On Tridiagonalization of Matrices

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ABSTRACT

We consider the question: Is every $n \times n$ complex matrix unitarily similar to a tridiagonal one? It is shown that the answer is negative if $n \geqslant 6$, and is affirmative if $n = 3$. Additionally, some positive partial answers and related results are given. For example, (1) every pair of (Hermitian) projections is simultaneously unitarily similar to a pair of tridiagonal matrices; (2) if $A - A^*$ has a rank one, then A is unitarily similar to a tridiagonal matrix.

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1. INTRODUCTION

Throughout, H denotes a finite-dimensional Hilbert space of dimension at least three and all scalars are complex. It is well known that every normal transformation A on H is (*unitarily*) *diagonalizable*, that is, there exists an orthonormal basis relative to which the matrix of A is diagonal. A square matrix $B = (b_{ij})$ is called *tridiagonal* if all its entries below the first subdiagonal and all those above the first superdiagonal are zero, that is, if $b_{ij} = 0$ for $|i - j| > 1$. The theory of Jordan canonical form shows that for every transformation T on H there exists a basis for H relative to which the matrix of T is tridiagonal (even bidiagonal). We will call T *tridiagonalizable* if there is an *orthonormal* basis relative to which the matrix of T is tridiagonal. The question: Is every transformation on a space of dimension n tridiagonalizable? is obviously equivalent to: Is every $n \times n$ complex matrix unitarily similar to a tridiagonal matrix? It is shown that the answer is negative if

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$n \geqslant 6$ and affirmative if $n = 3$. The situation when $n = 4$ or 5 remains unresolved. Additionally, we mention some positive partial answers and some related results. Throughout, $(\cdot|\cdot)$ denotes the inner product on H, and for vectors $e, f, g, \ldots$ of H, $\langle e, f, g, \ldots \rangle$ denotes their linear span. By a *reducing subspace* of a transformation T on H is meant a (vector) subspace M of H such that both M and $M^{\perp}$ are invariant under T. The norm on H is denoted by $\|\cdot\|$, and for a transformation T on H, $\|T\|$ denotes the corresponding operator norm of T, given by $\|T\| = \sup_{\|x\| \leqslant 1} \|Tx\|$.

2. ON TRIDIAGONALIZATION

THEOREM 2.1. *For every $n \geqslant 6$ here exists a complex $n \times n$ matrix which is not unitarily similar to a tridiagonal matrix.*

Proof. Let $n \geqslant 6$, and let $M(n)$ $[U(n), \Delta(n)]$ denote the set of complex $n \times n$ matrices [complex unitary $n \times n$ matrices, complex tridiagonal $n \times n$ matrices]. It is sufficient to show that the mapping $p: \Delta(n) \times U(n) \to M(n)$ defined by $p(T, U) = UTU^*$ is not surjective. Identifying $M(n)$ with $\mathbb{R}^{2n^2}$ in the usual way, the dimension of $M(n)$ as a real differentiable manifold is $2n^2$. Similarly, the dimension of $\Delta(n)$ is $6n - 4$. The real Lie group $\mathrm{U}(n)$ has as associated Lie algebra the real vector space of skew Hermitian matrices, so $\dim \mathrm{U}(n) = n^2$ [7, p. 108]. The mapping p is a polynomial mapping and so is smooth. Since $\dim[\Delta(n) \times \mathrm{U}(n)] = n^2 + 6n - 4 < 2n^2 = \dim M(n)$, every point of $\Delta(n) \times U(n)$ is a critical point of p, so by Sard's theorem [2, 16.23, p. 167] the image of $\Delta(n) \times \mathrm{U}(n)$ under p has measure zero in $M(n)$. Thus p is not surjective and the proof is complete. ∎

The preceding result was first proved, very recently, by B. Sturmfels [6]. The proof given above is a similar version, slightly simpler though no more elegant, and is due to J. L. Noakes. The author thanks these authors for allowing this proof to be included here.

So, at least if $\dim H \geqslant 6$, not every transformation on H is tridiagonalizable. Which transformations *are* tridiagonalizable?

Influenced by the case of normal transformations, one might entertain the idea that, if the transformation T on H is tridiagonalizable, then $T^*T - TT^*$ is in some sense "small." The following example shows that this measure of "smallness" cannot be rank. Note that $T^*T - TT^*$ can never be of rank one, since it is self-adjoint (that is, Hermitian) with zero trace.

