Orbit determination

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CONTENT

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- Given a number of observations of a celestial body, determine its orbit

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♢ Leonhard Euler (Basel 1707 – St. Petersburg 1783)

♢ Giuseppe Piazzi (Ponte in Valtellina 1746 – Napoli 1826)

♢ Pierre Simon Laplace (Beaumont-en-Auge 1749 – Paris 1827, France)

♢ Carl Friedrich Gauss (Brunswick 1777 – Göttingen 1855, Germany)

♢ Ottaviano Fabrizio Mossotti (Novara 1791– Pisa 1863, Italy).

Titius–Bode law

- Johann Daniel Tietz – Titius (1729–1796), Johann Elert Bode (1747–1826) found a relation for computing the semimajor axes of the planets:

$$a_n = 0.4 + 0.3 \cdot 2^n \ AU \ ,$$

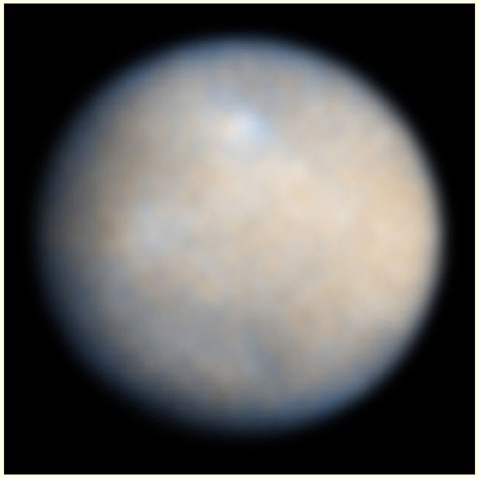
where Mercury: $n = -\infty$, Venus: $n = 0$, Earth: $n = 1$, Mars: $n = 2$, ???? (another planet?): $n = 3$, Jupiter: $n = 4$, Saturn: $n = 5$.

Planet	Titius–Bode Law (AU)	Distance (AU)
Mercury	0.4	0.39
Venus	0.7	0.72
Earth	1	1
Mars	1.6	1.52
Asteroids	2.8	2.77
Jupiter	5.2	5.20
Saturn	10	9.54
Uranus	19.6	19.19
Neptune	38.8	30.07

Ceres discovery

- Piazzi: on the night of 1 January 1801 a new, faint object is observed from the observatory of Palermo
- 19 observations over 42 days, but nothing after 12 February 1801
- In total the observed object made an arc of only 9^o on the celestial sphere
- Piazzi thought a small comet without tail, or a planet within Mars and Jupiter (remember Titius–Bode law)
- Franz von Zach (director of the Seeberg Observatory) published a "call for help" on the "Monatliche Correspondenz" (Monthly Correspondence). In the September issue Piazzi's observations were finally published
- Gauss (24 years old): in a little more than a month, he found a method to determine the orbit and to compute the ephemerides
- Using these ephemerides, von Zach observed Ceres on December 7, but stopped due to bad weather

- On 31 December 1801, von Zach and then Olbers (two days later) independently confirmed the recovery of Ceres; using Gauss' words, they were able to "restore the fugitive to the observations"
- Piazzi named the object **Ceres Ferdinandea** in honour of the roman goddess of grain and of King Ferdinand IV of Naples and Sicily. The Ferdinandea part was later dropped for political reasons.
- Ceres is the first asteroid discovered in the asteroidal belt (diameter 974 km, $a = 2.76$ AU, $e = 0.079$, $i = 10.58^o$)
- Ceres has been upgraded in 2006 to a dwarf planet

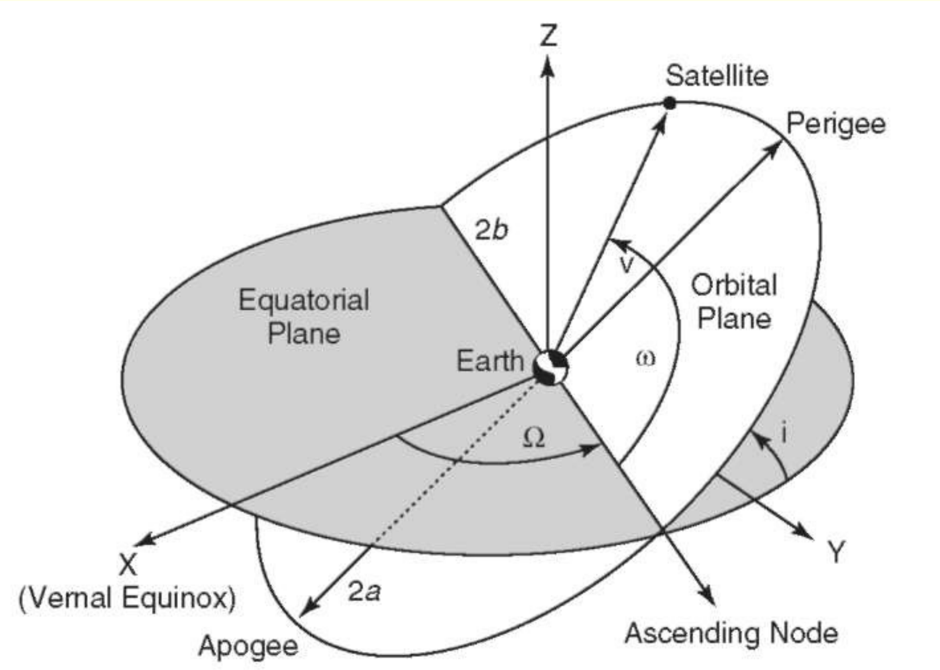


Elliptic elements

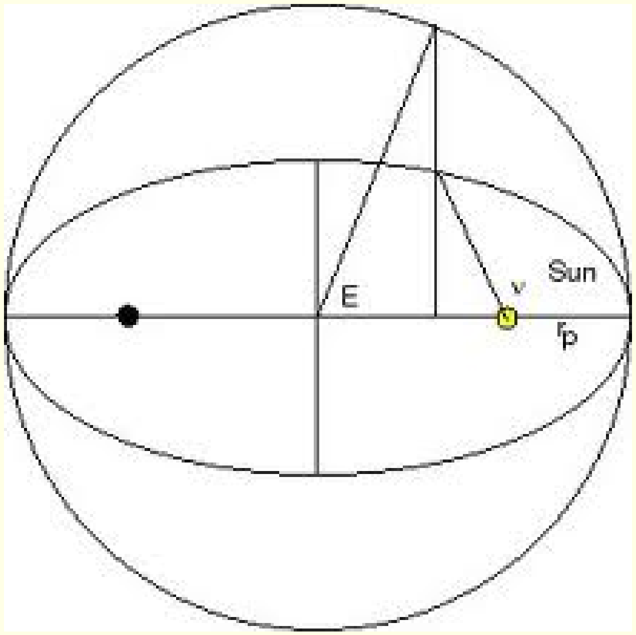
The motion of a celestial body is determined by means of 6 elliptic elements:

i **inclination,** $0 \le i < \pi$, Ω **longitude of the ascending node,** $0 \le \Omega < 2\pi$, ω **argument of perihelion,** $0 \le \omega < 2\pi$

a **semimajor axis** or p parameter, e **eccentricity,** M **mean anomaly** (v is the true anomaly)



- v is the **true anomaly**
- E is the **eccentric anomaly** (draw the circle with radius equal to a and connect the perpendicular across the satellite to the center of the ellipse)
- M is the **mean anomaly,** the angle that would have span by the satellite if the ellipse is run with uniform velocity, $M(t) = nt + M(0)$ with $n = \frac{2\pi}{T_{rev}}$



- Kepler's equation: $M = E + e\sin(E)$
- $r = a(1 - e\cos(E))$, $\operatorname{tg}\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\ \operatorname{tg}\frac{E}{2}$

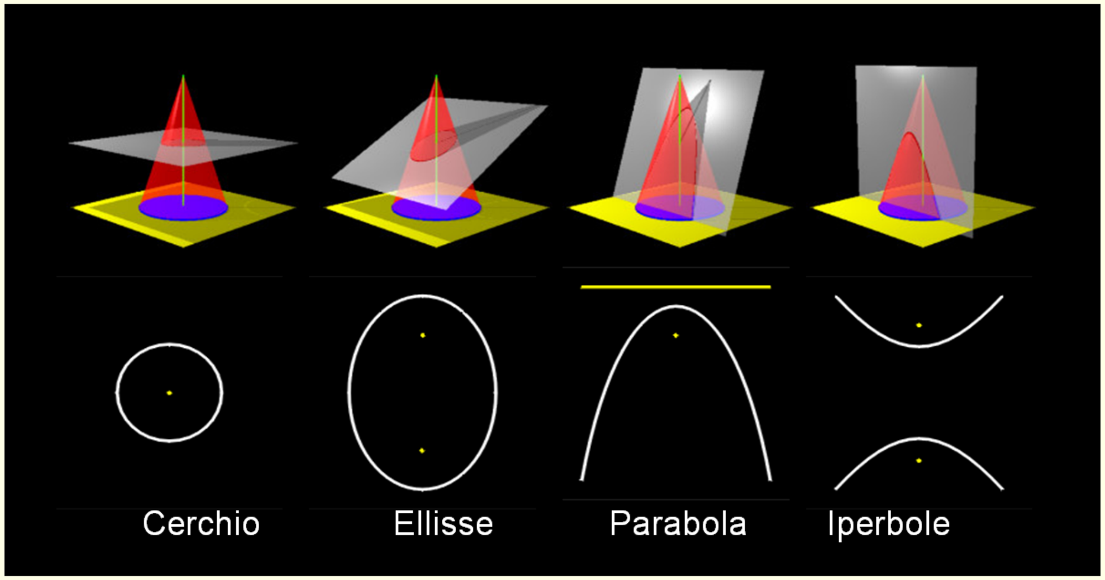
Two–body problem

- Simplified model for the study of of the dynamics; one considers only 2 celestial bodies (with masses M, m at distance d) interacting under Newton's law of gravitation:

$$F = -G\ Mm/d^2\ .$$

Example: Sun–Earth, Sun–Jupiter, Earth–Moon, Sun–asteroid, binary stars

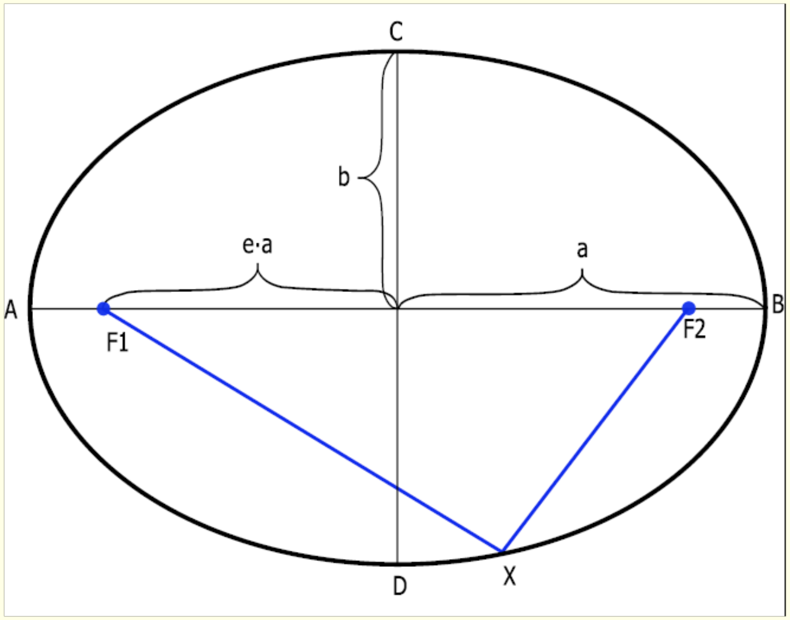
- The solution is a conic section: ellipse, hyperbole, parabola.



