Everything is under control

Optimal control and applications to aerospace problems

E. Trélat

Univ. Paris 6 (Labo. J.-L. Lions) and Institut Universitaire de France

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What is control theory?

Controllability
Steer a system from an initial configuration to a final configuration.

Optimal control
Moreover, minimize a given criterion.

Stabilization
A trajectory being planned, stabilize it in order to make it robust, insensitive to perturbations.

Observability
Reconstruct the full state of the system from partial data.
Application domains of control theory:

**Mechanics**

Vehicles (guidance, dampers, ABS, ESP, ...),
Aeronautics, aerospace (shuttle, satellites), robotics

**Biology, medicine**

Predator-prey systems, bioreactors, epidemiology, medicine (peacemakers, laser surgery)

**Electricity, electronics**

RLC circuits, thermostats, regulation, refrigeration, computers, internet and telecommunications in general, photography and digital video

**Economics**

Gain optimization, control of financial flux, Market prevision

**Chemistry**

Chemical kinetics, engineering process, petroleum, distillation, petrochemical industry
Here we focus on applications of control theory to problems of aerospace.
The orbit transfer problem with low thrust

Controlled Kepler equation

\[ \ddot{q} = -q \frac{\mu}{r^3} + \frac{F}{m} \]

\( q \in \mathbb{R}^3 \): position, \( r = |q| \), \( F \): thrust, \( m \) mass:

\[ \dot{m} = -\beta |F| \]

Maximal thrust constraint

\[ |F| = (u_1^2 + u_2^2 + u_3^2)^{1/2} \leq F_{\text{max}} \approx 0.1 N \]

Orbit transfer

from an initial orbit to a given final orbit.

Controllability properties studied in


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Modelization in terms of an optimal control problem

State: \( x(t) = \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix} \)

Control: \( u(t) = F(t) \)

Optimal control problem

\[
\dot{x}(t) = f(x(t), u(t)), \quad x(t) \in \mathbb{R}^n, \quad u(t) \in \Omega \subset \mathbb{R}^m, \\
x(0) = x_0, \quad x(T) = x_1,
\]

\[
\min C(T, u), \quad \text{where} \quad C(T, u) = \int_0^T f^0(x(t), u(t)) \, dt
\]
### Optimal control problem

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\]
\[
x(T) = x_1, \quad \min C(T, u), \quad \text{where } C(T, u) = \int_0^T f^0(x(t), u(t)) \, dt.
\]

### Pontryagin Maximum Principle

Every minimizing trajectory \( x(\cdot) \) is the projection of an extremal \( (x(\cdot), p(\cdot), p^0, u(\cdot)) \) solution of

\[
\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}, \quad H(x, p, p^0, u) = \max_{v \in \Omega} H(x, p, p^0, v),
\]

where \( H(x, p, p^0, u) = \langle p, f(x, u) \rangle + p^0 f^0(x, u) \).

An extremal is said normal whenever \( p^0 \neq 0 \), and abnormal whenever \( p^0 = 0 \).
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\[ u(t) = u(x(t), p(t)) \]

(locally, e.g. under the strict Legendre assumption: \( \frac{\partial^2 H}{\partial u^2}(x, p, u) \) negative definite)
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**Pontryagin Maximum Principle**

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Shooting method:

**Extremals** \((x, p)\) are solutions of

\[
\begin{align*}
\dot{x} &= \frac{\partial H}{\partial p}(x, p), \quad x(0) = x_0, \quad (x(T) = x_1), \\
\dot{p} &= -\frac{\partial H}{\partial x}(x, p), \quad p(0) = p_0,
\end{align*}
\]

where the optimal control maximizes the Hamiltonian.

**Exponential mapping**

\[
\exp_{x_0}(t, p_0) = x(t, x_0, p_0),
\]

(extendal flow)

\[\Rightarrow \text{Shooting method: determine } p_0 \text{ s.t. } \exp_{x_0}(t, p_0) = x_1.\]

**Remark**

- **PMP** = first-order necessary condition for optimality.
- Necessary / sufficient (local) second-order conditions: **conjugate points**.
  \[\Rightarrow \text{test if } \exp_{x_0}(t, \cdot) \text{ is an immersion at } p_0.\]
There exist other numerical approaches to solve optimal control problems:

- **direct methods**: discretize the whole problem
  \[\Rightarrow\] finite-dimensional nonlinear optimization problem with constraints

- Hamilton-Jacobi methods.

