1. Let A be of size $n \times n$ and of rank k, and B is a nonsingular submatrix of order k. Denote by R the submatrix of size $k \times n$ of the rows with B, and by C a submatrix of size $n \times k$ of the columns containing B. Prove that

$$A = CB^{-1}R.$$

2. A and B are matrices of rank 1 and $AB = BA \neq 0$. Prove that the rank of $A + B$ does not exceed 1.

3. A matrix A has r columns, and B has r rows. Prove that

$$r \geq \text{rank}(A) + \text{rank}(B) - \text{rank}(AB).$$

4. All singular values of a square matrix A are less than or equal to 1. Prove that

$$A^*(I - AA^*)^{1/2} = (I - A^*A)^{1/2}A^*.$$

5. Let

$$A = \begin{bmatrix} 1 \\ 2 \\ \dots \\ n \end{bmatrix} [1\ 1 \dots\ 1].$$

   (a) Find the positive singular values and corresponding singular vectors of A.

   (b) Find the normal pseudo-solution vector (a minimal-length vector providing the minimal possible residual) of the linear algebraic system with A as a coefficient matrix and the right-hand side vector $b = [1, 1, ..., 1]^t$.

6. Let A be an upper bidiagonal matrix with positive entries on the two diagonals, and the "see-saw"algorithm produces a sequence $A = A_0, A_1, ...$ of upper (for even k) and lower (for odd k) bidiagonal matrices and a sequence of orthogonal matrices $Q_0, Q_1, ...$ such that $A_{k+1} = A_K Q_k$, if k is even, and $A_{k+1} = Q_k A_k$, if k is odd. Denote by $a_i(k)$ and $b_i(k)$ the diagonal and off-diagonal entries of A_k. Show that Q_k can be chosen so that $a_i(k) > 0$ and $b_i(k) > 0$ for all k. With this choice, prove that

$$a_{i+1}(k)\, b_i(k) = a_i(k+1)\, b_i(k+1), \quad 1 \leq i \leq n-1,$$

and use this to prove that

$$\lim_{k\to\infty} a_i(k) \geq \lim_{k\to\infty} a_{i+1}(k), \quad 1 \leq i \leq n-1.$$

7. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find the best rank-1 approximation to A: (a) in the spectral norm; (b) in the Frobenius norm. Is it unique? What is the distance from A to the nearest nonsingular matrix?

8. Let A be a square matrix. Prove that $\lambda_{min}(A + A^*) \leq 2\sigma_{min}(A)$, where $\lambda_{min}(\cdot)$ и $\sigma_{min}(\cdot)$ denote the minimal eigenvalue and minimal singular value. Can $\lambda_{\min}$ in the right-hand side be replaced with $\sigma_{\min}$?

9. Prove that:

   (a) $\sigma_1(A) = \max\limits_{||u||_2 = ||v||_2 = 1} |u^* A v|$;

   (b) $f(A) = \sigma_1(A) + \sigma_2(A)$ is a unitarily invariant norm of matrix A.