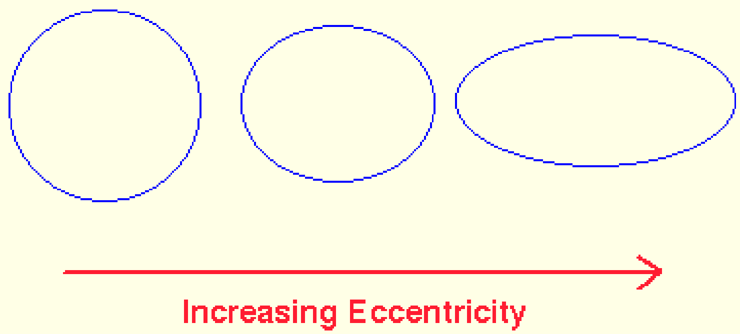
Kepler's laws (1)

- Three laws providing the solution of the two–body problem: all planets move along ellipses, they are faster close to the Sun, their period increases with the distance from the Sun.

- Ellipse: locus of points such that the sum of the distances from two fixed points F_1, F_2, called foci, is constant. The ellipse is characterized by the semimajor axis a and by the eccentricity e; b is the seminor axis.



- Different eccentricities: $e = 0$ circle, $0 < e < 1$ ellipse, $e = 1$ parabola, $e > 1$ hyperbole.
- Earth: 0.017, Jupiter: 0.048, Mercury: 0.206, Neptune: 0.008, Pluto: 0.249, Moon: 0.055, Europa: 0.0094, Tarvos: 0.5309



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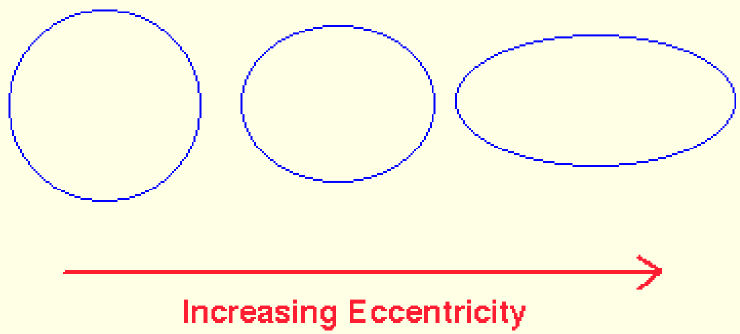
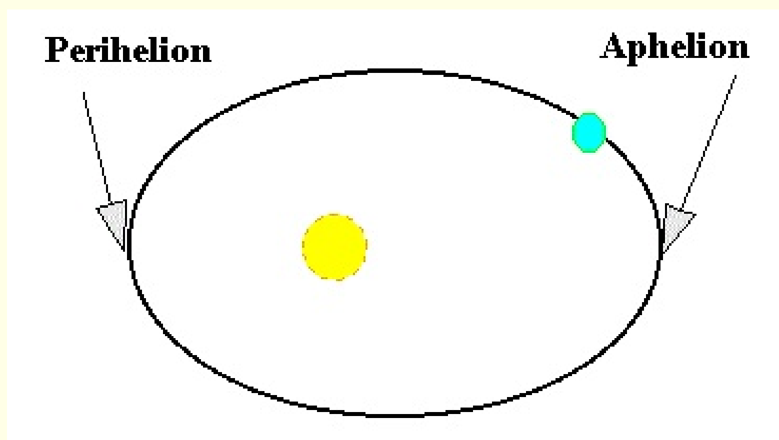


- **Perihelion** = closer to the Sun, **Aphelion** = farther from the Sun



Kepler's laws (2)

- Assume that a small body C (with zero–mass) orbits the Sun; let $\vec{r}_1, \vec{r}_2, \vec{r}_3 =$ be the positions of C w.r.t Sun at t_1, t_2, t_3
- $n = \frac{2\pi}{T_{rev}}$ mean motion
- $M = M(t) = nt + M(0) = nt =$ mean anomaly (assume $M(0) = 0$)

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$* *$

$I\ law$: $\vec{r}_1, \vec{r}_2, \vec{r}_3$ are coplanar and determine a conic section with parameter p and eccentricity e ($a = p/(1-e^2)$)

$II\ law$: The areal velocity is constant: $\dot{A}(t) = \frac{p^2}{(1-e^2)^{3/2}} n$

$III\ law$: $n^2 a^3 = constant$

$* *$

Remark: II, III laws $\Rightarrow A(t) = \sqrt{p}\ t$

Ceres–Gauss problem

- How can we compute the elliptic elements starting from observations from Earth? How many observations do we need?

- The orbit is determined by 6 elements: $(p, e, \omega, \Omega, i, M)$. Therefore we need 3 observations of 2 quantities, e.g. longitude and latitude, right ascension and declination

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- Difficulties:

◇ rotation of the Earth and revolution around the Sun

◇ equinox precession

◇ aberration due to the velocity of the observer

◇ nutation

◇ observational errors (telescope, clock, etc.)

◇ observational times are close (within the same night or spaced by few days)

- For $t = t_1, t_2, t_3$, let for $k = 1, 2, 3$:

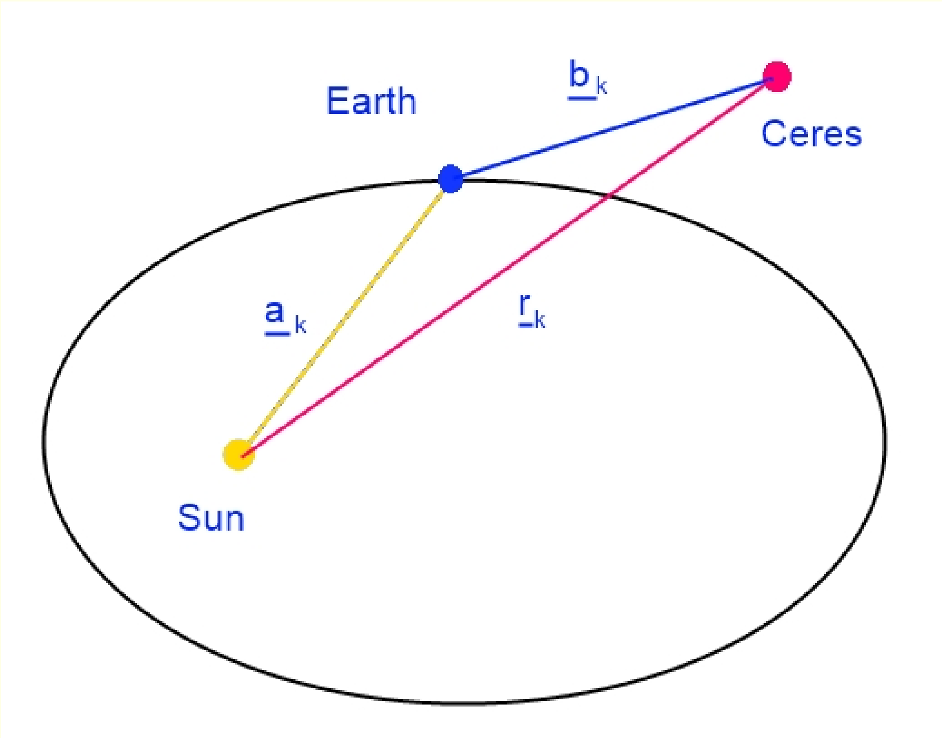
$\vec{a}_k$: position of the observer w.r.t. Sun (known from ephemerides)

$\vec{b}_k$: unit vector of C w.r.t. Earth (known from observations)

ρ_k : **unknown** distances C to Earth

$\rho_k \vec{b}_k$: geocentric position vectors of C (ρ_k unknowns)

$\vec{r}_k = \vec{a}_k + \rho_k \vec{b}_k$: position of C w.r.t. Sun



Gauss Problem: Let the times of observations be t_1, t_2, t_3. Let us define the vectors $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$, such that $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$ are independent and $|\vec{b}_k| = 1$. Find ρ_1, ρ_2, ρ_3, such that the vectors $\vec{r}_k = \vec{a}_k + \rho_k \vec{b}_k$ are coplanar and define a conic section $\mathcal{C}$ such that, denoting by $\vec{r}(t)$ the position vector evolving on $\mathcal{C}$ according to Kepler laws from the initial datum $\vec{r}(t_2) = \vec{r}_2$, one has $\vec{r}(t_1) = \vec{r}_1, \vec{r}(t_3) = \vec{r}_3$.

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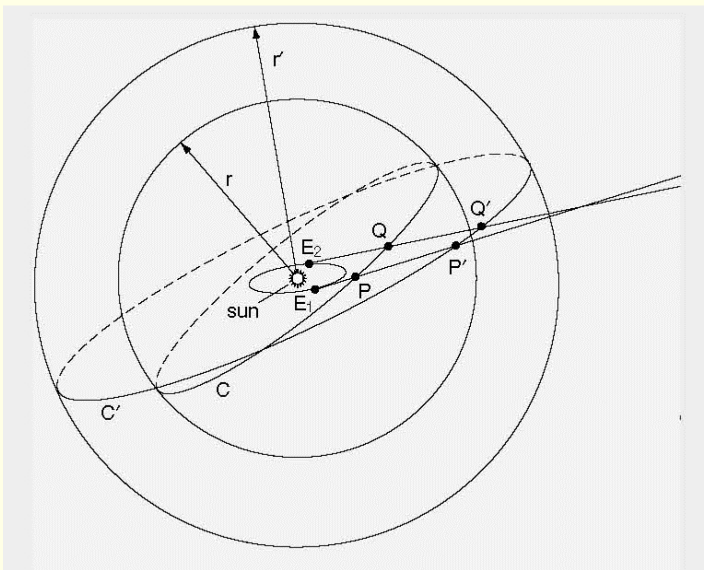
- Newton: one of the most difficult problems of mathematical astronomy
- C.F. Gauss, "Theoria motus corporum coelestium in sectionibus conicus solem ambientium" (1809, "Theory of the motion of heavenly bodies moving about the sun in conic sections")
- The method is based on (minimum number) 3 observations, it is iterative, it needs the solution of an implicit equation (> 3 use *least square method*)

"*... for it is now clearly shown that the orbit of a heavenly body may be determined quite nearly from good observations embracing only a few days; and this without any hypothetical assumption [non–zero eccentricity]*" - C.F. Gauss

Orbit determination for a circular orbit

- Orbit determination in case of (nearly) circular orbit: W. Olbers used 2 observations and computed the radius of a circular orbit which fits the 2 observations. Recall Kepler's III law relating a and T.
- The 2 observations define 2 lines of sight from the Earth's positions E_1, E_2 at times t_1, t_2. Any sphere of radius r (r') centered in the Sun intersects the lines of sight in 2 points P, Q (P', Q').
- Kepler's III law tells how large is the arc PQ in the time interval $t_2 - t_1$.