The shooting method is called an **indirect method**.

In the present aerospace applications, the use of shooting methods is privileged in general because of their very good numerical accuracy.

**BUT**: difficult to make converge... *(Newton method)*

To improve their performances and widen their domain of applicability, optimal control tools must be combined with other techniques:

- geometric tools \(\Rightarrow\) geometric optimal control
- continuation or homotopy methods
- dynamical systems theory
Orbit transfer, minimal time

Maximum Principle $\Rightarrow$ the extremals $(x, p)$ are solutions of

$$\dot{x} = \frac{\partial H}{\partial p}, \quad x(0) = x_0, \quad x(T) = x_1, \quad \dot{p} = -\frac{\partial H}{\partial x}, \quad p(0) = p_0,$$

with an optimal control saturating the constraint: $\|u(t)\| = F_{\text{max}}$.

$\rightarrow$ **Shooting method:** determine $p_0$ s.t. $x(T) = x_1$,

combined with a homotopy on $F_{\text{max}} \mapsto p_0(F_{\text{max}})$

---

**Heuristic on $t_f$:**

$$t_f(F_{\text{max}}) \cdot F_{\text{max}} \simeq \text{cste.}$$

(the optimal trajectories are "straight lines", Bonnard-Caillau 2009)
Orbit transfer, minimal time

\[ F_{\text{max}} = 6 \text{ Newton} \]

\[ P_0 = 11625 \text{ km}, \ |e_0| = 0.75, \ i_0 = 7^\circ, \ P_f = 42165 \text{ km} \]

Minimal time: 141.6 hours (\( \sim 6 \text{ days} \)). First conjugate time: 522.07 hours.
Main tool used: continuation (homotopy) method
→ continuity of the optimal solution with respect to a parameter $\lambda$

Theoretical framework (sensitivity analysis):

$$\exp_{x_0,\lambda}(T, p_0(\lambda)) = x_1$$

Local feasibility is ensured: in the absence of conjugate points.

Global feasibility is ensured: in the absence of abnormal minimizers.

this holds true for **generic** systems having more than 3 controls
(Chitour-Jean-Trélat, J. Differential Geom., 2006)
Recent work with EADS Astrium (now Airbus DS):

Minimal consumption transfer for launchers Ariane V and next Ariane VI (third atmospheric phase, strong thrust)

Objective: automatic and instantaneous software.

continuation on the curvature of the Earth (flat Earth $\rightarrow$ round Earth)


eclipse constraints $\rightarrow$ state constraints, hybrid systems

Optimal control

A challenge (urgent!!)

Collecting space debris:
- 22000 debris of more than 10 cm (cataloged)
- 500000 debris between 1 and 10 cm (not cataloged)
- millions of smaller debris

→ difficult mathematical problems combining optimal control, continuous / discrete / combinatorial optimization
(Max Cerf, PhD 2012)

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Around the geostationary orbit
**Optimal control**

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The circular restricted three-body problem

Dynamics of a body with negligible mass in the gravitational field of two masses \( m_1 \) and \( m_2 \) (primaries) having circular orbits:

### Equations of motion in the rotating frame

\[
\begin{align*}
\ddot{x} - 2\dot{y} &= \frac{\partial \Phi}{\partial x} \\
\dot{y} + 2\dot{x} &= \frac{\partial \Phi}{\partial y} \\
\ddot{z} &= \frac{\partial \Phi}{\partial z}
\end{align*}
\]

with

\[
\Phi(x, y, z) = \frac{x^2 + y^2}{2} + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} + \frac{\mu(1 - \mu)}{2},
\]

and

\[
\begin{align*}
r_1 &= \sqrt{(x + \mu)^2 + y^2 + z^2}, \\
r_2 &= \sqrt{(x - 1 + \mu)^2 + y^2 + z^2}.
\end{align*}
\]

Some references

American team: Koon, Lo, Marsden, Ross...

Spanish team: Gomez, Jorba, Llibre, Masdemont, Simo...

Optimal control and applications to aerospace problems
**Lagrange points**

Jacobi integral \( J = 2\Phi - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \) \( \rightarrow \) 5-dimensional energy manifold

Five equilibrium points:
- 3 collinear equilibrium points: \( L_1, L_2, L_3 \) (unstable);
- 2 equilateral equilibrium points: \( L_4, L_5 \) (stable).

