \neq 0$. Persistence of invariant tori with diophantine frequency of the integrable case, provided ε is sufficiently small, say $\varepsilon \leq \varepsilon_{KAM}$.*

- Results consistent with observations in spin–orbit problem and R3BP.

KAM theorem for SM: persistence of invariant curves with diophantine frequency, provided ε is sufficiently small, say $\varepsilon \leq \varepsilon_{KAM}$. For the golden mean torus with $\omega = \frac{\sqrt{5}-1}{2}$, then $\varepsilon_{KAM} = 0.838$, whereas the numerical threshold is $\varepsilon_{num} = 0.971635$ (agreement of 86%).

5. Dissipative KAM results

• Nearly–integrable Hamiltonian flow with dissipation, e.g.

$$\begin{aligned} \dot{x} &= y + \varepsilon f_y(y,x,t) \\ \dot{y} &= -\varepsilon f_x(y,x,t) - \mu(y-\eta)\ , \qquad y \in \mathbb{R},\ (x,t) \in \mathbb{T}^2\ . \end{aligned} \tag{4}$$

• Case $\varepsilon = 0$ and $\mu \neq 0$: the torus $\mathcal{T}_0 \equiv \{y = \eta\} \times \{(\theta,t) \in \mathbb{T}^2\}$ is a global attractor with frequency η: the general solution is given by

$$x(t) = x_0 + \eta\,(t-\tau) + \frac{1-\exp(-\mu(t-\tau))}{\mu}\,(v_0 - \eta)\ .$$

KAM theorem. *For ω diophantine, there exists $\varepsilon_{KAM} > 0$ such that for any $0 < \varepsilon \leq \varepsilon_{KAM}$ and $0 \leq \mu < 1$ there exists a quasi–periodic solution with frequency ω, if $\eta = \omega\ (1 + O(\varepsilon^2))$.*

• Dissipative KAM proof extended to general (i.e. conformally symplectic) mapping systems (results for maps imply results for flows).

6. Quasi–periodic (dissipative) attractors

Definition. A diffeomorphism $f = f_c$ defined on the phase space $\mathcal{M} \subseteq \mathbb{R}^n \times \mathbb{T}^n$ (endowed with a symplectic form Ω) is *conformally symplectic*, if there exists a function $\lambda : \mathcal{M} \to \mathbb{R}$ such that the pull-back f_c^* satisfies

$$f_c^* \Omega = \lambda \Omega$$

(i.e. a mapping contracting the area by a factor λ).

• **Invariance equation.** Look for $c \in \mathbb{R}^n$, $K : \mathbb{T}^n \to \mathcal{M}$, such that the following invariance equation is satisfied:

$$f_c \circ K(\theta) = K(\theta + \omega)\ . \qquad (INV)$$

(i.e. parametrize as $K(\theta) = (\omega + u(\theta) - u(\theta - \omega), \theta + u(\theta))$ for the SM with $u = O(\varepsilon)$).

• **Remark.** The solutions of (INV) are locally unique up to a shift. If $T_s(\theta) = \theta + s$ and if (K, c) is a solution, then $(K \circ T_s, c)$ is a solution $\forall s \in \mathbb{T}^n$.

• The **standard map** with $\varepsilon = 0$ is given by $(y', x') = (\lambda y + c, x + y')$; from $K(\theta) = (\omega, \theta)$ we get

$$f_c \circ K(\theta) = (\lambda\omega + c, \theta + \lambda\omega + c) \ , \qquad K(\theta + \omega) = (\omega, \theta + \omega) \ .$$

The invariance equation provides:

$$c = (1 - \lambda)\omega \ .$$

• For the **standard map** with $\varepsilon \neq 0$, recall $y_j = x_j - x_{j-1}$ and let $K(\theta) = (\omega + u(\theta) - u(\theta - \omega), \theta + u(\theta))$. Since:

$$\begin{aligned} f_c \circ K(\theta) \quad &= \quad (\lambda\omega + \lambda u(\theta) - \lambda u(\theta - \omega) + c + \varepsilon g(\theta + u(\theta)), \\ &\quad\ \theta + (1 + \lambda) u(\theta) - \lambda u(\theta - \omega) + \lambda\omega + c + \varepsilon g(\theta + u(\theta))) \end{aligned}$$

and

$$K(\theta + \omega) = (\omega + u(\theta + \omega) - u(\theta),\ \theta + \omega + u(\theta + \omega)) \ ,$$

from the second component we get the invariance equation:

$$u(\theta + \omega) - (1 + \lambda) u(\theta) + \lambda u(\theta - \omega) + (1 - \lambda)\omega - c = \varepsilon g(\theta + u(\theta)) \ .$$

- For a Lagrangian torus $K^*(\Omega) = 0$ and therefore the following holds true:

$$DK^T(\theta)\ J \circ K(\theta)\ DK(\theta) = 0\ , \qquad (LAGR)$$

J being the matrix representing Ω at x: $\Omega_x(u,v) = (u, J(x)v)$ for any u, v.

- **Reduction of the linearization.** In the neighborhood of an (approximate) invariant torus, there exists a change of coordinates such that the linearization of the invariance equation $f_c \circ K(\theta) = K(\theta + \omega)$ (INV) is a constant coefficient equation.

- Taking the derivative of (INV):

$$D(f_c \circ K)\ DK - DK \circ T_\omega\ =\ 0$$

and using $(LAGR)$ one obtains an equation of the form

$$Df_c \circ K(\theta) M(\theta) - M(\theta+\omega) \left(\begin{array}{cc} Id & S(\theta) \\ 0 & \lambda \end{array} \right) = 0 \qquad (RED)$$

for $M(\theta) = [DK(\theta)|\ J^{-1} \circ\ DK(\theta)\ N(\theta)]$ and for suitable functions N, S.

- **Approximate solution** (K, c) of (INV) with error E:

$$f_c \circ K - K \circ T_\omega = E \ .$$

By a Newton method, find **corrections** MW, σ, such that $K' = K + MW$ $(W = (W_1, W_2))$, $c' = c + \sigma$ provide an error quadratically smaller: take the linearization of (INV) and use (RED) to find

$$\begin{pmatrix} Id & S(\theta) \\ 0 & \lambda \end{pmatrix} W - W \circ T_\omega = -\widetilde{E} - \widetilde{A}\,\sigma$$

for suitable known $\widetilde{E} \equiv (\widetilde{E}_1, \widetilde{E}_2)$, $\widetilde{A} \equiv [\widetilde{A}_1 |\ \widetilde{A}_2]$, namely:

$$\begin{aligned} W_1 - W_1 \circ T_\omega &= -\widetilde{E}_1 - SW_2 - \widetilde{A}_1\,\sigma && (W_1) \\ \lambda W_2 - W_2 \circ T_\omega &= -\widetilde{E}_2 - \widetilde{A}_2\,\sigma && (W_2)\ . \end{aligned}$$

- (W_1) involves small divisors, solvable for ω diophantine.
- (W_2) always solvable for any $|\lambda| \neq 1$.

• Let $W_2 \equiv \overline{W}_2 + \sigma \widetilde{W}_2$:

$$\begin{aligned} \lambda \overline{W}_2 - \overline{W}_2 \circ T_\omega &= -\widetilde{E}_2 \\ \lambda \widetilde{W}_2 - \widetilde{W}_2 \circ T_\omega &= -\widetilde{A}_2 \ . \end{aligned}$$

Then:

$$W_1 - W_1 \circ T_\omega = -\widetilde{E}_1 - S\overline{W}_2 - (\widetilde{A}_1 + S\widetilde{W}_2)\sigma \ ,$$

where σ is chosen so that the r.h.s. has zero average, thanks to the assumption:

$$\det(\langle \widetilde{A}_1 + S\widetilde{W}_2 \rangle) \neq 0 \ .$$

Notice that:

$$W_1 - W_1 \circ T_\omega = W_1(\theta) - W_1(\theta + \omega) = \sum_k \widehat{W}_{1,k} \, e^{ik\theta}(1 - e^{ik\omega})$$

and $1 - e^{ik\omega} = 0$ for $k = 0$.

• In the nearly–integrable case the compatibility condition gives a relation between ω and c of the form $c = (1-\lambda)\omega \ (1 + O(\varepsilon^2))$.

• **Norms.**

Definition. Given $\rho > 0$, we denote by $\mathbb{T}^n_\rho$ the set

$$\mathbb{T}^n_\rho = \{z \in \mathbb{C}^n/\mathbb{Z}^n : \ |Im(z_i)| \leq \rho\} \ .$$

For $k \in \mathbb{N}$ let $\mathcal{A}_{\rho,k}$ the set of analytic functions in the interior of $\mathbb{T}^n_\rho$ and continuous with its k derivatives in $\mathbb{T}^n_\rho$ with the norm:

$$\|f\|_{\mathcal{A}_{\rho,k}} = \max_{0\leq i\leq k} \ \sup_{z\in\mathbb{T}^n_\rho} \ |D^i f(z)| \ . \tag{5}$$

Given $m > 0$ and denoting the Fourier series of $f = f(z)$ as $f(z) = \sum_{k\in\mathbb{Z}^n} \widehat{f}_k \exp(2\pi i k z)$, define H^m as

$$H^m = \Big\{ f : \mathbb{T}^n \to \mathbb{C} \ : \ \|f\|^2_m \equiv \sum_{k\in\mathbb{Z}^n} |\,\widehat{f}_k\,|^2 (1 + |k|^2)^m < \infty \Big\} \ . \tag{6}$$

Advantages of using Sobolev norms are that they apply to mappings with finite regularity and that all estimates in the proof can be followed by a straightforward numerical implementation.

- **Homological equation.** Let $\varphi : \mathbb{T}^n \to \mathbb{C}$, $\eta : \mathbb{T}^n \to \mathbb{C}$:

$$\varphi(\theta + \omega) - \lambda\varphi(\theta) = \eta(\theta) \ ,$$

where $\lambda \in \mathbb{C}$, $\omega \in \mathbb{R}^n$ are given.

Lemma. Let $|\lambda| \in [A, A^{-1}]$ for $0 < A < 1$, ω diophantine, $\eta \in \mathcal{A}_{\rho,k}$, $\rho > 0$, $k \in \mathbb{Z}$ (resp. $\eta \in H^m$, $m \geq \tau$) and

$$\int_{\mathbb{T}^n} \eta(\theta)\, d\theta = 0 \ .$$

Then, there is only one solution up to addition of a constant and if we choose it such that $\int_{\mathbb{T}^n} \varphi(\theta) d\theta = 0$, then we have ($C$ is a constant):

$$\begin{aligned} \|\varphi\|_{\mathcal{A}_{\rho-\delta,k+\ell}} &\leq C\delta^{-\tau-\ell}\frac{1}{\nu}\|\eta\|_{\mathcal{A}_{\rho,k}} \ , \qquad 0 < \ell < k \ , \\ \|\varphi\|_{H^{m-\tau}} &\leq C\frac{1}{\nu}\|\eta\|_{H^m} \ . \end{aligned}$$

• **Main Theorem.** Let ω be Diophantine and f_c be a family of conformally symplectic mappings. Let $K_0 : \mathbb{T}^n \to \mathcal{M}$, $c_0 \in \mathbb{R}^n$ be such that

$$f_{c_0} \circ K_0 - K_0 \circ T_\omega = E \ , \qquad f_{c_0}^* \Omega = \lambda_0 \Omega \ .$$

Assume that the compatibility condition holds:

$$\det(\langle \widetilde{A}_1 + S\widetilde{W_2} \rangle) \neq 0 \ . \qquad (7)$$

Then, in the analytic and in the Sobolev setting, there exists K_e, c_e, such that

$$f_{c_e} \circ K_e - K_e \circ T_\omega = 0$$

and

$$||K_e - K_0||_{\mathcal{A}_{\rho-\delta,k}} \leq C\nu^{-2}\delta^{-2\tau-1}||E||_{\mathcal{A}_{\rho,k}} \ , \qquad |c_e - c_0| \leq C\nu^{-2}\delta^{-2\tau-1}||E||_{\mathcal{A}_{\rho,k}}$$

or

$$||K_e - K_0||_{H^{m+\tau-1}} \leq C\nu^{-2}||E||_{H^m} \ , \qquad |c_e - c_0| \leq C\nu^{-2}||E||_{H^m} \ .$$

- **Uniqueness Theorem.** Let (K_1, c_1), (K_2, c_2) be solutions; let

$$\beta_A = \frac{\|K_2 - K_1\|_{\mathcal{A}_{\rho,k}}}{|c_2 - c_1|}, \quad \beta_H = \frac{\|K_2 - K_1\|_{H^m}}{|c_2 - c_1|} .$$

Assume the inequality

$$C\|K_2 - K_1\|_{\mathcal{A}_{\rho,k}} \left(1 + \frac{\|M\|^2_{\mathcal{A}_{\rho,k}}}{\beta_A}\right) < 1.$$

and in the Sobolev case:

$$C\|K_2 - K_1\|_{H^m} \left(1 + \frac{\|M\|^2_{H^m}}{\beta_H}\right) < 1 .$$

Then, there exists $s \in \mathbb{R}^n$ such that

$$K_2(\theta) = K_1(\theta + s) , \qquad c_1 = c_2 .$$

7. Break–down of quasi–periodic attractors

7.1 Sobolev's norms

• From the equations of the dissipative standard map, setting $\gamma = \omega(1-\lambda) - c$:

$$u(\theta+\omega) - (1+\lambda)u(\theta) + \lambda u(\theta-\omega) + \gamma = \varepsilon g(\theta + u(\theta)) \ .$$

• Close to the breakdown the smooth norms must blow up. Given the trigonometric polynomial $u^{(M)}(\vartheta) = \sum_{|k|\le M} \hat{u}_k \ e^{2\pi i k \vartheta}$, we consider the Sobolev norm defined as:

$$\|u^{(M)}\|_m = \left(\sum_{|k|\le M} (2\pi k)^m |\hat{u}_k|^2 \right)^{\frac{1}{2}} \ . \tag{8}$$

• A regular behavior of $\|u^{(M)}\|_m$ as parameters increase provides evidence of the existence of the invariant attractor; see Table showing ε_{crit} for $\omega_r = \frac{\sqrt{5}-1}{2}$.

λ	ε_{crit}
0.5	0.979215
0.9	0.972088

7.2 Periodic orbits and Arnold's tongues

• Parametric representation of the $2\pi\, p/q$–periodic orbit up to ε^{q-1}, being the order q undetermined. The drift c (equivalent to γ) must belong to an interval, say $[c_-^{(p,q)}, c_+^{(p,q)}]$, whose amplitude decreases as ε gets smaller: *Arnold's tongues* (inside the periodic orbit is stable).

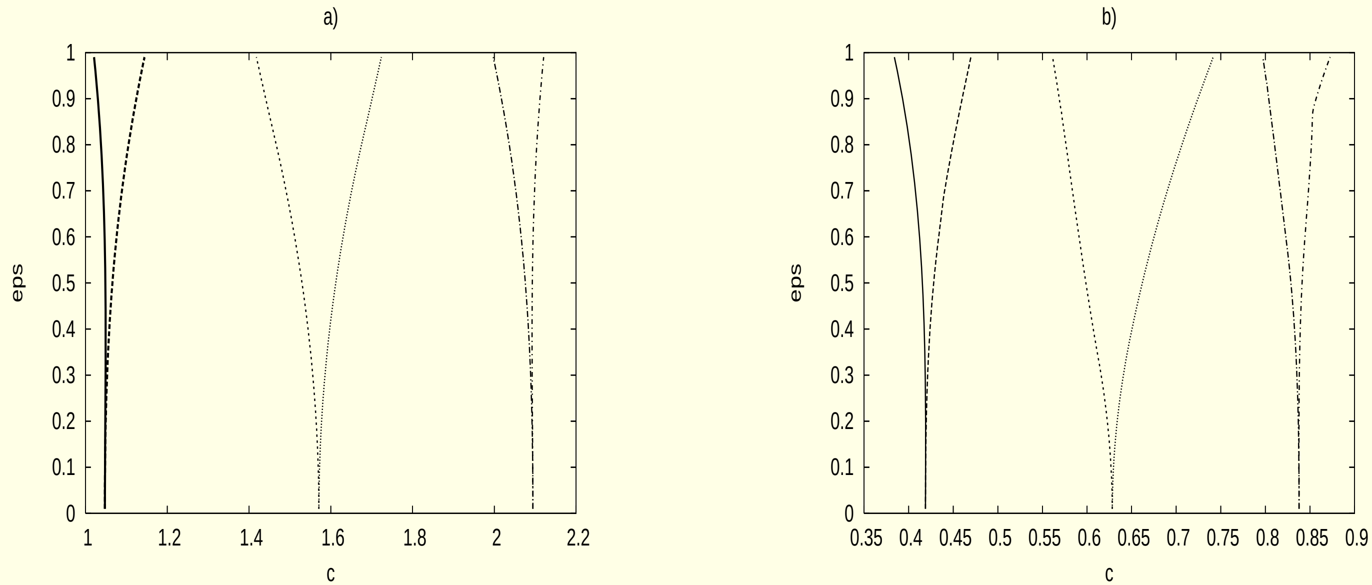

Figure 3: Arnold's tongues providing the drift c as a function of ε. Each panel shows the tongues associated to three different periodic orbits, precisely with frequencies $2\pi \cdot 1/3$ (left), $2\pi \cdot 1/2$ (center), $2\pi \cdot 2/3$ (right). a) $\lambda = 0.5$, b) $\lambda = 0.8$.

7.3 Approximation through periodic orbits

• Critical breakdown threshold: compute the sequence of periodic orbits $\mathcal{P}(\frac{p_j}{q_j}) \to \mathcal{T}(\omega)$ (analytically found through a constructive IFT).

• Fix the period $\frac{p_j}{q_j}$; for each set (ε, λ) there exists a whole interval of the drift parameter c corresponding to that periodic orbit.

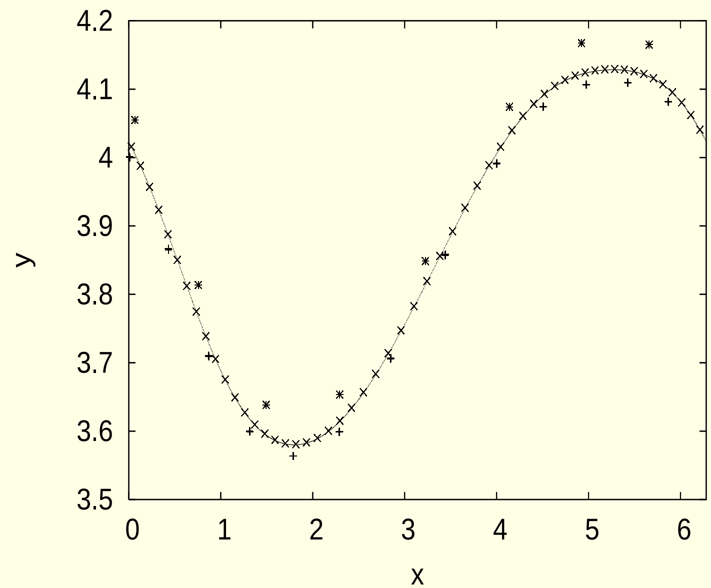

Figure 4: For $\lambda = 0.9$ and $\varepsilon = 0.5$ invariant attractor with frequency ω_r and periodic orbits: 5/8 ($*$), 8/13 ($+$), 34/55 ($\times$).

• Conservative standard map: the breakdown of an invariant curve is strictly related to the stability character of the approximating periodic orbits.

• Dissipative case: for fixed values of the parameters there is a whole interval of c which admits p/q–periodic orbits.

Select one of these periodic orbits and evaluate its stability by computing the monodromy matrix along a full cycle of the periodic orbit.

Let $\varepsilon_{p,q}(\lambda)$ the maximal value of $\varepsilon \in [0,1)$ for which the periodic orbit is stable.

• The results show that the stability value seems to decrease toward a given threshold as the order of the periodic approximant is increased, thus defining a breakdown threshold of the invariant attractor consistent with Sobolev's norms.

Table 1: Stability threshold $\varepsilon_{p,q}(\lambda)$ of the periodic orbits approximating $\omega_r = \frac{\sqrt{5}-1}{2}$.

p_j/q_j	$\varepsilon^{\omega_r}_{p,q}(\lambda = 0.9)$ $\varepsilon_{Sob} = [0.972]$	$\varepsilon^{\omega_r}_{p,q}(\lambda = 0.5)$ $\varepsilon_{Sob} = [0.979]$
$1/2$	0.999	0.999
$2/3$	0.999	0.999
$3/5$	0.999	0.999
$5/8$	0.993	0.992
$8/13$	0.981	0.987
$13/21$	0.980	0.983
$21/34$	0.976	0.980
$34/55$	0.975	0.979
$55/89$	0.974	0.979

8. Applications to Celestial Mechanics

☞ ROTATIONAL DYNAMICS: quasi–periodic attractors for the spin–orbit problem.

• Model: satellite $\mathcal{S}$, ellipsoid rotating about an internal spin–axis and revolving around a central body $\mathcal{P}$:

(i) S moves on a Keplerian orbit;

(ii) the spin–axis coincides with the smallest physical axis (principal rotation);

(iii) the spin–axis is perpendicular to the orbital plane (zero obliquity);

(iv) dissipative forces: tidal torque $\mathcal{T}$ depending linearly on the angular velocity of rotation.

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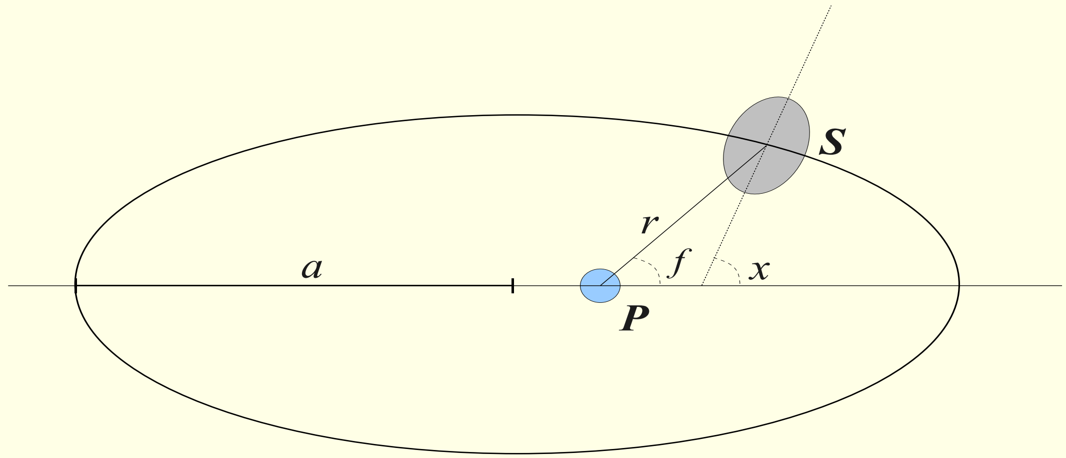
• Notation:

$A < B < C$ principal moments of inertia; $n = \frac{2\pi}{T_{rev}} \equiv 1$ mean motion; a semimajor axis; r orbital radius; f true anomaly; x angle between pericenter line and major axis of the ellipsoid.



$$\ddot{x} + \frac{3}{2}\,\frac{B-A}{C}\left(\frac{a}{r}\right)^3 \sin(2x-2f) = -\mu[\dot{x}-\eta]\ . \tag{9}$$

- $\varepsilon \equiv \frac{3}{2}\frac{B-A}{C}$ perturbing parameter; Moon–Mercury: $\varepsilon \simeq 10^{-4}$;
- r and f are known Keplerian functions of the time;
- Tidal torque T due to internal non–rigidity: as in [Correia–Laskar] average over one orbital period ($K \simeq 10^{-8}$ for Moon–Mercury):

$$\langle T\rangle = -\mu(e,K)\Big[\dot{x}-\eta(e)\Big]\ ,$$

with

$$\mu(e,K) = K\,\frac{1+3e^2+\frac{3}{8}e^4}{(1-e^2)^{9/2}}\ ,\quad \eta(e) = \frac{1+\frac{15}{2}e^2+\frac{45}{8}e^4+\frac{5}{16}e^6}{(1+3e^2+\frac{3}{8}e^4)(1-e^2)^{3/2}}\ .$$

• We are led to consider the following equation of motion for the dissipative spin–orbit problem:

$$\ddot{x} \ + \varepsilon V_x(x,t) = -\mu[\dot{x} - \eta] \ . \tag{10}$$

• The tidal torque vanishes provided

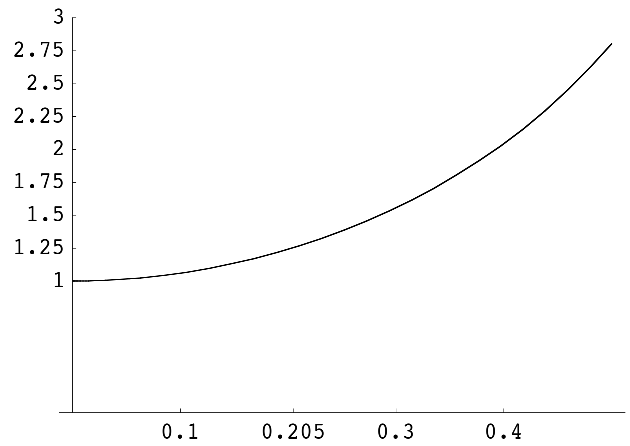
$$\dot{x} \equiv \eta(e) = \frac{1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6}{(1-e^2)^{\frac{3}{2}}(1 + 3e^2 + \frac{3}{8}e^4)} \ .$$

• It is readily shown that for circular orbits the angular velocity of rotation corresponds to the synchronous resonance, being $\dot{x} = 1$. For Mercury's eccentricity $e = 0.2056$, it turns out that $\dot{x} = 1.256$.

☞ SMD corresponds to the Poincarè map at times 2π, obtained integrating the spin–orbit problem with a leap–frog method.

☞ Mercury and the quasi–periodic attractors.

QUESTION: Why Mercury is not trapped in a 1:1 resonance? The frequency of the attractor depends on the eccentricity.



Graph of $e \to \eta(e)$

☞ By the Theorem there exist a quasi–periodic attractor associated to $e = 0.2056$ with $\omega \simeq \eta(e) + O(10^{-8}) \simeq 1.256$; the 3:2 periodic orbit is above, thus Mercury is trapped above.

☞ In particular, $\omega = 1.5$ for $e = 0.285$ compatible with the variations induced by perturbations on Mercury's eccentricity.

☞ Interplay periodic/quasi–periodic attractors.

• Frequencies of the attractors versus percentage of attracted initial data: integrating 10^3 random initial conditions (with $\varepsilon = 10^{-3}$, $K = 10^{-4}$), compute the occurrences of the attractors.

• For e small the 1/1 resonance dominates; increasing e there appear 3/2, 2/1, 5/2 and the natural frequency $\eta = \eta(e)$; for example $\eta(0.284927) = 1.5$.

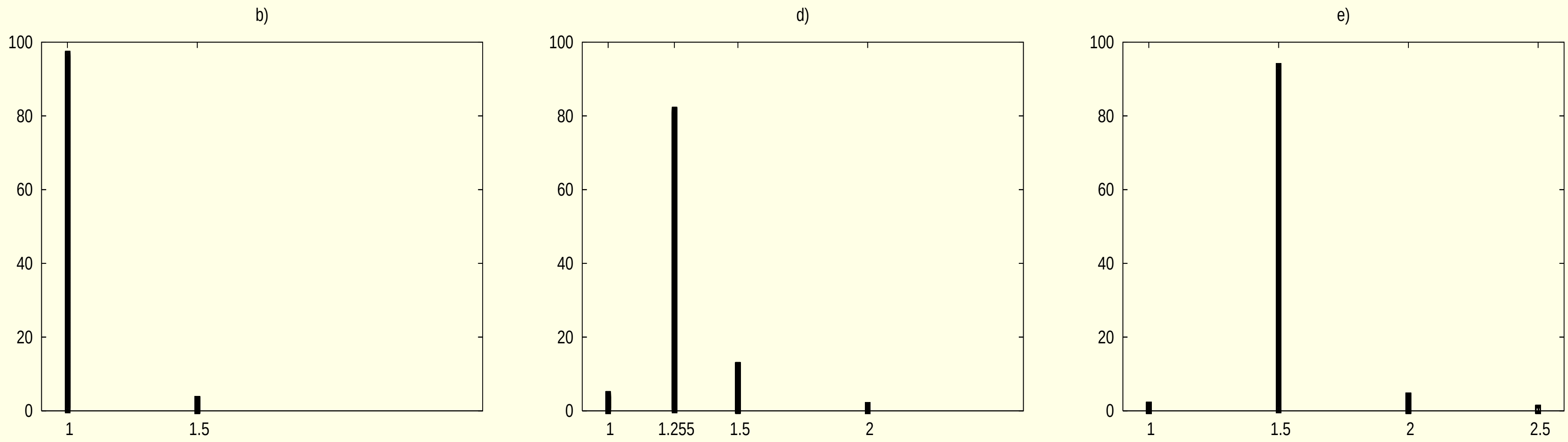


Figure 5: x–axis: frequency. y–axis: percentage of attracted initial data. $e = 0.0549$, $e = 0.2056$, $e = 0.284927$.

☞ ORBITAL DYNAMICS: Dissipative 3BP, equations of motion

• Planar, circular, restricted 3–body problem with dissipation. Masses of the primaries m, $1-m$. Coordinates in a synodic reference frame: $P=(x,y)$, $P_1=(-m,0)$, $P_2=(1-m,0)$. Equations of motion in a synodic frame:

$$\begin{aligned}\ddot{x} &= 2\dot{y}+x-(1-m)\frac{x+m}{r_1^3}-m\frac{x-1+m}{r_2^3}+F_x \\ \ddot{y} &= -2\dot{x}+y-(1-m)\frac{y}{r_1^3}-m\frac{y}{r_2^3}+F_y\ ,\end{aligned}$$

where $r_1^2=(x+m)^2+y^2$, $r_2^2=(x-1+m)^2+y^2$, $K=$ dissipative constant,

$$\begin{aligned}(F_x,F_y) &= -K(\dot{x}-y,\dot{y}+x) && (linear) \\ (F_x,F_y) &= -K(\dot{x}-y+\alpha\Omega y,\dot{y}+x-\alpha\Omega x) && (Stokes) \\ (F_x,F_y) &= -\frac{K}{{r_1}^2}(\dot{x}-y,\dot{y}+x) && (PR)\ .\end{aligned}$$

$\Omega=\Omega(r)\equiv r^{-3/2}$ is the Keplerian angular velocity at $r=\sqrt{x^2+y^2}$ from the origin of the synodic frame, $\alpha\in[0,1)$ ratio between the gas and Keplerian velocities.

☞ ORBITAL DYNAMICS: Dissipative 3BP, qualitative dynamics

• Use regularized equations: Levi–Civita transformation and introduction of the fictitious time to regularize collisions with the primaries.

• Periodic attractors are found in the linear and Stokes drag models; they do not occur within the PR–drag model $\rightarrow$ one needs to modify the model $\rightarrow$ add a third primary (e.g., Saturn–like), whose action balances that of the two main primaries.

• Global behavior of the dynamics through FLI: the norm of the tangent vector over a finite interval of time is an indicator of the dynamical character of a trajectory.

• The dissipation decreases the semi–major axis of orbits that collide with one of the primaries, even on very short time scales; a large fraction of initial conditions in the interior region collides with the Sun. In the exterior region, collisions with Jupiter. Increasing the dissipation or the integration time, a larger fraction of orbits ends on one of the primaries. The effect of the PR–drag is faster than Stokes and linear.

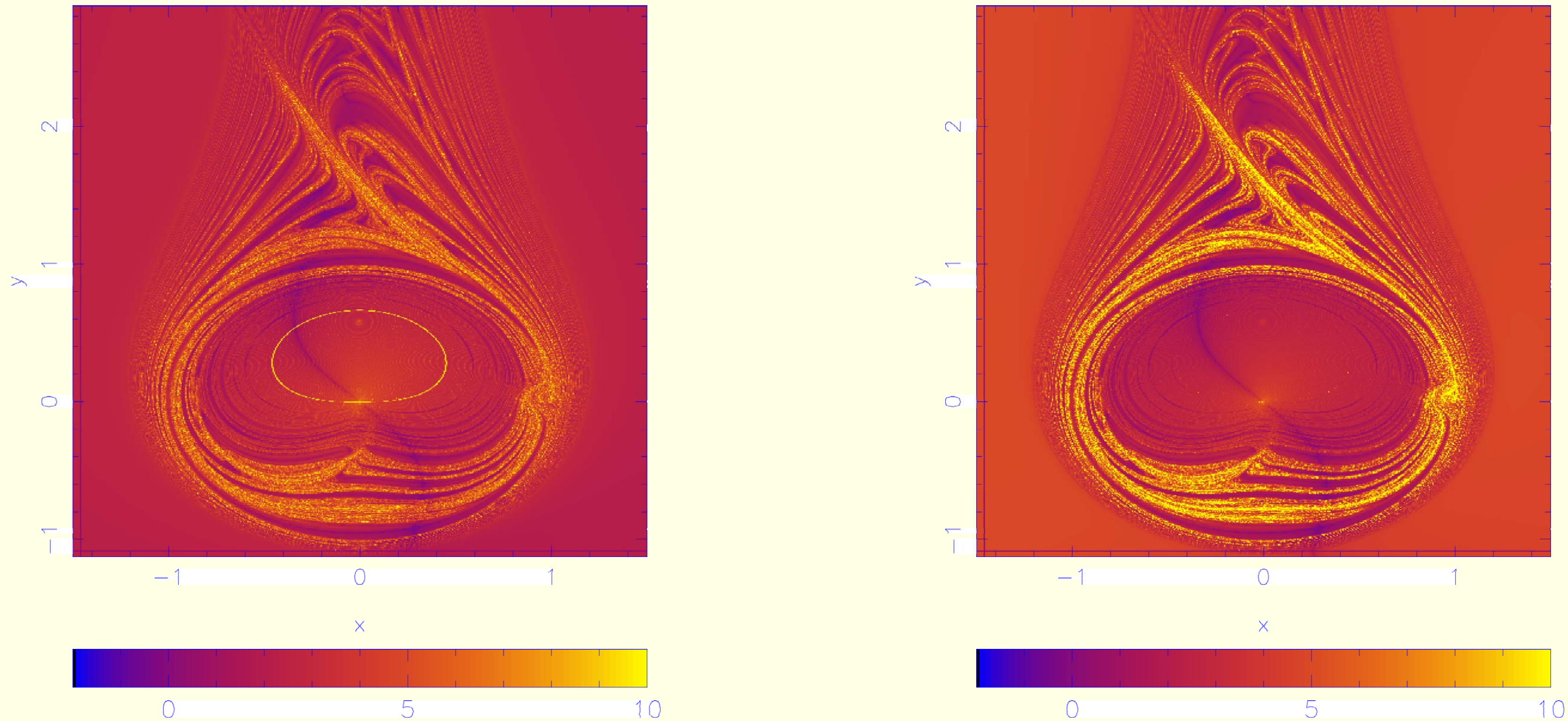


Figure 6: FLI map for the conservative case ($\dot{y}(0) = 0$; $\dot{x}(0)$ is computed from the Jacobi constant $C = 2.99047$ corresponding to the Lagrangian point L_4). Left: non regularized equations; right: regularized equations. Periodic (blue to red), quasi–periodic (orange), chaotic (yellow), collision orbits (black).

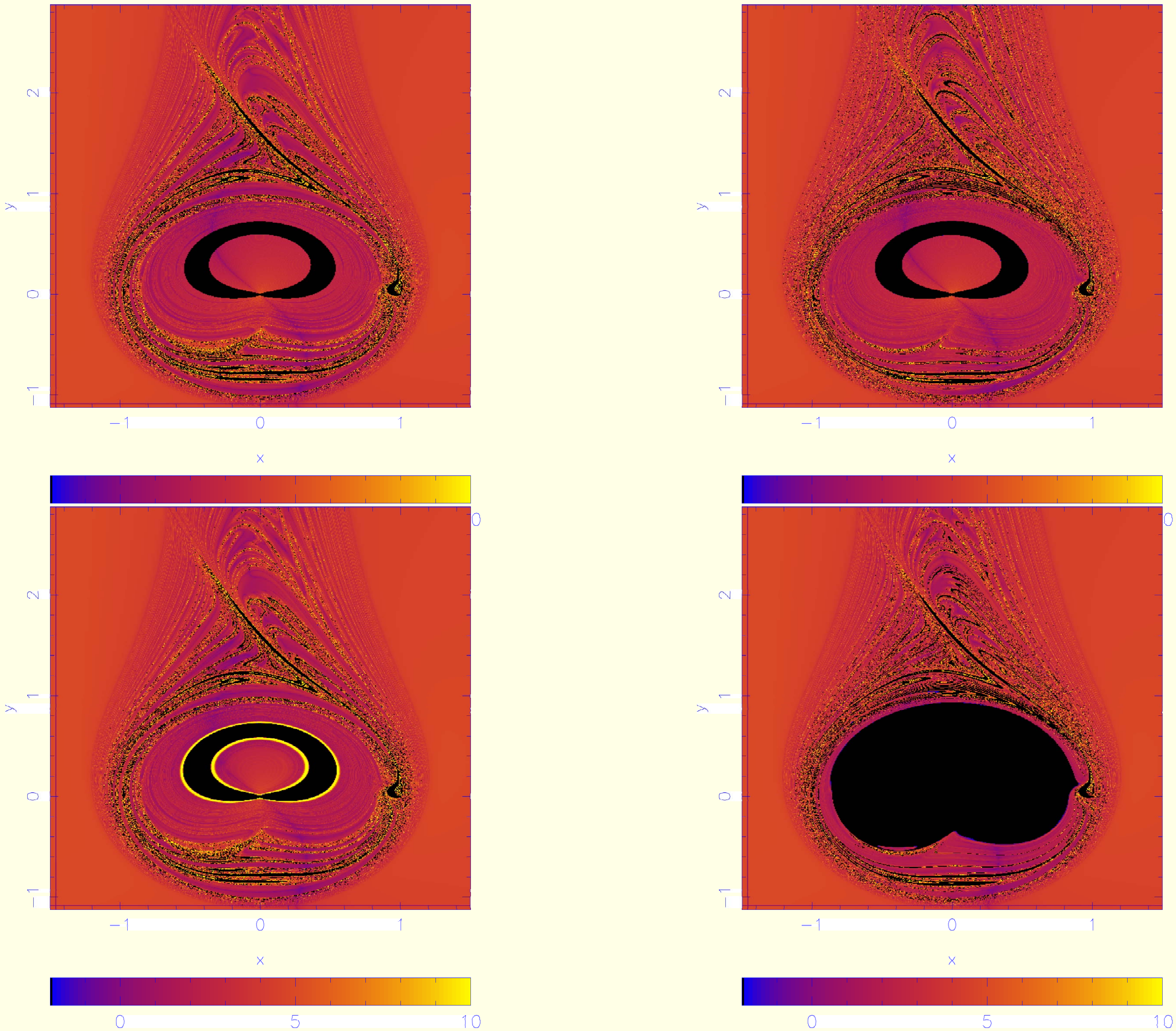


Figure 7: FLI map for the Stokes drag case with $\alpha = 0.995$ (top) and for the PR drag (bottom) with $T = 100$. Left: $K = 10^{-5}$, right: $K = 10^{-3}$.

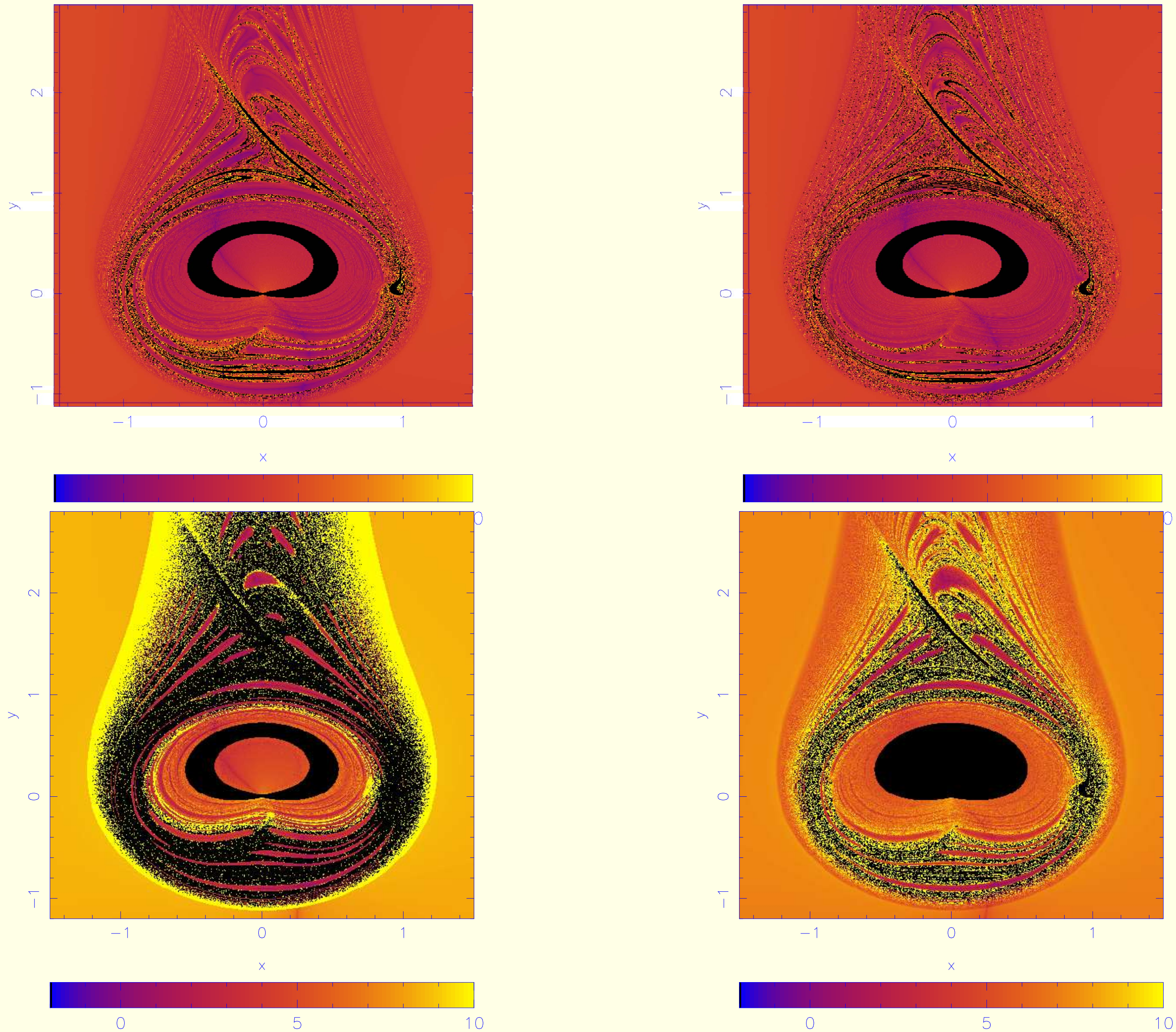


Figure 8: FLI map for the Stokes case with $\alpha = 0.995$. Top: $T = 100$, bottom: $T = 5000$. Left: $K = 10^{-5}$, right: $K = 10^{-3}$.

9. Nekhoroshev's theorem

• Original version of Nekhoroshev's theorem formulated under the *steepness* condition, later relaxed to the convex/quasi–convex hypotheses (J. Pöschel):

$$\mathcal{H}(y,x) = h(y) + \varepsilon f(y,x)\ , \qquad y \in Y \subset \mathbb{R}^n\ , \quad x \in \mathbb{T}^n\ . \tag{11}$$

• Define a **complex neighborhood** of $Y \times \mathbb{T}^n$ as $V_{r_0}Y \times W_{s_0}\mathbb{T}^n$, where

▷ $V_{r_0}Y =$ complex nbh. of radius r_0 around Y w.r.t. Euclidean norm $\|\cdot\|$

▷ $W_{s_0}\mathbb{T}^n \equiv \{x \in \mathbb{C}^n\ :\ \max_{1\leq j\leq n} |\mathrm{Im}\ x_j| < s_0\}$.

• Let $U_{r_0}Y \equiv V_{r_0}Y \cap \mathbb{R}^n$ be the real neighborhood of Y.

• For an analytic function on $V_{r_0}Y \times W_{s_0}\mathbb{T}^n$ with Fourier expansion $u(y,x) = \sum_{k\in\mathbb{Z}^n} \hat{u}_k(y) e^{ik\cdot x}$, let the **norm**

$$\|u\|_{Y,r_0,s_0} \equiv \sup_{y\in V_{r_0}Y} \sum_{k\in\mathbb{Z}^n} |\hat{u}_k(y)|\ e^{(|k_1|+\ldots+|k_n|)s_0}\ .$$

- **Definition.** Given $m>0$, then $h(y)$ is said m–**convex** if

$$(h''(y)v,v)\geq m\|v\|^2 \qquad \text{for all } v\in\mathbb{R}^n\ , \quad \text{for all } y\in U_{r_0}Y\ .$$

Given m, $\ell>0$, the unperturbed Hamiltonian is said m,ℓ–**quasi–convex** if for any $y\in U_{r_0}Y$ one of the following inequalities holds for any $v\in\mathbb{R}^n$:

$$|(\omega(y),v)|>\ell\|v\|\ , \qquad (h''(y)v,v)\geq m\|v\|^2\ . \tag{12}$$

Nekhoroshev's theorem [Pöschel]. Assume h is quasi–convex with $\sup_{y\in V_{r_0}Y}\|h''(y))\|\leq M$. Let $r_0\leq\frac{4\ell}{m}$, $A\equiv\frac{11M}{m}$, $\varepsilon_0\equiv\frac{mr_0^2}{2^{10}A^{2n}}$; if for $s_0>0$, one has $\|f\|_{Y,r_0,s_0}\varepsilon\leq\varepsilon_0$, then for any initial condition $(y_0,x_0)\in Y\times\mathbb{T}^n$ the following estimates hold:

$$\|y(t)-y_0\|\leq\frac{r_0}{A}\Big(\frac{\varepsilon}{\varepsilon_0}\Big)^{\frac{1}{2n}} \qquad \text{for } |t|\leq\frac{A^2s_0}{\Omega_0}\ e^{\frac{s_0}{6}(\frac{\varepsilon_0}{\varepsilon})^{\frac{1}{2n}}}\ ,$$

where $\Omega_0\equiv\sup_{\|y-y_0\|\leq\frac{r_0}{A}}\|\omega(y)\|$.

- The proof is based on three main ingredients: the construction of a suitable normal form, the use of the convexity/quasi–convexity, the analysis of the geography of the resonances.

- Why quasi–convexity is necessary?

- **Counterexample** to Nekhoroshev theorem if quasi-convexity is violated:

$$\mathcal{H}(y_1,y_2,x_1,x_2)=\frac{y_1^2}{2}-\frac{y_2^2}{2}-\varepsilon\sin(x_1+x_2)\ ;$$

Hamilton's equations:

$$\begin{aligned}\dot y_1 &= \varepsilon\cos(x_1+x_2) \qquad & \dot y_2=\varepsilon\cos(x_1+x_2)\\ \dot x_1 &= y_1 & \dot x_2=-y_2\ .\end{aligned}$$

A solution is

$$x_1(t)=-x_2(t)=x_0+y_0t+\frac{1}{2}\varepsilon t^2\ , \qquad y_1(t)=y_2(t)=y_0+\varepsilon t\ ;$$

Nekhoroshev's exponential estimates do not hold.

10. Effective estimates in Celestial Mechanics

• The stability estimates provided by Nekhoroshev's theorem are particularly relevant in Celestial Mechanics. In fact, they can be used to provide bounds on the elliptic elements for an exponentially long time, possibly comparable with the age of the solar system, namely 5 billion years.

• Effective estimates have been developed for

♢ the three–body problem;

♢ the triangular Lagrangian points;

♢ the resonant D'Alembert problem;

♢ the perturbed Euler rigid body.

11. Exponential estimates for dissipative systems

• Consider the following n–dimensional, time–dependent vector field

$$\begin{aligned} \dot{x} &= \omega(y) + \varepsilon h_{10,y}(y,x,t) + \mu f_{01}(y,x,t) \\ \dot{y} &= -\varepsilon h_{10,x}(y,x,t) + \mu\left(g_{01}(y,x,t) - \eta(y,x,t)\right)\ , \end{aligned} \tag{13}$$

where $y \in \mathbb{R}^n$, $(x,t) \in \mathbb{T}^{n+1}$, $\varepsilon \in \mathbb{R}_+$, $\mu \in \mathbb{R}_+$, ω and η are real–analytic, h_{10}, f_{01}, g_{01} are periodic, real–analytic.

Having fixed $y_0 \in \mathbb{R}^n$, denote by $A \subset \mathbb{R}^n$ an open neighborhood of y_0.

• Assume that for some $K \in \mathbb{Z}_+$ the vector function $\omega = \omega(y)$ satisfies the *non–resonance condition*

$$|\omega(y)\cdot k + m| > 0 \quad \text{for all } y \in D \subset A\ , \quad (k,m) \in \mathbb{Z}^{n+1}\backslash\{0\}\ , |k| + |m| \le K\ . \tag{14}$$

Theorem. *Consider the vector field* (13) *defined on* $A \times \mathbb{T}^{n+1}$ *and let* $y \in D$ *be such that* $\omega = \omega(y)$ *satisfies* (14)*. Let* $\rho_0 > 0$, $\tau_0 > 0$*; there exist* $\varepsilon_0 > 0$, $\mu_0 > 0$*, such that for any* $\varepsilon \leq \varepsilon_0$, $\mu \leq \mu_0$*:*

$$\|y(t) - y(0)\| \leq \rho_0 \qquad \text{for } |t| \leq C_t e^{K\tau_0} \ ,$$

for some positive constant C_t*.*

• Proof: is based on the construction, up to an optimal order N, of a double coordinate change of variables:

$$(X, Y) = \Delta_d^{(N)} \Delta_c^{(N)}(x, y) \ ,$$

where we refer to $\Delta_c^{(N)}$ as the conservative transformation and to $\Delta_d^{(N)}$ as the dissipative transformation.

• The proof can be extended to the resonant case.

• The drift $\eta = \eta(y, x, t)$ must be chosen in order that the compatibility conditions (required to solve the homological equations) are satisfied.

12. Conclusions and perspectives

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- Role of the dissipation in the selection of the resonances (capture into resonance or separatrix crossing).