EXAMPLE. Let $n \geqslant 3$, and let $m \in \mathbf{Z}$ satisfy $2 \leqslant m \leqslant n$. Let C be the $n \times n$ matrix

$$C = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ c_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & c_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & c_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_{n-1} & 0 \end{pmatrix},$$

where

$$c_j = \min\{\sqrt{j}, \sqrt{m-1}\}.$$

Here $C^*C - CC^*$ has rank m, since

$$C^*C - CC^* = \operatorname{diag}(1, 1, \ldots, 1, 0, \ldots, 0, -(m-1)),$$

where there are $m - 1$ ones.

Besides rank, another measure of the "size" of $T^*T - TT^*$ is its norm. Of course, if T is tridiagonalizable and $T^*T - TT^*$ is nonzero, then T can be rescaled, preserving tridiagonalizability, so as to make the value of $\|T^*T - TT^*\|$ any given positive number. On the other hand, for any T satisfying $\|T\| = 1$, we have $\|T^*T - TT^*\| \leqslant 1$. For,

$$\|T^*T - TT^*\| = \sup_{\|x\| = 1} |((T^*T - TT^*)x|x)|,$$

and for $\|x\| = 1$ we have

$$|((T^*T - TT^*)x|x)| = |\|Tx\|^2 - \|T^*x\|^2| \leqslant 1,$$

since $\|Tx\| \leqslant 1$ and $\|T^*x\| \leqslant 1$. Can $\|T^*T - TT^*\| = 1$ with $\|T\| = 1$ and T tridiagonalizable? The answer is affirmative, and the transformation $T/\|T\|$, with T the transformation on $\mathbf{C}^n$ whose matrix relative to the usual basis is C as in the preceding example, establishes it to be so. Another example is the transformation T_1 on $H \oplus H$ given by $T_1(x, y) = (ix + y, x - iy)/2$, that is, with operator-entried matrix representation

$$\frac{1}{2}\begin{pmatrix} iI & I \\ I & -iI \end{pmatrix},$$

where I is the identity transformation on H (T_1 is tridiagonalizable, since both $2\,\mathrm{Re}\,T_1$ and $2\,\mathrm{Im}\,T_1$ are symmetries and thus simultaneously tridiagonalizable; see Corollary 3.2.1). Actually, slightly more is true.

PROPOSITION 2.2. *For every $\alpha \in [0,1]$ there exists a tridiagonalizable transformation T on H satisfying $\|T\| = 1$ and $\|T^*T - TT^*\| = \alpha$.*

Proof. If $\alpha = 0$ take $T = I$. Suppose $\alpha \in (0,1]$. Let S be any tridiagonalizable transformation satisfying $\|S\| = \|S^*S - SS^*\| = 1$. Define the function $f: \mathbb{R} \to \mathbb{R}$ by $f(\xi) = \|S - \xi I\|$. Then, for every $\xi, \eta \in \mathbb{R}$ we have $|f(\xi) - f(\eta)| \leqslant |\xi - \eta|$, so f is continuous on $\mathbb{R}$. Also $f(\xi) \geqslant |1 - |\xi||$ for every $\xi \in \mathbb{R}$. Since $f(0) = 1 \leqslant 1/\sqrt{\alpha}$ and $f(1 + 1/\sqrt{\alpha}) \geqslant 1/\sqrt{\alpha}$, we have $f(\xi_0) = 1/\sqrt{\alpha}$ for some $\xi_0 \in \mathbb{R}$ by the intermediate-value theorem. Let $T = (S - \xi_0 I)/\|S - \xi_0 I\|$. Then T is tridiagonalizable, it has norm one, and

$$\|T^*T - TT^*\| = \frac{\|S^*S - SS^*\|}{\|S - \xi_0 I\|^2} = \frac{1}{f(\xi_0)^2} = \alpha.$$

If a transformation T on H is tridiagonalized by some orthonormal basis, then the same basis tridiagonalizes T^* and both $\mathrm{Re}\,T = (T + T^*)/2$ and $\mathrm{Im}\,T = (T - T^*)/2i$. Conversely, if an orthonormal basis simultaneously tridiagonalizes both $\mathrm{Re}\,T$ and $\mathrm{Im}\,T$, it tridiagonalizes $T = \mathrm{Re}\,T + i\,\mathrm{Im}\,T$. This shows that our earlier question is equivalent to "Which pairs of self-adjoint transformations are simultaneously tridiagonalizable?" (and to the equivalent question concerning pairs of Hermitian matrices). The latter formulation would seem to present a more tractable problem than the original, primarily because, for a given self-adjoint transformation A, all tridiagonalizing orthonormal bases can be described. This is the substance of the famous Lanczos algorithm [4, 5]. Some positive partial results to this reformulation are given in the next section, but before considering these, observe that if $\dim H = 3$ then things are easy.

PROPOSITION 2.3. *If* $\dim H = 3$, *every transformation* T *on* H *is tridiagonalizable.*

Proof. Let $f_1 \in H$ be a unit eigenvector of T. The dimension of $\langle f_1, Tf_1, T^*f_1 \rangle$ is at most two. Let $f_3 \in \langle f_1, Tf_1, T^*f_1 \rangle^{\perp}$ be a unit vector. Then $(f_3 \mid f_1) = 0$. Let $\{f_1, f_2, f_3\}$ be an orthonormal basis for H. Then $\{f_1, f_2, f_3\}$ tridiagonalizes T, since $(Tf_1 \mid f_3) = 0$ and $(Tf_3 \mid f_1) = (f_3 \mid T^*f_1) = 0$. ∎

COROLLARY 2.3.1. *If* $\dim H = 3$, *every pair of self-adjoint transformations on* H *is simultaneously tridiagonalizable.*

3. SIMULTANEOUS TRIDIAGONALIZATION

We now describe some results concerning simultaneous tridiagonalization of pairs of self-adjoint transformations and related results. In the following, by a *projection* we mean a self-adjoint idempotent transformation, and by a *symmetry* we mean a self-adjoint unitary transformation. First we show that tridiagonal unitary matrices are in fact quasidiagonal in the sense of Watters [8].

PROPOSITION 3.1. *If* U *is a tridiagonal* $n \times n$ *unitary matrix, then* U *has the block diagonal form* $\operatorname{diag}(u_1, u_2, \dots, u_k)$, *where* u_i *is either* 1×1 *or* 2×2 *for each* $i = 1, 2, \dots, k$.

Proof. The proof is by induction on n. For $n = 1$ or 2 there is nothing to prove. Suppose

$$U = \begin{pmatrix} a_1 & b_1 & 0 \\ c_1 & a_2 & b_2 \\ 0 & c_2 & a_3 \end{pmatrix}.$$

Note that $|a_1|^2 + |b_1|^2 = |a_1|^2 + |c_1|^2 = 1$, so $|b_1| = |c_1|$. Also, $|c_1|^2 + |a_2|^2 + |b_2|^2 = |b_1|^2 + |a_2|^2 + |c_2|^2 = 1$, so $|b_2| = |c_2|$. Suppose $b_1 = c_1 = 0$ is false. Then both b_1 and c_1 are nonzero, and since $a_1\bar{c}_1 + b_1\bar{a}_2 = 0$, we have $a_1\bar{c}_1 = -b_1\bar{a}_2$, so $|a_1| = |a_2|$. Now $|a_1|^2 + |b_1|^2 = |b_1|^2 + |a_2|^2 + |c_2|^2 = 1$ gives $c_2 = 0$. Thus $b_2 = c_2 = 0$. Hence the result is true for $n = 3$.

Assume the result is true for every $n < m$. Let

$$U = \begin{pmatrix} a_1 & b_1 & 0 & 0 & \cdots & 0 & 0 \\ c_1 & a_2 & b_2 & 0 & \cdots & 0 & 0 \\ 0 & c_2 & a_3 & b_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{m-1} & b_{m-1} \\ 0 & 0 & 0 & 0 & \cdots & c_{m-1} & a_m \end{pmatrix}.$$

The above argument again gives either $b_1 = c_1 = 0$ or $b_2 = c_2 = 0$. The argument is completed by using the induction assumption and the fact that if

$$\begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}$$

is unitary, then U_2 is unitary. ■

COROLLARY 3.1.1. *If P is a tridiagonal (Hermitian) projection matrix, then P has the block diagonal form* $\operatorname{diag}(p_1, p_2, \ldots, p_k)$, *where p_i is either 1×1 or 2×2 for each $i = 1,2,\ldots,k$.*

Proof. $U = 2P - I$ is unitary. ■

The preceding proposition and its corollary show that tridiagonality is a fairly strong imposition on a unitary or projection matrix. In view of this one might be led to believe that simultaneous tridiagonality of a pair of unitaries or a pair of projections is a nontrivial imposition. For projections this is not the case; for unitaries it is (see the next example).

THEOREM 3.2. *Every pair of projections on H is simultaneously tridiagonalizable.*

Proof. Recall that we are assuming $\dim H \geqslant 3$. The proof is by induction on $n = \dim H$ and uses the fact that for every pair of projections on H there exists a projection different from 0 and I commuting with both. The latter result was first proved in [1] (another proof is given in [3]) as follows. Let E and F be projections on H. Define C (the *closeness operator*) by $C = 1 - E - F + EF + FE = EFE + (1-E)(1-F)(1-E)$. It is enough to show that there is a nonscalar transformation commuting with both E and F (since the commutant of $\{E, F\}$ is generated as a von Neumann algebra by the projections it contains). Now C commutes with both E and F, so if C is nonscalar we are done. Assume that C is scalar. We may assume $E \neq 0$. Choose a nonzero vector x belonging to the range of E. Obviously, F leaves $\langle x, Fx \rangle$ invariant. Also, E leaves $\langle x, Fx \rangle$ invariant, since $Ex = x$ and $EFx = EFEx = Cx$, where Cx is a scalar multiple of x. Thus the projection with range $\langle x, Fx \rangle$ commutes with both E and F and is nonscalar, since $\langle x, Fx \rangle$ is neither (0) nor H.

The result is true for $n = 3$, by Corollary 2.3.1. Assume the result is true for every $n < m$, and let P and Q be projections on H, where $\dim H = m$. Let $R \neq 0, I$ be a projection commuting with both P and Q. Let M be the range of R. By commutativity, M is a nontrivial reducing subspace of P and of Q. Relative to the decomposition $H = M \oplus M^{\perp}$ of H, P is represented by an operator-entried matrix, say

$$\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix},$$

where $P_1 : M \to M$ is the restriction of P to M and $P_2 : M^{\perp} \to M^{\perp}$ is the restriction of P to $M^{\perp}$. Both P_1 and P_2 are projections. Similarly Q has the representation

$$\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$$

with both Q_1 and Q_2 projections. There is an orthonormal basis $\{x_1, x_2, \ldots, x_k\}$ of M which simultaneously tridiagonalizes P_1 and Q_1. This is obvious if $\dim M < 3$, and it follows from the induction assumption otherwise. Similarly, there is an orthonormal basis $\{y_1, y_2, \ldots, y_l\}$ of $M^{\perp}$ which simultaneously tridiagonalizes P_2 and Q_2. Clearly the orthonormal basis $\{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_l\}$ of H simultaneously tridiagonalizes P and Q. ■

COROLLARY 3.2.1. *Every pair of symmetries on H is simultaneously tridiagonalizable.*

Proof. If V is a symmetry, $(V + I)/2$ is a projection. ■

It follows fairly easily from Proposition 3.1 that if two unitaries U_1 and U_2 are simultaneously tridiagonalizable, then there exist eigenvectors f_1 and f_2 of U_1 and U_2 respectively such that $(f_1 \mid f_2) = 0$. It follows that not every pair of unitaries is simultaneously tridiagonalizable, as the following example shows.

EXAMPLE. Let $n \geq 3$, and define the $n \times n$ matrix P by $P = (1/n)J$, where J is the matrix whose every entry is one. Put $V = 2P - I$. Let $\omega_1, \omega_2, \ldots, \omega_n$ be distinct complex numbers of modulus one, and let U_1 $[U_2]$ be the unitary whose matrix relative to the usual basis of $\mathbb{C}^n$ is $\mathrm{diag}(\omega_1, \omega_2, \ldots, \omega_n)$ $[V\mathrm{diag}(\omega_1, \omega_2, \ldots, \omega_n)V]$. The eigenvectors of U_1 are the nonzero scalar multiples of the usual basis vectors, and the eigenvectors of U_2 are the nonzero scalar multiples of the column vectors of V. Since V has no zero entries, U_1 and U_2 are not simultaneously tridiagonalizable.

The following proposition is used to show that, if T acts on H and if T and T^* differ by a rank-one transformation, then T is tridiagonalizable.

PROPOSITION 3.3. *If P is a rank-one projection and A is a self-adjoint transformation on H, then P and A are simultaneously tridiagonalizable.*

Proof. Let e be a unit vector spanning the range of P, and let $W(e)$ be the cyclic invariant subspace of A generated by e. Put $q = \dim W(e)$. Then $\{e, Ae, A^2e, \ldots, A^{q-1}e\}$ is a basis for $W(e)$. Now $W(e)$ is invariant under both A and P; in fact $PW(e)^{\perp} = 0$. Thus, relative to the decomposition $H = W(e) \oplus W(e)^{\perp}$, since A is self-adjoint, A and P are represented by operator-entried matrices of the forms, respectively,

$$\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}$$

with B and C self-adjoint transformations on $W(e)$ and $W(e)^{\perp}$ respectively and with Q a rank-one projection on $W(e)$. Since C is diagonalizable, it suffices to show that P and A are simultaneously tridiagonalizable if $W(e) = H$. But this is fairly obvious: The orthonormal basis $\{g_1, g_2, \ldots, g_n\}$ obtained by applying the Gram-Schmidt process to $\{e, Ae, \ldots, A^{n-1}e\}$ simultaneously tridiagonalizes A and P. For, if $1 \leq i \leq n-2$, then $g_i \in \langle e, Ae, \ldots, A^{i-1}e \rangle$, so Ag_i and Pg_i (a scalar multiple of $g_1 = e$) both belong to $\langle e, Ae, \ldots, A^i e \rangle = \langle g_1, g_2, \ldots, g_{i+1} \rangle$. Since A and P are self-adjoint, the result follows. ∎

COROLLARY 3.3.1. *If T is a transformation on H such that the linear span of T and T^* contains a rank-one projection, then T is tridiagonalizable.*

Proof. Let P be a rank-one projection such that $P = aT + bT^*$ for some scalars a and b. Here a and b are not both zero. Write $T = A + iB$ with both A and B self-adjoint. Then $P = (a+b)A + i(a-b)B$, and $a+b$ and $a-b$ are not both zero. Suppose $a + b \neq 0$. By the preceding proposition some orthonormal basis simultaneously tridiagonalizes P and B. The same basis tridiagonalizes

$$A = \frac{1}{a+b}P - i\left(\frac{a-b}{a+b}\right)B$$

and hence T. If $a - b \neq 0$ a similar argument gives the result. ■

COROLLARY 3.3.2. *If T is a transformation on H such that $T - T^*$ has rank at most one, then T is tridiagonalizable.*

Proof. We may suppose that $T - T^*$ has rank one. Then $T - T^*$ is a nonzero scalar multiple of a rank-one projection, and the result follows from Corollary 3.3.1. ■

We conclude with some results valid on four-dimensional space.

PROPOSITION 3.4. *If A is a self-adjoint transformation and P is a projection on H, where* $\dim H = 4$, *then A and P are simultaneously tridiagonalizable.*

Proof. Let f_1 be a unit eigenvector of A. If f_1 is an eigenvector of P then, relative to the decomposition $H = \langle f_1\rangle \oplus \langle f_1\rangle^{\perp}$, A and P are represented by operator-entried matrices of the forms, respectively,

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix},$$

where a_2 and p_2 are self-adjoint transformations on $\langle f_1\rangle^{\perp}$. By Corollary 2.3.1, a_2 and p_2 are simultaneously tridiagonalizable and the desired result easily follows. Suppose f_1 is not an eigenvector of P. Choose a unit vector $f_2 \in \langle f_1\rangle^{\perp}$ such that $\langle f_1, f_2\rangle = \langle f_1, Pf_1\rangle$; then choose a unit vector $f_3 \in \langle f_1, f_2\rangle^{\perp}$ such that $Af_2 \in \langle f_1, f_2, f_3\rangle$. Finally, choose a unit vector $f_4 \in \langle f_1, f_2, f_3\rangle^{\perp}$. Then $\{f_1, f_2, f_3, f_4\}$ tridiagonalizes P, since $\langle f_1, f_2\rangle$ is an invariant and thus a reducing subspace of P, and this orthonormal basis obviously tridiagonalizes A. ■

PROPOSITION 3.5. *If U is a unitary and P is a rank-one projection on H, where* $\dim H = 4$, *then U and P are simultaneously tridiagonalizable.*

Proof. Let e be a unit vector spanning the range of P, and let M be any two-dimensional reducing subspace of U. There must exist a unit vector $f_1 \in M$ orthogonal to e; otherwise $M \cap \langle e \rangle^{\perp} = (0)$. Similarly, there exists a unit vector $f_4 \in M^{\perp}$ orthogonal to e. Let $\{f_1, f_2\}$ be an orthonormal basis for M and let $\{f_3, f_4\}$ be an orthonormal basis for $M^{\perp}$. Then $\{f_1, f_2, f_3, f_4\}$ tridiagonalizes U, and also P, since $Pf_1 = Pf_4 = 0$. ■

The following example shows that we cannot replace "rank-one projection" by "rank-two projection" in the preceding proposition.

EXAMPLE. Let $\{e_1, e_2, e_3, e_4\}$ be the usual basis for $\mathbf{C}^4$, and let U be the unitary whose matrix relative to this basis is $\operatorname{diag}(\omega_1, \omega_2, \omega_3, \omega_4)$, where ω_1, ω_2, ω_3, and ω_4 are distinct complex numbers of modulus one. The reducing subspaces of U are precisely those subspaces of the form $\langle\{e_i : i \in \mathscr{E}\}\rangle$ for some subset $\mathscr{E}$ of $\{1,2,3,4\}$. Let P be the projection with range $M = \langle(-2,1,1,0),(-1,-4,2,3)\rangle$. Relative to the usual basis the matrix of P is

$$\frac{1}{10}\begin{pmatrix} 7 & -2 & -4 & -1 \\ -2 & 7 & -1 & -4 \\ -4 & -1 & 3 & 2 \\ -1 & -4 & 2 & 3 \end{pmatrix}.$$

Since every entry of this matrix is nonzero, U and P have no common reducing subspaces except (0) and $\mathbf{C}^4$. Now $M^{\perp} = \langle(1,1,1,1),(0,1,-1,2)\rangle$, and for every two-dimensional reducing subspace N of U we have $M^{\perp} \cap N = (0)$, since if any two of $\alpha, \alpha+\beta, \alpha-\beta, \alpha+2\beta$ $(\alpha, \beta \in \mathbf{C})$ are zero, then $\alpha = \beta = 0$.

By Proposition 3.1 and its corollary, and bearing in mind that U and P have no common reducing subspaces except (0) and C^4, if there were an orthonormal basis $\{f_1, f_2, f_3, f_4\}$ simultaneously tridiagonalizing U and P, then, relative to this basis, the matrices of U and P would either have the forms, respectively,

$$\begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$$

or these interchanged. Here " * " denotes a possibly nonzero entry. Consider the first possibility. In this case the matrix of P cannot have either of the forms

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

since then either f_1 or f_4 would belong to a two-dimensional reducing subspace of U and to $M^{\perp}$. Thus, since P has rank two, its matrix would have to be diag(1,0,0,1). This contradicts U and P having no common reducing subspaces except (0) and $\mathbf{C}^4$.

Finally, consider the second possibility. In this case, both f_1 and f_4 would be eigenvectors of U, so there would exist a permutation π of $\{1,2,3,4\}$ such that $f_1 \in \langle e_{\pi(1)} \rangle$, $f_4 \in \langle e_{\pi(4)} \rangle$, and $\langle f_2, f_3 \rangle = \langle e_{\pi(2)}, e_{\pi(3)} \rangle$. But then $Pe_{\pi(1)} \in \langle e_{\pi(1)}, e_{\pi(2)}, e_{\pi(3)} \rangle$ and this contradicts the fact that the matrix of P relative to the usual basis has no zero entries.

Hence U and P are not simultaneously tridiagonalizable.

Note that, with U and P as in the above example, the unitary U and the symmetry $V = 2P - 1$ are not simultaneously tridiagonalizable.

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