Singular limits and large time profiles for viscous Hamilton-Jacobi equations under Dirichlet conditions

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Aim of the talk:

Pb 1: describe the singular limit

$$\begin{cases} \lambda u_\lambda - \Delta u_\lambda + |\nabla u_\lambda|^q = f(x) & \text{in } \Omega \\ u_\lambda = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

- What happens as $\lambda \to 0$?

Pb. 2: Large time behavior of the viscous H-J equation

$$\begin{cases} u_t - \Delta u + |\nabla u|^q = f(x) & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), & [\text{or } u = g \in C(\partial\Omega)] \\ u(0) = u_0 & \text{in } \Omega\,. \end{cases} \tag{2}$$

- What happens as $t \to \infty$? Convergence Vs Blow-up rate/profile of $u(t)$

Range of exponent: $1 < q \leq 2$.

Note: No sign condition on u_0 and on f

A well known fact: the limit problem

$$\begin{cases} -\Delta u + |\nabla u|^q = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

may have no solutions.

There are two main reasons for failure of existence:

• the regularity of f (local effects: is the integrability of nonlinear term consistent with the linear potential of f) ?
(see deep results in [Hansson-Mazja-Verbitsky], [Mazja-Verbitsky],...)

• the *size* of f (global effects: also related to the Hamilton-Jacobi-Bellmann character).

Note: both reasons depend on superlinearity (and coerciveness) of first order terms. Indeed, the problem

$$-\Delta u + H(x, Du) = f$$

has solution for every f provided H has (at most) linear growth

(if f is smooth, even linear+$\log^\alpha$, $\alpha \le 1$; actually, if $H \lesssim h(|Du|)$, $\int^\infty \frac{ds}{h(s)} = \infty$)

Ex: when $q=2$ existence depends on eigenvalues [Kazdan-Kramer]:

$$\begin{cases} -\Delta u + |\nabla u|^2 = f(x) \\ u_{|_{\partial\Omega}} = 0 \end{cases} \qquad v = e^{-u} - 1 \qquad \begin{cases} -\Delta v = -f(x)(v+1) \\ v_{|_{\partial\Omega}} = 0 \end{cases}$$

If $f \le -\lambda_1$ (first eigenvalue), then there is no solution.

More precisely: when $q=2$ there exists a solution if and only if $\lambda_1(-\Delta + f, \Omega) > 0$.

On the other hand:

- If $\|f\|_\infty$ is sufficiently small, $\exists$ a solution:

$$\begin{cases} -\Delta u + |\nabla u|^q = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(e.g. [Alvino-Lions-Trombetti], [Hansson-Mazja-Verbitsky], [Ferone-Murat], [Maderna-Pagani-Salsa])

- if we add $\lambda\, u$, with $\lambda > 0$, there always exists a solution. (see [Boccardo-Murat-Puel] for general results).

Therefore: let $f \in L^\infty(\Omega)$,

- If $\lambda > 0$, there always exists a solution of

$$\begin{cases} \lambda u - \Delta u + |\nabla u|^q = f(x) & \text{in } \Omega \\ u = 0\,. \end{cases} \tag{3}$$

- There always exists a solution of

$$\begin{cases} u_t - \Delta u + |\nabla u|^q = f(x) & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), & u(0) = u_0 \qquad \text{in } \Omega\,. \end{cases} \tag{4}$$

- What happens to the solutions of (3) when $\lambda \to 0$?
- What happens to the solutions of (4) when $t \to \infty$?

There is a deep relation between these two limits (so-called ergodic behaviour) related to the stochastic interpretation of (3), (4)

[Bensoussan-Frehse, Arisawa-PL Lions, Evans, Alvarez-Bardi, Lions-Souganidis, Barles-Souganidis...]

The typical behavior in the periodic case:

- $\exists$ a unique constant c_0 such that the problem

$$- \Delta v + |\nabla v|^q + c_0 = f(x) \tag{5}$$

  has a solution (periodic).

- One has

$$\lim_{\lambda \to 0} \lambda\, u_\lambda = \lim_{t \to \infty} \frac{u(t)}{t} = c_0$$

- Moreover, in the periodic case one has

$$u(x, t) - c_0 t \to v$$

  where v is a solution of (5).