(see Szebehely 1967)

Extension of a Lyapunov theorem (Moser) \( \Rightarrow \) same behavior than the linearized system around Lagrange points.
Lagrange points in the Earth-Sun system

From Moser’s theorem:
- \( L_1, L_2, L_3 \): unstable.
- \( L_4, L_5 \): stable.
Lagrange points in the Earth-Moon system

- $L_1, L_2, L_3$: unstable.
- $L_4, L_5$: stable.
Points L4 and L5 (stable) in the Sun-Jupiter system: Trojan asteroids
Examples of objects near Lagrange points

Sun-Earth system:

Point L1: SOHO

Point L2: JWST

Point L3: planet X...
From a Lyapunov-Poincaré theorem, there exist:

- a 2-parameter family of periodic orbits around $L_1$, $L_2$, $L_3$
- a 3-parameter family of periodic orbits around $L_4$, $L_5$

Among them:

- planar orbits called Lyapunov orbits;
- 3D orbits diffeomorphic to circles called halo orbits;
- other 3D orbits with more complicated shape called Lissajous orbits.

(see Richardson 1980, Gomez Masdemont Simo 1998)
Examples of the use of halo orbits:

Orbit of SOHO around L1

(requires control by stabilization)

Invariant manifolds

Invariant manifolds (stable and unstable) of periodic orbits: 4-dimensional tubes \((S^3 \times \mathbb{R})\) inside the 5-dimensional energy manifold. (they play the role of separatrices)

→ invariant "tubes", kinds of "gravity currents" ⇒ low-cost trajectories
Invariant manifolds

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\[ \rightarrow \text{invariant ”tubes”, kinds of ”gravity currents”} \Rightarrow \text{low-cost trajectories} \]
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Cartography ⇒ design of low-cost interplanetary missions
Meanwhile...

Back to the Moon

⇒ lunar station: intermediate point for interplanetary missions

**Challenge:** design low-cost trajectories to the Moon and flying over all the surface of the Moon.

Mathematics used:
- dynamical systems theory
- differential geometry
- ergodic theory
- control
- scientific computing
- optimization
Eight Lissajous orbits

(PhD thesis of G. Archambeau, 2008)
Periodic orbits around $L_1$ et $L_2$ (Earth-Moon system) having the shape of an eight:

$\Rightarrow$ Eight-shaped invariant manifolds:
Invariant manifolds of Eight Lissajous orbits

We observe numerically that they enjoy two nice properties:

1) Stability in long time of invariant manifolds

→ global structure conserved

(numerical validation by computation of local Lyapunov exponents)

Invariant manifolds of a halo orbit:

→ chaotic structure in long time

E. Trélat

Optimal control and applications to aerospace problems
Invariant manifolds of Eight Lissajous orbits

We observe numerically that they enjoy two nice properties:

2) Flying over almost all the surface of the Moon

Invariant manifolds of an eight-shaped orbit around the Moon:

- oscillations around the Moon
- global stability in long time
- minimal distance to the Moon: 1500 km.

Partnership between EADS Astrium (les Mureaux, France) and FSMP (Fondation Sciences Mathématiques de Paris). Kick off in May 2014.

- Planning low-cost "cargo" missions to the Moon (using gravity currents) → Maxime Chupin, ongoing PhD
- Interplanetary missions: compromise between low cost and long transfer time; gravitational effects (swing-by)
- Collecting space debris (urgent!)
- Optimal design of space vehicles
- Optimal placement problems (vehicle design, sensors)
- Inverse problems: reconstructing a thermic, acoustic, electromagnetic environment (coupling ODE’s / PDE’s)
- Robustness problems
- ...
Invariant manifolds of eight-shaped Lissajous orbits

\( \Phi(\cdot, t) \): transition matrix along a reference trajectory \( x(\cdot) \)

\( \Delta > 0 \).

Local Lyapunov exponent

\[
\lambda(t, \Delta) = \frac{1}{\Delta} \ln \left( \text{maximal eigenvalue of } \sqrt{\Phi(t + \Delta, t)\Phi^T(t + \Delta, t)} \right)
\]

Simulations with \( \Delta = 1 \) day.
LLE of an eight-shaped Lissajous orbit:

LLE of an halo orbit:

LLE of an invariant manifold of an eight-shaped Lissajous orbit:

LLE of an invariant manifold of an halo orbit: