Eigenvalues of large Toeplitz matrices: the asymptotic approach

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Abstract

The main goal of this course is to give an introduction to the modern state of the art in the asymptotic analysis of eigenvalues (and other spectral characteristics) of large Toeplitz matrices. We discuss the finite section method and stability, Szegö's limit theorems, and asymtotics of determinants. We also embark on the description of the limiting sets of the spectrum in some cases and on the asymptotics of the extreme eigenvalues. Eventually we turn to theorems about the asymptotics of individual eigenvalues and eigenvectors in several important special cases. In conclusion we formulate some open problems that might be of impact for further research.

Content of the Course.

1. Infinite Toeplitz matrices.
   Wiener-Hopf factorization. Description of spectra of Toeplitz operators. The cases of rational, continuous, piecewise continuous symbols.

2. Finite section method, stability.
   Asymptotic inverses. Stability criteria. The cases of real and even symbols.

3. Szegö's limit theorems.
   The fist Szegö limit theorem. Tyrtyshnikov's generalization. The strong Szegö limit theorem (asymtotics of determinants). The case of complex-valued symbols.

4. Limiting spectral set of sequences of Toeplitz matrices.
   The tridiagonal case. The theorem of Schmidt and Spitzer (polynomial case). The theorem of Day (rational case). Results of Widom concerning canonical distribution.

5. Asymptotics of extreme eigenvalues.
   Classical results - the smooth case. A case of a symbol with power singularity.

6. Asymptotics of inner eigenvalues - real-valued rational symbols.
   Widom's formula for Toeplitz determinants. Asymptotic analysis with the help of Widom's formula. Asymptotics of eigenvalues. Asymptotics of eigenvectors. The case of even rational symbols.

7. Asymptotics of inner eigenvalues - symbols having a power singularity.
   The conjecture of Dai, Geary, Kadanoff. Asymptotics of determinants that are uniform in spectral parameters. Asymptotics of eigenvalues - a proof of the conjecture of Dai, Geary, Kadanoff. Asymptotics of eigenvectors.

8. Open problems.

Knowledge desirable to follow the lessons of the course:

1. Standard courses of Functional Analysis and Complex Variables.
2. A. Böttcher, S. Grudsky. Toeplitz Matrices, Asymptotic Linear Algebra, and Functional Analysis. Hindustan Book Agency, New Delhi, 2000 and Birkhäuser Verlag, Basel,Boston, Berlin, 2000.
3. A. Böttcher, B. Silbermann. Introduction to Large Truncated Toeplitz Matrices. Universitext, Springer-Verlag, New York, 1999.

Additional Literature

1. Albrecht Böttcher and Sergei M. Grudsky. Spectral Properties of Banded Toeplitz Matrices. SIAM, Philadelphia, 2005, pp. 411.
2. Evgenii E. Tyrtyshnikov. A unifying approach to some old and new theorems on distribution and clustering. Linear Algebra and its Applications 232 (1996), 1-43.
3. Albrecht Böttcher, Sergei Grudsky and Moshe Schwartz. Some problems concerning the test functions in the Szegö and Avram-Parter theorems. Operator Theory: Advances and Applications 187 (2008), 81-93.
4. Albrecht Böttcher, Sergei M. Grudsky and Egor A. Maksimenko. Pushing the envelope of the test functions in the Szegö and Avram-Parter theorems. Linea Algebra and its Applications 429 (2008), 346-366
5. Albrecht Böttcher, Sergei M. Grudsky and Egor A. Maksimenko. The Szegö and Avram-Parter theorems for general test functions. Comptes Rendus Mathématique, Académie des Sciences, Paris, Ser.I, 346 (2008), 749-752.
6. Albrecht Böttcher, Sergei M. Grudsky, Egor A. Maksimenko and Jérémie Unterberger. The first order asymptotics of the extreme eigenvectors of certain Hermitian Toeplitz matrices. Integr. Equ. Oper. Theory 63 (2009), 165-180.
7. Albrecht Böttcher, Sergei M. Grudsky, Egor A. Maksimenko. On the structure of the eigenvectors of large Hermitian Toeplitz band matrices. Operator Theory: Advances and Applications 210 (2010), 15-36.
8. Albrecht Böttcher, Sergei M. Grudsky, Egor A. Maksimenko. Inside the eigenvalues of certain Hermitian Toeplitz band matrices. J. of Computational and Applied Mathematics 233 (2010), 2245-2264.