Astronomical effects

- As seen from Earth, the motion of C is determined by taking into account:

◇ the rotation of the Earth on its axis

◇ the non–uniform orbit of the Earth around the Sun

◇ the precession of the equinoxes with a period of 25 000 years $\Rightarrow$ slow drift of the rotation axis over the period of observation, larger than the precision required by Gauss (1 arc second)

◇ the nutation, i.e. the periodic shift of Earth's rotation axis, connected to the orbit of the Moon

◇ the aberration, i.e. a shift of the apparent position of C relative to the true one, due to the combined effect of the finite velocity of the light and the velocity of the observer

◇ the diffraction of light, i.e. the apparent position of C is modified by the diffraction of light in the atmosphere, which bends the rays.

Carl Friedrich Gauss (1777 – 1855)

- Born from a low-middle class family; the father was a gardener, the uncle (from the mother) was very brilliant and helped Gauss in culture; child prodigy
- In primary school the teacher asked to sum all integer numbers from 1 to 100; Gauss gave the answer within seconds. Probably he recognized that adding terms from opposite sides gives: $1 + 100 = 101$, $2 + 99 = 101$, $3 + 98 = 101$, ..., for a total sum of $50 \times 101 = 5050$
- Became famous after the discovery of Ceres; director of the observatory of Göttingen
- Contributions in analysis, number theory, statistics, differential geometry, astronomy, geodesy, magnetism, optics
- A. von Humboldt: "Who is the the greatest mathematician of Germany"; Laplace: "Pfaff"; v H.: "What about Gauss?"; L.: "Gauss is the greatest mathematician of the world!"

Overview of Gauss method

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◇ **Step 4:** Compute the elements e, p, ω

Gauss method, coplanarity: STEP 1

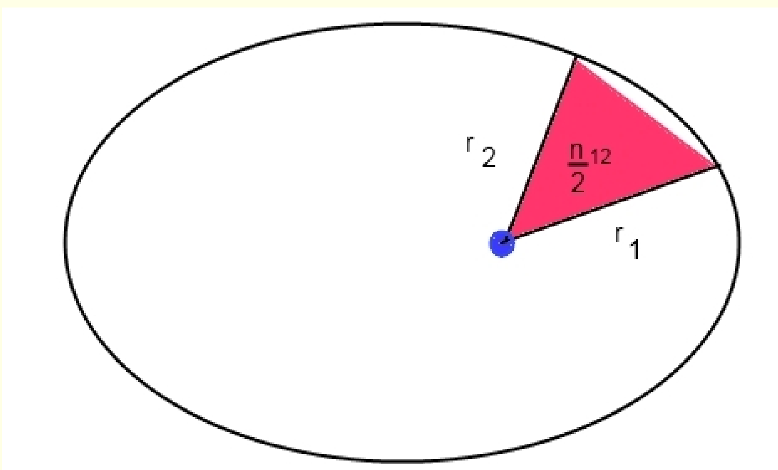
- Assume that C moves on a conic section
- Input data: geocentric longitudes and latitudes, Earth–Sun distances, ecliptical longitudes of the Earth $\Rightarrow \vec{a}_k, \vec{b}_k$ are **known,** while ρ_k are **unknown**
- Coplanarity of the vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3$: there exist α, β s.t. $\vec{r}_2 = \alpha\vec{r}_1 + \beta\vec{r}_3$
- Multiply by $\vec{r}_3$ or $-\vec{r}_1$:

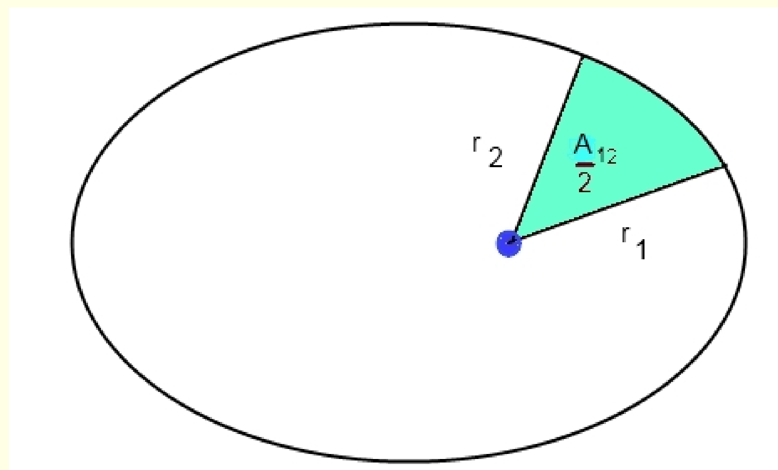
$$\begin{aligned} \vec{r}_2 \wedge \vec{r}_3 &= \alpha\, \vec{r}_1 \wedge \vec{r}_3 \Rightarrow \alpha = \frac{\vec{r}_2 \wedge \vec{r}_3}{\vec{r}_1 \wedge \vec{r}_3} = \frac{n_{23}}{n_{13}} \\ -\beta\, \vec{r}_3 \wedge \vec{r}_1 &= -\vec{r}_2 \wedge \vec{r}_1 \Rightarrow \beta = \frac{\vec{r}_1 \wedge \vec{r}_2}{\vec{r}_1 \wedge \vec{r}_3} = \frac{n_{12}}{n_{13}}\ , \end{aligned}$$

where n_{pq} is related to the area spanned by $\vec{r}_p$, $\vec{r}_q$.

Gauss method, n_{pq}, A_{pq}: STEP 1

- $\frac{n_{pq}}{2}$ = area triangle spanned by $\vec{r}_p$, $\vec{r}_q$
- Recall: $\alpha = \frac{n_{23}}{n_{13}}$, $\beta = \frac{n_{12}}{n_{13}}$
- $\frac{A_{pq}}{2}$ = area conic sector.
- $\eta_{pq} \equiv \frac{A_{pq}}{n_{pq}}$
- f_{pq} = angle between $\vec{r}_p$, $\vec{r}_q$





Gauss method, ρ_k: STEP 2

- Since $\vec{r}_k = \vec{a}_k + \rho_k \vec{b}_k$, coplanarity becomes:

$$\alpha(\vec{a}_1 + \rho_1 \vec{b}_1) - (\vec{a}_2 + \rho_2 \vec{b}_2) + \beta(\vec{a}_3 + \rho_3 \vec{b}_3) = \vec{0}$$

- Assume that that $\vec{r}_k$ are coplanar and not parallel, and that $\vec{b}_k$ are linear independent $\Rightarrow \vec{b}_1 \wedge \vec{b}_2 \cdot \vec{b}_3 \neq 0$

- Take the scalar product of coplanarity with

$$\vec{c}_1 = \frac{\vec{b}_2 \wedge \vec{b}_3}{\vec{b}_1 \wedge \vec{b}_2 \cdot \vec{b}_3}\ , \quad \vec{c}_2 = \frac{\vec{b}_3 \wedge \vec{b}_1}{\vec{b}_1 \wedge \vec{b}_2 \cdot \vec{b}_3}\ , \quad \vec{c}_3 = \frac{\vec{b}_1 \wedge \vec{b}_2}{\vec{b}_1 \wedge \vec{b}_2 \cdot \vec{b}_3}$$

Assuming $\alpha\beta \neq 0$, it follows:

$$\begin{aligned} \rho_1 &= -\vec{c}_1 \cdot \vec{a}_1 + \frac{1}{\alpha}\vec{c}_1 \cdot \vec{a}_2 - \frac{\beta}{\alpha}\vec{c}_1 \cdot \vec{a}_3 \\ \rho_2 &= \alpha\vec{c}_2 \cdot \vec{a}_1 - \vec{c}_2 \cdot \vec{a}_2 + \beta\vec{c}_2 \cdot \vec{a}_3 \\ \rho_3 &= -\frac{\alpha}{\beta}\vec{c}_3 \cdot \vec{a}_1 + \frac{1}{\beta}\vec{c}_3 \cdot \vec{a}_2 - \vec{c}_3 \cdot \vec{a}_3 \end{aligned} \tag{1}$$

Gauss method, P, Q: STEP 2

- Inserting α and β in the second of (1):

$$\rho_2 = -\vec{c}_2 \cdot \vec{a}_2 + \frac{\vec{c}_2 \cdot \vec{a}_1 n_{23} + \vec{c}_2 \cdot \vec{a}_3 n_{12}}{n_{12} + n_{23}} \frac{n_{12} + n_{23}}{n_{13}} \ , \tag{2}$$

which is a function of α, β or of P, Q as defined below.

- Define

$$P \equiv \frac{n_{12}}{n_{23}} \ , \quad Q \equiv 2r_2^3 \left(\frac{n_{12} + n_{23}}{n_{13}} - 1 \right)$$

- Then:

$$\rho_2 = -\vec{c}_2 \cdot \vec{a}_2 + \frac{\vec{c}_2 \cdot \vec{a}_1 + \vec{c}_2 \cdot \vec{a}_3 \ P}{P+1} (1 + \frac{Q}{2r_2^3})$$

- Let $t_{pq} = t_q - t_p$, using $A_{pq}/A_{rs} = t_{pq}/t_{rs}$ from Kep II law, one finds:

$$P = \frac{\beta}{\alpha} = \frac{t_{12}}{t_{23}} \frac{\eta_{23}}{\eta_{12}} \ , \qquad Q = \frac{t_{12} t_{23} \ r_2^2}{r_1 r_3 \eta_{12} \eta_{23} \ \cos f_{12} \cos f_{23} \cos f_{13}} \tag{3}$$

Gauss equation: STEP 2

- One obtains:

$$\alpha = \frac{1}{1+P}(1+\frac{Q}{2r_2^3}) \ , \qquad \beta = \frac{P}{1+P}(1+\frac{Q}{2r_2^3}) \ ,$$

- Using ρ_2 (eq. 2) and P, Q (eq. 3), one finds an implicit equation for ρ_2 of degree 8 ($r_2 = r_2(\rho_2)$):

$$\rho_2 = G(P, Q, \rho_2)$$
$$G(P, Q, \rho_2) \equiv -\vec{c}_2 \cdot \vec{a}_2 + \frac{\vec{c}_2 \cdot \vec{a}_1 + \vec{c}_2 \cdot \vec{a}_3 \ P}{P+1}(1+\frac{Q}{2r_2^3}) \ .$$

- Given $\vec{a}_k$, $\vec{b}_k$ (known from ephemerides and observations) and an initial guess $P = P_0$, $Q = Q_0$, solve Gauss equation by a Newton's method

Gauss method, ρ_1, ρ_3: STEP 2

- After solving

$$\begin{aligned} \rho_2 &= G(P, Q, \rho_2) \\ G(P, Q, \rho_2) &\equiv -\vec{c}_2 \cdot \vec{a}_2 + \frac{\vec{c}_2 \cdot \vec{a}_1 + \vec{c}_2 \cdot \vec{a}_3\ P}{P+1}\left(1 + \frac{Q}{2r_2^3}\right) \end{aligned}$$

compute ρ_1, ρ_3 by means of

$$\begin{aligned} \rho_1 &= -\vec{c}_1 \cdot \vec{a}_1 + \frac{P+1}{1 + \frac{Q}{2r_2^3}} \vec{c}_1 \cdot \vec{a}_2 - P\vec{c}_1 \cdot \vec{a}_3 \\ \rho_3 &= -\frac{1}{P}\vec{c}_3 \cdot \vec{a}_1 + \frac{P+1}{P(1 + \frac{Q}{2r_2^3})} \vec{c}_3 \cdot \vec{a}_2 - \vec{c}_3 \cdot \vec{a}_3 \ . \end{aligned}$$

Gauss method, map: STEP 3

- Iterate Steps 1, 2 by introducing Gauss map $F \equiv F_{t_{12},t_{23}} : (P, Q) \to (P', Q')$ with

$$
\begin{aligned}
P' &\equiv \frac{t_{12}}{t_{23}} \frac{\eta_{23}}{\eta_{12}} \\
Q' &\equiv \frac{t_{12} t_{23} \; r_2^2}{r_1 r_3 \, \eta_{12} \eta_{23} \, \cos f_{12} \cos f_{23} \cos f_{13}} \; .
\end{aligned}
$$

Proposition: A conic section $\mathcal{C}$ on which a Keplerian motion $t \to \vec{r}(t)$ takes place is a solution of Gauss problem $\iff$ there exists a fixed point (P, Q) of the Gauss map, with $\mathcal{C}$ being its associated conic section.

- Gauss problem: looking for a non–trivial fixed point of the Gauss map (trivial solution: Earth's orbit if $m_E = 0$)

- We describe the following algorithm: start from an initial approximation (P_0, Q_0), find the solution, iterate (P_k, Q_k) until finding a fixed point

Gauss method, fixed point: STEP 3

- Find the fixed point according to the following procedure:

◇ Let

$$P_0 = \frac{t_{12}}{t_{23}}\ , \qquad Q_0 = t_{12}t_{23}$$

◇ Compute $\rho_{2,0}$ from Gauss equation

◇ Find the corresponding values for $\rho_{1,0}$, $\rho_{3,0}$; hence we have $r_{k,0}$, $\eta_{pq,0}$, $f_{pq,0}$

◇ Iterate defining P_1, Q_1 through

$$P_1 = \frac{\beta}{\alpha} = \frac{t_{12}}{t_{23}}\frac{\eta_{23,0}}{\eta_{12,0}}\ , \qquad Q_1 = \frac{t_{12}t_{23}\ r_{2,0}^2}{r_{1,0}r_{3,0}\eta_{12,0}\eta_{23,0}\ \cos f_{12,0}\cos f_{23,0}\cos f_{13,0}}$$

◇ Look for a non–trivial fixed point of Gauss map, which provides the solution

Gauss method, approximations: STEP 3

- Let $t_{13} = \varepsilon$, $t_{12} = \tau_{12}\,\varepsilon$, $t_{23} = \tau_{23}\,\varepsilon$
- $n_{kl} = A_{kl} + O(\varepsilon^3) = \sqrt{p}\,\tau_{kl}\varepsilon + O(\varepsilon^3)$
- The approximations on P, Q are:

$$
\begin{aligned}
P &= \frac{n_{12}}{n_{23}} = \frac{\sqrt{p}\ \tau_{12}\varepsilon + O(\varepsilon^3)}{\sqrt{p}\ \tau_{23}\varepsilon + O(\varepsilon^3)} = \frac{\tau_{12}}{\tau_{23}} + O(\varepsilon^2) \\
Q &= 2r_2^3\ (\frac{n_{12}+n_{23}}{n_{13}} - 1) = \varepsilon^2\,\tau_{12}\tau_{23}\ f_0(\varepsilon) + O(\varepsilon^3)
\end{aligned}
$$

with $f_0(\varepsilon) \to 1$ as $\varepsilon \to 0$.

- α, β are $O(\varepsilon^2)$, since

$$
\alpha = \frac{1}{1+P}\ (1 + \frac{Q}{2r_2^3})\ , \qquad \beta = \frac{P}{1+P}\ (1 + \frac{Q}{2r_2^3})\ .
$$

Gauss method, approximation for ρ_k: STEP 3

- Since $P = \frac{\beta}{\alpha}$, $Q = 2r_2^3(\alpha+\beta-1)$, then $\alpha+\beta-1 = O(\varepsilon^3)$ (as the order of Q)
- One has that $\vec{a}_p - \vec{a}_q = O(\varepsilon^2)$.

- Therefore ρ_k are determined up to $O(\varepsilon)$, as it follows from the following formulae:

$$\begin{aligned}
\rho_1 &= -\frac{\alpha+\beta-1}{\alpha}\vec{c}_1\cdot\vec{a}_1 + \frac{1}{\alpha}\vec{c}_1\cdot(\vec{a}_2-\vec{a}_1) - \frac{\beta}{\alpha}\vec{c}_1\cdot(\vec{a}_3-\vec{a}_1)\\
\rho_2 &= \alpha\vec{c}_2\cdot(\vec{a}_1-\vec{a}_2) + (\alpha+\beta-1)\vec{c}_2\cdot\vec{a}_2 + \beta\vec{c}_2\cdot(\vec{a}_3-\vec{a}_2)\\
\rho_3 &= -\frac{\alpha}{\beta}\vec{c}_3\cdot(\vec{a}_1-\vec{a}_3) + \frac{1}{\beta}\vec{c}_3\cdot(\vec{a}_2-\vec{a}_3) - \frac{\alpha+\beta-1}{\beta}\vec{c}_3\cdot\vec{a}_3\ .
\end{aligned}$$

Gauss method, e, p, ω: STEP 4

- Compute e, p, ω using the polar equation

$$r = \frac{p}{1 + e\cos(\theta - \omega)}$$

as

$$\begin{aligned} e &= -\frac{r_3 - r_1}{r_3\cos(\theta_3 - \omega) - r_1\cos(\theta_1 - \omega)} \\ p &= r_1\, r_3\, \frac{\cos(\theta_3 - \omega) - \cos(\theta_1 - \omega)}{r_3\cos(\theta_3 - \omega) - r_1\cos(\theta_1 - \omega)} \end{aligned}$$

and ω as $(\theta_{ij} = \theta_j - \theta_i)$

$$\begin{aligned} A &= r_2(r_3 - r_1) + r_1(r_2 - r_3)\cos\theta_{12} + r_3(r_1 - r_2)\cos\theta_{23} \\ B &= r_1(r_2 - r_3)\sin\theta_{12} - r_3(r_1 - r_2)\sin\theta_{23} \\ \tan(\theta_2 - \omega) &= -\frac{A}{B}\,. \end{aligned}$$

Gauss method: summary

- Compute $\vec{c}_1$, $\vec{c}_2$, $\vec{c}_3$ by means of $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$
- Define P_0, Q_0 by means of t_{12}, t_{23}, t_{13}
- Solve the implicit equation for ρ_2
- Compute ρ_1, ρ_3
- Iterate to compute successive approximations
- Compute the orbital elements

Gauss algorithm: details

Assume that $\vec{a}_k$, $\vec{b}_k$, t_k are given for $k = 1, 2, 3$.

- Compute $t_{12} = t_2 - t_1$, $t_{23} = t_3 - t_2$, $t_{13} = t_3 - t_1$
- Compute $\vec{z}_1 = \vec{b}_2 \wedge \vec{b}_3$, $\vec{z}_2 = \vec{b}_1 \wedge \vec{b}_3$, $\vec{z}_3 = \vec{b}_1 \wedge \vec{b}_2$
- Compute $D_0 = \vec{z}_3 \cdot \vec{b}_3$
- Compute $\vec{c}_k = \frac{\vec{z}_k}{D_0}$
- Compute $P_0 = \frac{t_{12}}{t_{23}}$, $Q_0 = t_{12}\, t_{23}$
- Compute $\tilde{A} = -\vec{c}_2 \cdot \vec{a}_2 + \frac{\vec{c}_2 \cdot \vec{a}_1 + \vec{c}_2 \cdot \vec{a}_3\ P}{P+1}$
- Compute $\tilde{B} = \frac{\vec{c}_2 \cdot \vec{a}_1 + \vec{c}_2 \cdot \vec{a}_3\ P}{P+1} \frac{Q}{2}$
- Find the roots of the implicit equation $\rho_2 = \tilde{A} + \frac{\tilde{B}}{r_2^3}$, where $r_2 = |\vec{a}_2 + \rho_2 \vec{b}_2|$
- Solve the equations for ρ_1, ρ_3
- Compute $\vec{r}_k = \vec{a}_k + \rho_k \vec{b}_k$
- Determine the elliptic elements or refine the preliminary orbit

Ottaviano Fabrizio Mossotti (1791-1863)

- Born in Novara, studied at the University of Pavia and started to work in 1813 at the Observatory of Brera
- He needed to go abroad due to political reasons, since he was a supporter of the revolutionary Filippo Buonarroti of the liberal party
- He went to Switzerland, England and Argentina; he came back to Italy in 1835
- In 1840 he was appointed professor at the University of Pisa, where he taught Mathematical Physics and Celestial Mechanics
- he was named senator of the Italian kingdom in 1861
- He contributed to mathematics, celestial dynamics, fluid dynamics, optics.

Leonhard Euler (1707-1783)

- Born from a pastor of the Reformed Church, his father was friend of the Bernoulli family. Arago: ”Euler calculated without apparent effort, as men breathe, or as eagles sustain themselves in the wind”
- He contributed to the problem of orbit determination, but after 3 days of computations he got sick and lost the right eye
- Contributions in analysis, graph theory, mechanics, fluid dynamics, astronomy (lunar theory), optics

”On 7 September 1783, after having enjoyed some calculations on his blackboard concerning the laws of ascending motion for aerostatic machines for which the recent discovery was the rage of Europe, he dined with Mr. Lexell and his family, spoke of Herschel's planet [Uranus] and the mathematics concerning its orbit and a little while later he had his grandson come and play with him and took a few cups of tea, when all of a sudden the pipe that he was smoking slipped from his hand and **he ceased to calculate and live.** ”

(Eulogy to Mr. Euler by the Marquis de Condorcet, The Euler Society, March 2005)

Pierre Simon Laplace (1749-1827)

- Son of a farm-labourer, Laplace owed his education to the interest excited in some wealthy neighbours by his abilities
- His masterpiece is *Celestial Mechanics* (hence the name of this discipline)
- Contributions in analysis, probability, astronomy, celestial mechanics
- He studied the stability of the solar system and came to the conclusion of *absolute determinism:* "We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom"

Overview of Laplace method

- Determine $\vec{r}$, $\vec{v}$ at a given time (t_2) and then compute the orbit.
- If $f = f(t)$, let $f'(t) = \frac{df(t)}{dt}$, $f''(t) = \frac{d^2 f(t)}{dt^2}$, ...

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◇ **Step 1:** From the equations of motion, derive an implicit equation for ρ_2

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◇ **Step 3:** Determine the distance $\vec{r}(t_2)$ and the velocity $\vec{v}(t_2)$

◇ **Step 4:** Compute the elements, given the position and the velocity

Laplace method, implicit equation for ρ: STEP 1

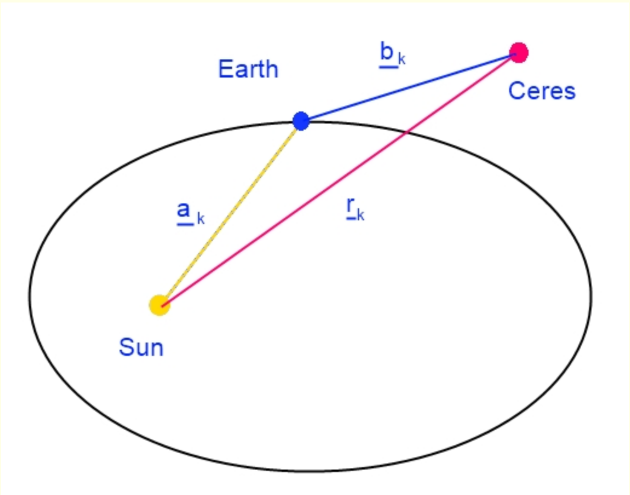
- For $t = t_k$, $k = 1, 2, 3$, let $\vec{r}_k = \vec{a}_k + \rho_k \vec{b}_k$ be the position of C w.r.t. Sun:

$\vec{a}_k$: position of the observer w.r.t. Sun (known from ephemerides)

$\vec{b}_k$: unit vector of C w.r.t. Earth (known from observations)

ρ_k : **unknown** distances C to Earth

$\rho_k \vec{b}_k$: geocentric position vectors of C (ρ_k unknowns)



Laplace method, implicit equation for ρ: STEP 1

- Assume that Earth and C have zero–mass ($m_E = m_C = 0$), that $Gm_{Sun} = 1$ and that Earth and C move on Keplerian orbits:

$$\vec{a}'' = -\frac{\vec{a}}{a^3} , \qquad \vec{r}\,'' = -\frac{\vec{r}}{r^3} .$$

- Geocentric position and velocity are given by

$$\vec{r} = \vec{a} + \rho\vec{b} , \qquad \vec{v} = \vec{a}' + \rho'\vec{b} + \rho\vec{b}' .$$

- The equation of motion of C becomes

$$-\frac{\vec{a}}{a^3} + \rho''\vec{b} + 2\rho'\vec{b}' + \rho\vec{b}'' = -\frac{\vec{a} + \rho\vec{b}}{r^3} .$$

or

$$\rho(\vec{b}'' + \frac{\vec{b}}{r^3}) + 2\rho'\vec{b}' + \rho''\vec{b} = -(\frac{1}{r^3} - \frac{1}{a^3})\vec{a} .$$

Laplace method, implicit equation for ρ: STEP 1

- Multiplying by $\vec{b} \wedge \vec{b}'$, $\vec{b} \wedge \vec{b}''$ write ρ, ρ' as:

$$\rho = \frac{d_1}{d}(\frac{1}{r^3} - \frac{1}{a^3}) \; , \qquad \rho' = \frac{d_2}{d}(\frac{1}{r^3} - \frac{1}{a^3}) \; ,$$

where if $d(t) \neq 0$ (depending on $\vec{b}$, $\vec{b}'$, $\vec{b}''$):

$$d = \vec{b} \wedge \vec{b}' \cdot \vec{b}'' \; , \quad d_1 = -\vec{b} \wedge \vec{b}' \cdot \vec{a} \; , \quad d_2 = -\frac{1}{2}\vec{b} \wedge \vec{a} \cdot \vec{b}'' \; .$$

- The first equation is an **implicit** equation for ρ of the form:

$$\begin{aligned} \rho &= L(\frac{d_1}{d}, \rho) \; , \quad L(x, \rho) \equiv x(\frac{1}{h(\rho)} - \frac{1}{a^3}) \; , \\ h(\rho) &\equiv r^3 = |\vec{a} + \rho\vec{b}|^3 = (a^2 + 2\vec{a}\vec{b}\rho + \rho^2)^{3/2} \; . \end{aligned}$$

Laplace method, main statement: STEP 1

- If t_2 is the time of **mean** observation, then

$$\vec{r}_2 = \vec{a}_2 + \rho_2 \vec{b}_2 \ , \qquad \vec{v}_2 = \vec{a}_2' + \rho_2' \vec{b}_2 + \rho_2 \vec{b}_2' \ ,$$

where $\vec{a}_2$ and $\vec{a}_2'$ are given by ephemerides.

- The unknown orbit is completely determined by the equation of motion and by the initial condition $\vec{r}(t_2) = \vec{r}_2$, $\vec{r}\,'(t_2) = \vec{v}_2$.

- The result is summarized in the following proposition.

Laplace method, main statement: STEP 1

- **Proposition:** Let $\mathcal{C}$ be a conic section with Keplerian motion $t \to \vec{r}(t)$. Let t_2 be the time of mean observation. Assume $d(t_2) \neq 0$ and let $t \to \vec{a}(t)$ be a fixed Keplerian motion on some conic section $\mathcal{C}_E$. Then, the position and velocity vectors at time t_2 may be expressed as functions of $\vec{b}_2$, $\vec{b}'_2$ as well as ρ_2, ρ'_2 at $t = t_2$:

$$\vec{r}_2 = \vec{a}_2 + \rho_2 \vec{b}_2 \ , \qquad \vec{v}_2 = \vec{a}'_2 + \rho'_2 \vec{b}_2 + \rho_2 \vec{b}'_2$$

with

$$\rho_2 = \frac{d_1}{d} \left(\frac{1}{r_2^3} - \frac{1}{a_2^3} \right) \ , \qquad \rho'_2 = \frac{d_2}{d} \left(\frac{1}{r_2^3} - \frac{1}{a_2^3} \right) \ ,$$

and

$$d = \vec{b}_2 \wedge \vec{b}'_2 \cdot \vec{b}''_2 \ , \quad d_1 = -\vec{b}_2 \wedge \vec{b}'_2 \cdot \vec{a}_2 \ , \quad d_2 = -\frac{1}{2} \vec{b}_2 \wedge \vec{a}_2 \cdot \vec{b}''_2 \ .$$

Laplace method, computation of $\vec{b}'_2$: STEP 2

- Algorithm for the computation of $\vec{b}'_2$, $\vec{b}''_2$ from λ, β and their derivatives. Let λ and β be the geocentric longitude and latitude of C; then:

$$\begin{aligned} b_1(t) &= \cos\lambda(t)\cos\beta(t) \\ b_2(t) &= \sin\lambda(t)\cos\beta(t) \\ b_3(t) &= \sin\beta(t)\ . \end{aligned}$$

- Let $\lambda_2 \equiv \lambda(t_2)$, $\beta_2 \equiv \beta(t_2)$; taking the derivatives in $t = t_2$, one obtains $(b'_{2,1} = db_1(t_2)/dt)$

$$\begin{aligned} b'_{2,1} &= -\lambda'_2 \sin\lambda_2 \cos\beta_2 - \beta'_2 \cos\lambda_2 \sin\beta_2 \\ b'_{2,2} &= \lambda'_2 \cos\lambda_2 \cos\beta_2 - \beta'_2 \sin\lambda_2 \sin\beta_2 \\ b'_{2,3} &= \beta'_2 \cos\beta_2\ . \end{aligned}$$

Laplace method, computation of $\vec{b}_2''$: STEP 2

- For the computation of $\vec{b}_2'' \equiv (b''_{2,1}, b''_{2,2}, b''_{2,3})$ one has:

$$
\begin{aligned}
b''_{2,1} &= -\lambda_2'' \sin\lambda_2 \cos\beta_2 - \beta_2'' \cos\lambda_2 \sin\beta_2 \\
&- (\lambda_2')^2 \cos\lambda_2 \cos\beta_2 + \\
&+ 2\lambda_2'\beta_2' \sin\lambda_2 \sin\beta_2 - (\beta_2')^2 \cos\lambda_2 \cos\beta_2 \\
b''_{2,2} &= \lambda_2'' \cos\lambda_2 \cos\beta_2 - \beta_2'' \sin\lambda_2 \sin\beta_2 \\
&- (\lambda_2')^2 \sin\lambda_2 \cos\beta_2 + \\
&- 2\lambda_2'\beta_2' \cos\lambda_2 \sin\beta_2 - (\beta_2')^2 \sin\lambda_2 \cos\beta_2 \\
b''_{2,3} &= \beta_2'' \cos\beta_2 - (\beta_2')^2 \sin\beta_2 \ .
\end{aligned}
$$

- $\vec{b}_2'$, $\vec{b}_2''$ depend on 4 parameters, namely the first and second derivatives of latitude and longitude at $t = t_2$: λ_2', λ_2'', β_2', β_2''

Laplace method, values for λ_2', λ_2'', β_2', β_2'': STEP 2

- Approximate values for λ_2', λ_2'', β_2', β_2'' can be found by quadratic interpolation between the observed values λ_1, λ_2, λ_3, β_1, β_2, β_3.

- Using Taylor expansion up to the second order:

$$\begin{aligned} \lambda_1 &= \lambda_2 - \lambda_2'\, t_{12} + \frac{1}{2}\lambda_2''\, t_{12}^2 + R_1 \\ \lambda_3 &= \lambda_2 + \lambda_2'\, t_{23} + \frac{1}{2}\lambda_2''\, t_{23}^2 + R_3 \ , \end{aligned}$$

where $t_{pq} \equiv t_q - t_p$ and R_1, R_3 are the remainder functions.

- Similarly for β_1, β_3 with remainder functions S_1, S_3.

- Solve these equations to find λ_2', λ_2'', β_2', β_2''.

Laplace method, values for λ_2', λ_2'', β_2', β_2'': STEP 2

- Set $t_{13}=\varepsilon$. The remainders R_1, R_3, S_1, S_3 are $O(\varepsilon^3)$:

$$\begin{aligned}
\lambda_2' &= -\frac{t_{23}}{t_{12}\,t_{13}}(\lambda_1-R_1)-\frac{t_{12}-t_{23}}{t_{12}\,t_{23}}\lambda_2+\frac{t_{12}}{t_{13}\,t_{23}}(\lambda_3-R_3)\\
\lambda_2'' &= \frac{2}{t_{12}\,t_{13}}(\lambda_1-R_1)-\frac{2}{t_{12}\,t_{23}}\lambda_2+\frac{2}{t_{13}\,t_{23}}(\lambda_3-R_3)\\
\beta_2' &= -\frac{t_{23}}{t_{12}\,t_{13}}(\beta_1-S_1)-\frac{t_{12}-t_{23}}{t_{12}\,t_{23}}\beta_2+\frac{t_{12}}{t_{13}\,t_{23}}(\beta_3-S_3)\\
\beta_2'' &= \frac{2}{t_{12}\,t_{13}}(\beta_1-S_1)-\frac{2}{t_{12}\,t_{23}}\beta_2+\frac{2}{t_{13}\,t_{23}}(\beta_3-S_3)\,.
\end{aligned}$$

- This allows to compute $\vec{b}'$, $\vec{b}''$ and therefore d, d_1, d_2, so that we can solve the implicit equation for ρ by a Newton's method.

Laplace method, interpolation formulae: STEP 2

- Inserting $R_1 = R_3 = S_1 = S_3 = 0$ one obtains approximate values for $\lambda'_{2,0}$, $\lambda''_{2,0}$, $\beta'_{2,0}$, $\beta''_{2,0}$ **(interpolation formulae):**

$$
\begin{aligned}
\lambda'_{2,0} &= -\frac{t_{23}}{t_{12}\ t_{13}}\lambda_1 - \frac{t_{12}-t_{23}}{t_{12}\ t_{23}}\lambda_2 + \frac{t_{12}}{t_{13}\ t_{23}}\lambda_3 \\
\lambda''_{2,0} &= \frac{2}{t_{12}\ t_{13}}\lambda_1 - \frac{2}{t_{12}\ t_{23}}\lambda_2 + \frac{2}{t_{13}\ t_{23}}\lambda_3 \\
\beta'_{2,0} &= -\frac{t_{23}}{t_{12}\ t_{13}}\beta_1 - \frac{t_{12}-t_{23}}{t_{12}\ t_{23}}\beta_2 + \frac{t_{12}}{t_{13}\ t_{23}}\beta_3 \\
\beta''_{2,0} &= \frac{2}{t_{12}\ t_{13}}\beta_1 - \frac{2}{t_{12}\ t_{23}}\beta_2 + \frac{2}{t_{13}\ t_{23}}\beta_3\ .
\end{aligned}
$$

- These quantities represent an approximation of λ'_2, β'_2, λ''_2, β''_2, being $\lambda'_2 = \lambda'_{2,0} + O(\varepsilon^2)$, $\beta'_2 = \beta'_{2,0} + O(\varepsilon^2)$, $\lambda''_2 = \lambda''_{2,0} + O(\varepsilon)$, $\beta''_2 = \beta''_{2,0} + O(\varepsilon)$.

Laplace method, more observations: STEP 2

• **Remark:** With m observations at t_1,...,t_m, $m \geq 3$, one can compute interpolating polynomials $\tilde{\lambda}(t)$, $\tilde{\beta}(t)$ of degree $m-1$:

$$
\begin{aligned}
\tilde{\lambda}(t) &= \frac{(t-t_2)...(t-t_m)}{(t_1-t_2)...(t_1-t_m)}\lambda_1 + ... + \frac{(t-t_1)...(t-t_{m-1})}{(t_m-t_1)...(t_m-t_{m-1})}\lambda_m \ , \\
\tilde{\beta}(t) &= \frac{(t-t_2)...(t-t_m)}{(t_1-t_2)...(t_1-t_m)}\beta_1 + ... + \frac{(t-t_1)...(t-t_{m-1})}{(t_m-t_1)...(t_m-t_{m-1})}\beta_m \ .
\end{aligned}
$$

• Taking the derivatives, for instance in $t = t_2$, one obtains:

$$
\begin{aligned}
\lambda_2' &= \tilde{\lambda}'(t_2) + O(\varepsilon^{n+2}) \ , \quad \beta_2' = \tilde{\beta}'(t_2) + O(\varepsilon^{n+2}) \ , \\
\lambda_2'' &= \tilde{\lambda}''(t_2) + O(\varepsilon^{n+1}) \ , \quad \beta_2'' = \tilde{\beta}''(t_2) + O(\varepsilon^{n+1}) \ ,
\end{aligned}
$$

where $n = m - 3$.

• More than 3 observations are used *only* to compute the interpolating polynomials.

Laplace method, distance and velocity: STEP 3

- Determine distance and velocity through the following algorithm
- **Theorem (Laplace Algorithm).** Given $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$, let $\mathcal{C}$ be a conic section such that $d(t_2) \neq 0$, $\rho(t_k) \neq 0$ for $k = 1, 2, 3$ and $\frac{\partial}{\partial \rho} L(d_1/d, \rho_2) \neq 1$. For $n = m - 3$, let

$$\begin{aligned} \lambda_2' &= \tilde{\lambda}'(t_2) + O(\varepsilon^{n+2}) \ , \quad \beta_2' = \tilde{\beta}'(t_2) + O(\varepsilon^{n+2}) \\ \lambda_2'' &= \tilde{\lambda}''(t_2) + O(\varepsilon^{n+1}) \ , \quad \beta_2'' = \tilde{\beta}''(t_2) + O(\varepsilon^{n+1}) \end{aligned}$$

and let $\vec{b}_n' = (b_{n,1}', b_{n,2}', b_{n,3}')$ be defined as:

$$\begin{aligned} b_{n,1}' &= -\tilde{\lambda}'(t_2) \sin \lambda_2 \cos \beta_2 - \tilde{\beta}'(t_2) \cos \lambda_2 \sin \beta_2 \\ b_{n,2}' &= \tilde{\lambda}'(t_2) \cos \lambda_2 \cos \beta_2 - \tilde{\beta}'(t_2) \sin \lambda_2 \sin \beta_2 \\ b_{n,3}' &= \tilde{\beta}'(t_2) \cos \beta_2 \ . \end{aligned}$$

Laplace method, main statement: STEP 3

- Similarly for $\vec{b}''_n$.
- Let

$$d_n = \vec{b_2} \wedge \vec{b}'_n \cdot \vec{b}''_n \ , \qquad d_{n,1} = -\vec{b_2} \wedge \vec{b}'_n \cdot \vec{a_2} \ , \qquad d_{n,2} = -\frac{1}{2}\vec{b_2} \wedge \vec{a_2} \cdot \vec{b}''_n \ .$$

Then, there exists U ngbh. of $x \equiv d_1/d$ and V of ρ_2 s.t. if $d_{n,1}/d_n \in U$, there exists $\rho_n \in V$ such that $\rho_2 = \rho_n + O(\varepsilon^{n+1})$. Defining

$$\rho'_n = \frac{d_{n,2}}{d_n}\left(\frac{1}{|\vec{a} + \rho_n \vec{b}|^3} - \frac{1}{a^3}\right)$$

and

$$\vec{r}_n = \vec{a}_2 + \rho_n \vec{b}_2 \ , \qquad \vec{v}_n = \vec{a}'_2 + \rho'_n \vec{b}_2 + \rho_n \vec{b}'_n \ ,$$

the Keplerian solution of the eq. of motion with initial data $\vec{r}(t_2) = \vec{r}_n$, $\vec{r}\,'(t_2) = \vec{v}_n$ defines a conic section $\mathcal{C}_n$ s.t. $\mathcal{C} = \mathcal{C}_n + O(\varepsilon^{n+1})$.

Laplace method, elliptic elements: STEP 4

- Computation of the elements given the position $\vec{r}_2$ and velocity $\vec{v}_2$
- Let $(\vec{i}, \vec{j}, \vec{k})$ = ecliptic frame, $\vec{m}$ = normal to the orbit plane, $\vec{n}$ unit vector of $\vec{k} \wedge \vec{m}$, $\vec{n}' = \vec{m} \wedge \vec{n}$
- Assume that $\vec{M} \equiv \vec{r} \wedge \vec{v} \neq \vec{0}$. Let $(\vec{n}, \vec{n}', \vec{m})$ as before; let θ = argument of latitude:

$$\cos\theta = \frac{\vec{r}\cdot\vec{n}}{|\vec{r}|}\ , \qquad \sin\theta = \frac{\vec{r}\cdot\vec{n}'}{|\vec{r}|}\ .$$

- To compute p, e, ω, use the polar equation:

$$r(\theta) = \frac{p}{1 + e\cos(\theta - \omega)}\ .$$

with

$$\dot{\theta}(t) = \frac{p^{1/2}}{r^2(t)} = p^{-3/2}\cdot[1 + e\cos(\theta(t) - \omega)]^2\ .$$

Laplace method, elliptic elements: STEP 4

- The components along $\vec{n}$ and $\vec{n'}$ of $\vec{r}(t)$ at time t are

$$\vec{r}(t)\cdot\vec{n} = r(t)\cos\theta(t)\ , \qquad \vec{r}(t)\cdot\vec{n}' = r(t)\sin\theta(t)\ ;$$

taking the derivatives:

$$\begin{aligned}\vec{v}(t)\cdot\vec{n} &= \dot{r}(t)\cos\theta(t) - r(t)\sin\theta(t)\dot{\theta}(t)\\ \vec{v}(t)\cdot\vec{n}' &= \dot{r}(t)\sin\theta(t) + r(t)\cos\theta(t)\dot{\theta}(t)\ .\end{aligned}$$

- Moreover, from the polar equation one has

$$\dot{r}(t) = \frac{p\ e\sin\left(\theta(t)-\omega\right)}{[1+e\cos\left(\theta(t)-\omega\right)]^2}\ \ \dot{\theta}(t) = p^{-1/2}e\sin\left(\theta(t)-\omega\right)\ .$$

Laplace method, elliptic elements: STEP 4

- Casting together we obtain

$$
\begin{aligned}
|\vec{v}|^2 &= p^{-1}(1 + 2e\cos(\theta - \omega) + e^2) \\
\vec{r} \cdot \vec{v} &= r p^{-1/2} e \sin(\theta - \omega) \ , \quad e\cos(\theta - \omega) = \frac{p}{r} - 1 \ .
\end{aligned}
$$

- Integrals of motion: angular momentum and energy in terms of e, p:

$$
\begin{aligned}
|\vec{r} \wedge \vec{v}| &= p^{1/2} = |\vec{r}_2 \wedge \vec{v}_2| \equiv M \\
\frac{v^2}{2} - \frac{1}{r} &= -\frac{1 - e^2}{2p} = \frac{v_2^2}{2} - \frac{1}{r_2} \equiv E \ ;
\end{aligned}
$$

therefore we get

$$
p = M^2 \ , \qquad e = (1 + 2E\ M^2)^{1/2} \ .
$$

- Let $f_2 \equiv \theta_2 - \omega$ be the true anomaly associated to $\vec{r}_2$; setting $\vec{r} = \vec{r}_2$, $\vec{v} = \vec{v}_2$ we obtain

$$
\cos f_2 = \frac{p - r_2}{r_2 e} \ , \qquad \sin f_2 = \frac{\vec{r}_2 \cdot \vec{v}_2\, p^{1/2}}{r_2 e} \ \Rightarrow \ \omega \ .
$$

Laplace method: summary

- Compute λ_2', λ_2'', β_2', β_2'' by means of λ_1, λ_2, λ_3, β_1, β_2, β_3
- Compute $\vec{b}_2'$, $\vec{b}_2''$ by means of λ_k, λ_k', λ_k'', β_k, β_k', β_k'', $k = 1, 2, 3$
- Compute d, d_1, d_2 at $t = t_2$ by means of $\vec{b}_2$, $\vec{b}_2'$, $\vec{b}_2''$, $\vec{a}_2$
- Solve the implicit equation for ρ_2, compute ρ_2'
- Compute $\vec{r}_2$, $\vec{v}_2$ with $\vec{r}_2 = \vec{a}_2 + \rho_2 \vec{b_2}$, $\vec{v}_2 = \vec{a}_2' + \rho_2' \vec{b_2} + \rho_2 \vec{b_2}'$ ($\vec{a}_2$, $\vec{a}_2'$ from ephemerides)
- Compute the elements of the orbit corresponding to the equations of motion with initial data $\vec{r}_2$, $\vec{v}_2$ at $t = t_2$

Application to (3) Juno

- Juno: 240 km diameter, mass $2 \cdot 10^{19}$ kg.
- Input data to start the procedure (see [Gauss]):

1) the epochs of three observations, say t_1, t_2, t_3;

2) the Earth–Sun (log) distances at the above epochs (from ephemerides);

3) the ecliptical longitudes of the Earth at times t_j $(j = 1, 2, 3)$ (from ephemerides);

4) the geocentric ecliptical longitudes of the body at times t_j $(j = 1, 2, 3)$ (from observations);

5) the geocentric ecliptical latitudes of the body at times t_j $(j = 1, 2, 3)$ (from observations).

- Corrections for fixed star aberration, time aberration, precession of the equinox, nutation, diurnal motion are included in the initial data referring to October 1804 ([Gauss]).

Application to (3) Juno

Epoch	Log(distance) AU Earth–Sun distance	Longitude Earth
Oct. 5.458644	9.9996826	12^o 28' 27.76"
Oct. 17.421885	9.9980979	24^o19'49.05"
Oct. 27.393077	9.9969678	34^o16'9.65"

Epoch	Longitude of C	Latitude of C
Oct. 5.458644	354^o44'31.60"	-4^o59'31.06"
Oct. 17.421885	352^o34'22.12"	-6^o21'55.07"
Oct. 27.393077	351^o34'30.01"	-7^o17'50.95"

- The output are the 6 elements $(a, e, i, \omega, \Omega, M)$, where M is referred to 1/1/1805 for the meridian of Paris.

Application to (3) Juno

- Table: first line Gauss after 3 iterations; second line (*GL*) Gauss or Laplace; third line (*Astr. data*) NASA values (M refers to different epochs).

- Difference between 3 and 1, 2 is due to the different computational framework (two or more body problem), to the epoch of computation, to the correction for aberrations and to eventual observational errors.

- Difference between the original results by Gauss (1) and our computer programs (2) are due to computational precision (we used double precision) and to a higher iterations (we iterated 100 times, instead of 3 as in Gauss)

	a	e	i	ω	Ω	M
Gauss	2.645080	0.245316	13.1123	241.1724	171.130	349.5701
GL	2.644619	0.245049	13.1155	241.1547	171.132	349.5678
Astr. data	2.667332	0.258614	12.9717	247.9220	170.129	

A comparison of the methods

i) In Gauss the vectors $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ need not to be coplanar. On the contrary, Laplace requires that the heliocentric position vectors of the observer are coplanar, since it starts with the equations of motion.

ii) In Laplace mass and radius of the Earth are zero. For more physical results, modify the equations of motion as follows ($\mu_E = m_E/m_{Sun}$):

$$\vec{a}'' = -(1+\mu_E)\frac{\vec{a}}{a^3} \ , \qquad \vec{r}\,'' = -\frac{\vec{r}}{r^3} \ ,$$

Consequently, one should take

$$\rho = \frac{d_1}{d}(\frac{1}{r^3} - (1+\mu_E)\frac{1}{a^3}) \ , \quad \rho' = \frac{d_2}{d}(\frac{1}{r^3} - (1+\mu_E)\frac{1}{a^3}) \ ,$$

where $d = \vec{b} \wedge \vec{b}' \cdot \vec{b}''$, $d_1 = -\vec{b} \wedge \vec{b}' \cdot \vec{a}$, $d_2 = -\frac{1}{2}\vec{b} \wedge \vec{a} \cdot \vec{b}''$.

iii) Laplace gets a first good approximation and stops; Gauss also finds a first good approximation and then he improves it by iterating.

Tests on Gauss and Laplace

- Implement Gauss and Laplace for 10^5 initial conditions, varying randomly longitude and latitude of C, and the three times of observation.

- First, compute the *true* orbital elements of the conic, obtained letting the program iterate several times until convergence is reached. Result: a_t, e_t, i_t.

- Implement the methods without iterating the algorithms (as in the original Laplace method) with results (a_G, e_G, i_G) for Gauss and (a_L, e_L, i_L) for Laplace.

- Define relative errors and make a test over 10^5 samples.

Gauss and Laplace: relative error

- To have a measure of the **relative error** let (G and L refer to Gauss and Laplace):

$$\varepsilon_{a,G} \equiv |\frac{a_G - a_t}{a_t}| \ , \quad \varepsilon_{e,G} \equiv |\frac{e_G - e_t}{e_t}| \ , \quad \varepsilon_{i,G} \equiv |\frac{i_G - i_t}{i_t}| \ ,$$
$$\varepsilon_{a,L} \equiv |\frac{a_L - a_t}{a_t}| \ , \quad \varepsilon_{e,L} \equiv |\frac{e_L - e_t}{e_t}| \ , \quad \varepsilon_{i,L} \equiv |\frac{i_L - i_t}{i_t}| \ .$$

- Introduce the difference between the relative errors as

$$\Delta a \equiv \varepsilon_{a,G} - \varepsilon_{a,L} \ , \quad \Delta e \equiv \varepsilon_{e,G} - \varepsilon_{e,L} \ , \quad \Delta i \equiv \varepsilon_{i,G} - \varepsilon_{i,L} \ .$$

- If one of the above quantities is negative, Gauss provides better results than Laplace and viceversa.

Gauss and Laplace compared

- Table: first line = number of times such that $\Delta a < 0$, $\Delta e < 0$, $\Delta i < 0$ and all 3 values simultaneously negative; second line = number of times for which they are positive.

- Laplace: better estimate of a, being $\Delta a > 0$ for 59 095 trajectories (G better for 40 905 orbits); G and L equivalent for e, while Gauss prevails for errors in i.

- The last column denotes the number of orbits for which all quantities Δa, Δe, Δi have simultaneously the same sign, providing therefore the correct result for all orbital elements. In this case, Gauss method gives more than twice times the best results when compared to Laplace algorithm.

	Δa	Δe	Δi	$\Delta a \& \Delta e \& \Delta i$
Gauss	40 905	49 402	71 979	28 436
Laplace	59 095	50 598	28 021	13 837

Dependence on the observation times

- Consider the first 10 000 numbered asteroids and 615 Kuiper belt objects.

- Apply G and L for different time intervals t_{12} and t_{23}, where the central time t_2 is the real observational time as provided by the astronomical data.

- Starting from $(a, e, i, \omega, \Omega, M)$ at t_2, and given the time intervals t_{12} and t_{23}, compute the geocentric longitude and latitude at times t_1, t_2, t_3 by means of the coordinates of the object and that of the Earth.

- Apply G and L (with iteration until convergence).

- To be sure that a given method converges in a significant range around t_2 (and not only for time t_2), define $t_{ij}^n \equiv t_{ij} + n/2$, where $n = 0, \pm 1, \pm 2$; if the method converges for t_{12}^n and t_{23}^n $(n = 0, \pm 1, \pm 2)$, then we say that the method is successful, otherwise we decide that the method fails.

- Consider several choices of t_{ij} from 3 to 90 days as well as two observations within the same night.

Equal time intervals

- The first percentage of successful results refers to asteroids, the second to Kuiper.

t_{12}	t_{23}	G	L
3^d	3^d	Ast 99.86 / Kui 79.67	Ast 99.00/ Kui 93.33
5^d	5^d	Ast 99.87/ Kui 93.33	Ast 98.90/ Kui 93.98
10^d	10^d	Ast 99.78/ Kui 93.98	Ast 98.73/ Kui 94.63
15^d	15^d	Ast 99.58/ Kui 94.47	Ast 98.54/ Kui 94.63
30^d	30^d	Ast 99.45/ Kui 94.63	Ast 98.17/ Kui 94.63
60^d	60^d	Ast 98.77/ Kui 94.63	Ast 96.00/ Kui 94.63
90^d	90^d	Ast 96.80/ Kui 94.63	Ast 94.32/ Kui 94.63

Different time intervals

- Notice that the time intervals t_{12}, t_{23} are now different.

t_{12}	t_{23}	G	L
10^d	30^d	Ast 99.60/ Kui 94.63	Ast 98.01/ Kui 94.63
5^d	10^d	Ast 99.82/ Kui 94.47	Ast 98.63/ Kui 94.63
1^h	5^d	Ast 99.77/ Kui 7.32	Ast 98.82/ Kui 93.17
5^h	5^d	Ast 99.87/ Kui 17.40	Ast 98.86/ Kui 93.66
1^h	10^d	Ast 99.80/ Kui 17.40	Ast 98.60/ Kui 94.31
5^h	10^d	Ast 99.81/ Kui 53.17	Ast 98.55/ Kui 94.30
1^h	30^d	Ast 99.68/ Kui 63.25	Ast 97.59/ Kui 94.63
5^h	30^d	Ast 99.70/ Kui 83.85	Ast 97.61/ Kui 94.63

Dependence on the observation times: remarks

- Concerning asteroids, Gauss provides the best result, while Laplace is better for Kuiper.

- For equal time intervals $t_{12} = t_{23}$, the number of successful cases within the asteroidal belt increases as the time interval decreases, while (again) the opposite conclusion can be drawn for the Kuiper belt objects.

- One might expect that whenever the time interval $\varepsilon = t_{13}$ among the observations is sufficiently small (say $\varepsilon < \bar{\varepsilon}$), Gauss converges.

- Of course $\bar{\varepsilon}$ depends on $\mathcal{C}$, $\tau_{12}(= t_{12}/\varepsilon)$, $\tau_{23}(= t_{23}/\varepsilon)$ (and t_2), implying that smaller is ε, greater is the number of converging orbits for fixed values of τ_{12}, τ_{23}. On the other hand, the dependence of $\bar{\varepsilon}$ on τ_{12}, τ_{23} implies that t_{12}, t_{23} cannot be chosen *too* small, otherwise $\mathcal{C}$ (as well as its approximants $\mathcal{C}_n$) is badly determined.

Different parameter regions

- To see the distribution of the previous results as functions of a, e, i, compute the percentages of successful results of the first 10 000 asteroids, considering four regions in a, e, i (each one with 2 500 objects).

- Each parameter region is composed by 2 500 objects belonging to the first 10 000 numbered asteroids.

- Parameter regions: $0 \leq a < 2.341$ AU, $2.341 \leq a < 2.6144$ AU, $2.6144 \leq a < 3.0053$ AU, $3.0053 \leq a < 100$ AU

- Conclusion: Gauss seems to be independent on the parameter regions, while Laplace depends on the a, e, i - parameter regions.

- Percentage of successful results; time intervals: $t_{12} = 1^h$ and $t_{23} = 5^d$.

	G	L
$0 \leq a < 2.341$	99.56	97.36
$2.341 \leq a < 2.6144$	99.96	98.48
$2.6144 \leq a < 3.0053$	99.96	99.52
$3.0053 \leq a < 100$	99.92	99.92
$0 \leq e < 0.094$	99.92	99.60
$0.094 \leq e < 0.140244$	99.92	99.56
$0.140244 \leq e < 0.187321$	99.64	98.52
$0.187321 \leq e < 1$	99.64	97.60
$0 \leq i < 3.2185$	99.72	98.68
$3.2185 \leq i < 6.0218$	99.84	98.36
$6.0218 \leq i < 10.918$	99.72	99.08
$10.918 \leq i < 360$	99.80	99.16

- Percentage of successful results; time intervals $t_{12} = 10^d$ and $t_{23} = 10^d$.

	G	L
$0 \leq a < 2.341$	99.56	96.88
$2.341 \leq a < 2.6144$	99.80	98.40
$2.6144 \leq a < 3.0053$	99.88	99.72
$3.0053 \leq a < 100$	99.88	99.92
$0 \leq e < 0.094$	99.92	99.56
$0.094 \leq e < 0.140244$	99.84	99.48
$0.140244 \leq e < 0.187321$	99.80	98.92
$0.187321 \leq e < 1$	99.56	96.96
$0 \leq i < 3.2185$	99.84	99.08
$3.2185 \leq i < 6.0218$	99.76	97.92
$6.0218 \leq i < 10.918$	99.76	98.84
$10.918 \leq i < 360$	99.76	99.08

Milani method: Too short arcs

- Observations are sometimes performed in 1 night, giving 3-5 positions over 1-2 hours $\Rightarrow$ very short arc

- Classical methods might fail to find the preliminary orbit, since the $\vec{b}_k$'s must be independent

- An **attributable** is a vector $A = (\alpha, \delta, \dot{\alpha}, \dot{\delta})$ representing the topocentric angular position and velocity of the body at time t

- When the preliminary orbit is not found or iterations do not converge:

Too Short Arc (TSA); a new paradigm for orbit determination is needed

- A TSA provides a number of positions with deviations from alignment compatible with a random observational error

- The attributables do not provide ρ and $\dot{\rho}$, but they are constrained if we assume C belonging to the Solar System and not being an Earth's satellite

Constraints on $(\rho, \dot\rho)$

heliocentric 2–body energy:

$$E_S(\dot\rho, \rho) = \frac{1}{2}\|\dot{\vec r}\|^2 - \frac{k^2}{\|\vec r\|}$$

geocentric 2–body energy:

$$E_E(\dot\rho, \rho) = \frac{1}{2}\|\dot{\vec r} - \dot{\vec a}\|^2 - \frac{k^2\mu}{\|\vec r - \vec a\|}\ , \qquad \mu = \frac{m_E}{m_{Sun}}$$

radius of the sphere of influence of the Earth (i.e. distance from L_2):

$$R_{SI} = a\left(\frac{\mu}{3}\right)^{\frac{1}{3}} = 0.010044\ AU$$

radius of the Earth:

$$R_E \simeq 4.2 \cdot 10^{-5}\ AU$$

Admissible region

- Introduce an **admissible region:** for a given attributable the values $(\rho, \dot\rho)$ must belong to a compact admissible region as follows

- Define the following domains:

$D_1 = \{(\rho, \dot\rho): \ E_E \geq 0\}$, i.e. C not satellite of E

$D_2 = \{(\rho, \dot\rho): \ \rho \geq R_{SI}\}$, i.e. orbit of C not controlled by E

$D_3 = \{(\rho, \dot\rho): \ E_S \leq 0\}$, i.e. C belongs to the Solar System

$D_4 = \{(\rho, \dot\rho): \ \rho \geq R_E\}$, i.e. C is outside the Earth.

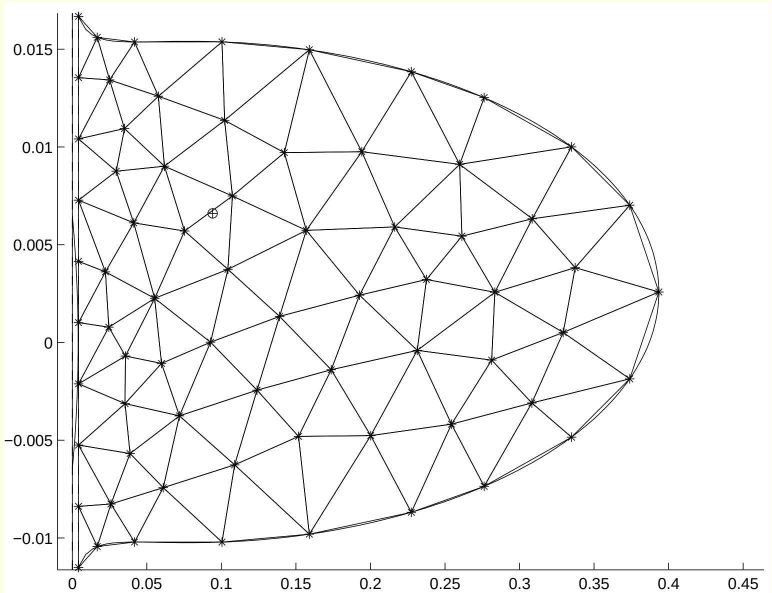
- **Definition.** Given an attributable A, we define an admissible region as

$$D = \{D_1 \bigcup D_2\} \bigcap D_3 \bigcap D_4 \ .$$

- Provide an analytical and topological description of the admissible region.

Sampling the admissible region

• Sample D with a number of points by a Delaunay triangulation; the nodes are selected as the points $(\rho_i, \dot{\rho}_i)$, $i = 1, ..., N$, sampling the admissible region, with the sides and the triangles providing a geometric structure.



• Triangulation of the admissible region: each node generates a Virtual Asteroid.

Virtual Asteroids

- When a single orbit solution is not available, or it is not enough to represent the possible orbits, it is replaced replaced by a swarm of **Virtual Asteroids (VAs)**
- The VAs share the reality of the physical asteroid, in that only one of them is real, but we do not know which one. Additional observations allow to decrease the number of VAs still compatible
- The VAs are identified through $X_i = (\alpha, \delta, \dot{\alpha}, \dot{\delta}, \rho_i, \dot{\rho}_i)$, $i = 1, ..., N$ and a covariance matrix Γ_A (where $A = (\alpha, \delta, \dot{\alpha}, \dot{\delta})$ associated to the mean observation time of the first very short arc
- To each VA one associates an *attribution penalty* by comparing the uncertainty of the VA with the attributable at another time

Propagate to another time

- Given a triangulation, take a VA in the admissible region at the first time with associated covariance matrix (with rank 4)
- Propagate to the time of the second attributable and compute the attribution penalty
- Note that the attribution penalty is a measure of the probability that another attributable, computed from an independently detected TSA, belongs to the same object
- Select the VAs with better attribution penalty, compute an orbit with constrained differential corrections. If they converges we have 5 parameter solutions (LOV solutions), that can be used as starting guesses for full differential corrections

Milani method: Warnings

- **Warning 1, Asteroid identification:** the correspondence between very short arcs and the physical objects is not known; provide an algorithm to know whether 2 TSAs belong to the same object *(attribution and linkage)*
- **Warning 2, Multiple solutions:** in some cases there can be multiple orbits.

Milani method: algorithm

Assume that 2 attributables belonging to the same object are available.

1. The admissible region of the first attributable is computed and sampled by a Delaunay triangulation, providing a set of VAs

2. The predictions for the time of the second arc, computed from the VAs of the first, are compared with the second arc attributable

3. For the VAs with lower attribution penalty compute a preliminary orbit

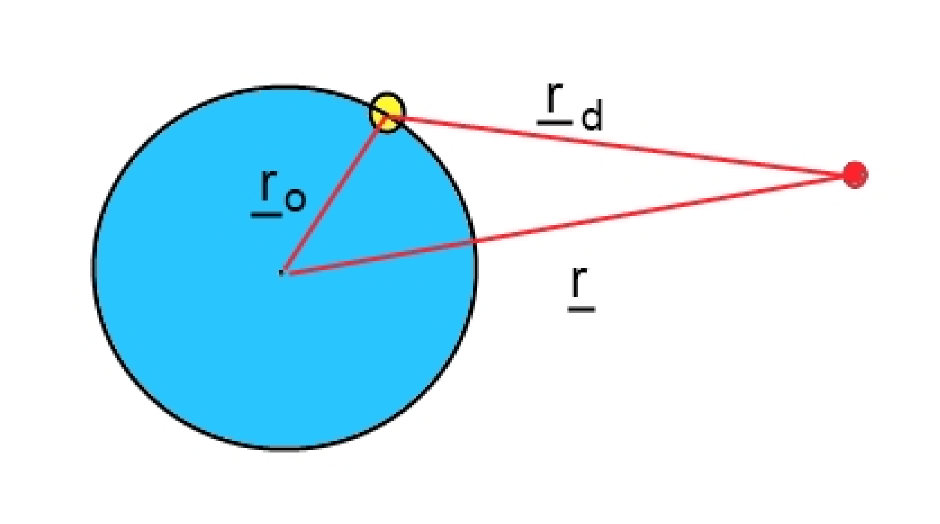
4. Use these preliminary orbits as first guess in the iterations; when convergence, they provide solutions fitting both very short arcs

5. Propagate these solutions and compare with the attributables of the third arc, fourth arc, etc.

6. The orbit resulting from the fit to all very short arcs is used with its covariance for additional attributions, when more observations are available.

Orbit determination for space debris

- Modify the definition of admissible region taking into account the orbits of the Earth's satellites
- $\vec{r}$ = geocentric position of a space debris, $\vec{r}_0$ = geocentric position of the observer, $\vec{r}_d = \rho\vec{b}$ topocentric position of the space debris



Debris admissible region

- Geocentric two-body energy:

$$E_E(\rho, \dot{\rho}) = \frac{1}{2} \, \|\dot{\vec{r}}\|^2 - \frac{m_E}{\vec{r}}$$

- Lower bound for the space debris distance from the position of the observer on the Earth:

$$\rho_{min} = 2R_E \simeq 12\,756 \ km$$

- Upper bound for the space debris distance from the position of the observer on the Earth

$$\rho_{max} = 20R_E \simeq 127\,560 \ km$$

- $D_1 = \{(\rho, \dot{\rho}) : \ E_E \leq 0\}$ (it is a satellite of the Earth)
- $D_2 = \{(\rho, \dot{\rho}) : \ \rho_{min} < \rho < \rho_{max}\}$ (the distance from the observer is in the interval (ρ_{min}, ρ_{max}))
- Admissible region: $D_1 \bigcap D_2$

Debris algorithm

1. The admissible region of the first attributable is computed and sampled by a Delaunay triangulation, providing a set of Virtual debris (VD) objects
2. The predictions for the time of the second arc, computed from the VD of the first, are compared with the second arc attributable
3. For the VD, such that the attribution penalty is low, a preliminary orbit is computed
4. The above preliminary orbits are used as first guesses in constrained differential corrections