The constant c_0 is so-called *ergodic constant*
(connection with control of diffusion processes and ergodicity of deterministic/stochastic dynamics in invariant domains)

Dynamic interpretation: solutions of the evolution problem can be represented in terms of diffusion processes.
Ex: (purely heat equation)

$$u(x,t) = E_x \left\{ \int_0^{t \perp \tau_x} f(X_\tau)\, d\tau + u_0(X_t) \right\} ,$$

where X_t is a standard Brownian motion starting with $X_0 = x$.
Traditionally the "ergodic behavior" means that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(X_t) dt \quad \text{is constant w.r.t. } X_0 = x$$

(and equal to the mean of f w.r.t. to the invariant measure). Such a constant is the ergodic constant.
On the other hand we have

$$u_\lambda = E_x \left\{ \int_0^\infty f(X_t)\, \mathrm{e}^{-\lambda t}\, dt \right\}$$

and it is known that

$$\lim_{\lambda\to 0} \lambda \underbrace{\int_0^\infty f(X_t) e^{-\lambda t} dt}_{u_\lambda(x)} = \lim_{T\to\infty} \frac{1}{T} \underbrace{\int_0^T f(X_t) dt}_{u(x,T)}$$

- From a PDE's point of view, the fact that

$$\lambda\, u_\lambda \to \quad \text{constant}\,, \qquad \frac{u(t)}{t} \to \quad \text{constant}$$

  is due to gradient bounds:
  u may blow-up but ∇u remains bounded.

- The constant c_0 has a role in homogenization for first order problems (see [Evans], [Lions-Souganidis], [Alvarez-Bardi]).
  Ex:

$$-\varepsilon \Delta u_\varepsilon + H(\nabla u_\varepsilon, \frac{x}{\varepsilon}) = 0$$

  Setting $u_\varepsilon = u_0 + \varepsilon v(\frac{x}{\varepsilon})$, if we solve (for a unique constant c_0)

$$c_0 - \Delta v + H(\nabla_x u_0 + \nabla v, y) = 0 \qquad y = \frac{x}{\varepsilon}$$

  then $c_0 = \overline{H}(\nabla u_0)$ is the so-called "effective Hamiltonian" .

What happens in case of Dirichlet boundary conditions?

Is it possible that $u(x,t) \sim c_0 t + v(x)$, as in the periodic case ?:

(and which problem c_0 and v should solve?)

Some hints:

- There are cases when $u(t)$ is bounded and cases when $u(t)$ blows-up.

  ex: in some cases, there is a stationary solution (if $\|f\|_\infty$ is small)

  (cfr. [Souplet-Zhang '06])

- u is always controlled from above:

$$u_t - \Delta u + |\nabla u|^q = f \quad \Rightarrow \quad u_t - \Delta u \leq f$$

  Therefore, if blow-up occurs, this means $u \to -\infty$.

- Heuristics: if $u(x,t) \sim c_0 t + v(x)$, then we should have:

  (i) $c_0 \leq 0$.

  (ii) Since $u = 0$ on the lateral boundary, in the limit we should have

$$v(x) = u(x,t) - c_0 t = -c_0 t \to +\infty \qquad \text{on } \partial\Omega.$$

  (iii) Therefore, v should satisfy

$$\begin{cases} -\Delta v + |\nabla v|^q + c_0 = f(x) & \text{in } \Omega, \\ \lim_{x \to \partial\Omega} v(x) = +\infty\,, \end{cases}$$

blow-up at infinity $\longrightarrow$ large solutions of elliptic PDE

Pb: Existence/uniqueness of (c_0, v)?

The ergodic problem

Thm [Lasry-Lions '89]: Let $1 < q \leq 2$. There exists a unique constant c_0 such that the problem

$$\begin{cases} -\Delta v + |\nabla v|^q + c_0 = f(x) & \text{in } \Omega\,, \\ v(x) \to +\infty \quad \text{as} & x \to \partial\Omega \end{cases} \tag{6}$$

admits a solution $v \in W^{2,p}_{loc}$ $(\forall p < \infty)$. Moreover the solution v is unique up to an additive constant.

- The condition $1 < q \leq 2$ is sharp in order that large solutions exist. If $q > 2$ a similar ergodic result holds for the maximal solutions v, but in that case v is bounded

Rmk: The constant c_0 is characterized by [LL] as the ergodic constant of a control problem with state constraint:

$$c_0 = \lim_{T\to\infty} \inf_{\mathcal{A}} \mathbb{E}_x \frac{1}{T}\int_0^T \left\{ f(X_t) + c|a(X_t)|^{\frac{q}{q-1}} \right\} dt$$

where

$$\begin{cases} dX_t = a(X_t)dt + \sqrt{2}dB_t \\ X_0 = x \end{cases} \qquad a(\cdot) \in \mathcal{A} \iff X_t \in \Omega \quad \forall t > 0 \ a.s.$$

As we will see: the behaviour of u_λ and of $u(t)$ will be described in terms of (c_0, v)

A remarkable fact: Existence of stationary solutions of the Dirichlet pb. depends on the ergodic constant c_0 of the state constraint pb.

$\longrightarrow$ $c_0 = c_0(f)$ plays a similar role as an eigenvalue

Theorem

There exists a solution of the Dirichlet problem

$$\begin{cases} -\Delta\varphi + |\nabla\varphi|^q = f(x) & \text{in } \Omega \\ \varphi = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{7}$$

if and only if $c_0(f) > 0$.

- Moreover, c_0 can also be characterized as

$$c_0 = \sup\{c \in \mathbb{R} \,:\, \exists \quad \text{subsolutions of} \quad -\Delta\varphi + |\nabla\varphi|^q + c = f\}$$

(recall PL Lions [Arma '80]: if $\exists$ a subsolution $\Rightarrow$ $\exists$ a solution)

- When $q = 2$ we have $c_0 = \lambda_1(-\Delta + f)$ (cfr. [Kazdan-Kramer])
- The same holds for any boundary data $g \in L^\infty(\partial\Omega)$. Actually, $c_0 = c_0(f)$ is independent of boundary values.

Consider now the behavior of u_λ sol. of

$$\begin{cases} \lambda u_\lambda - \Delta u_\lambda + |\nabla u_\lambda|^q = f(x) & \text{in } \Omega\,, \\ u_\lambda = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{8}$$

We can now distinguish the two situations in terms of $c_0(f)$:

(i) $c_0 > 0 \iff$ there exists a (unique) solution of the limit problem

$$\begin{cases} -\Delta \varphi + |\nabla \varphi|^q = f(x) & \text{in } \Omega \\ \varphi = 0 \quad \text{on } \partial\Omega \end{cases} \tag{9}$$

then $\|u_\lambda\|_\infty$ is bounded and we have stability: $u_\lambda \to \varphi$.

(ii) $c_0 \le 0 \iff$ there exists no solution of (9) $\to$ blow-up

Theorem (P. '10)

Let $1 < q \leq 2$, and $f \in L^\infty$. Let u_λ be sol. of (8).

(i) *If $c_0 > 0$ then u_λ converges uniformly to stationary solution.*

(ii) *If $c_0 \leq 0$, then*

$$\begin{cases} u_\lambda(x) \to -\infty & \text{for every } x \in \Omega, \\ \lambda\, u_\lambda \to c_0 & \text{locally uniformly in } \Omega, \end{cases}$$

and

$$v_\lambda := u_\lambda + \|u_\lambda\|_\infty \to v_0 \quad \text{locally uniformly,}$$

where c_0 is the unique constant such that

$$\begin{cases} -\Delta v + |\nabla v|^q + c_0 = f(x) & \text{in } \Omega, \\ \lim\limits_{x \to \partial\Omega} v(x) = +\infty\,, \end{cases}$$

admits a solution, and v_0 is the unique solution such that $\min\limits_{\Omega} v_0(x) = 0$.

Rough explanation in terms of stochastic control:

if X_t is a process satisfying the SDE

$$\begin{cases} dX_t = a(X_t) + \sqrt{2}dB_t\,, \\ X_0 = x \in \Omega\,, \end{cases}$$

the sol. u_λ of (1) can be represented as

$$u_\lambda(x) = \inf_{\mathcal{A}} E_x \left\{ \int_0^{\tau_x} \left[f(X_t) + \gamma_q |a(X_t)|^{\frac{q}{q-1}} \right] \mathrm{e}^{-\lambda t}\, dt \right\}$$

where E_x is the expectation conditioned to $X_0 = x$, τ_x is the exit time from Ω.

When $\lambda \to 0$, u_λ remains bounded unless $\tau_x \to \infty$ $\to$ state constraint pb.

Indeed, if f is very negative inside, the control will try to push the process in the interior to realize the minimum. This leads the exit time problem towards the state constraint problem.

Proof of this result relies on various steps:

$$\begin{cases} \lambda u_\lambda - \Delta u_\lambda + |\nabla u_\lambda|^q = f(x) & \text{in } \Omega\,, \\ u_\lambda = 0 \quad \text{on } \partial\Omega. \end{cases}$$

- $\|\lambda\, u_\lambda\|_\infty \le \|f\|_\infty$ by max. principle.
- Interior gradient estimates (Bernstein's technique):

  $|\nabla u_\lambda|$ is (locally) uniformly bounded.

  $\to$ hence $\lambda\, u_\lambda$ must converge to a constant c.
- If there exists a sol. φ of the limit problem, then $\|u_\lambda\|_\infty \le 2\|\varphi\|_\infty$. By standard compactness (e.g. [Boccardo-Murat-Puel]), $u_\lambda \to \varphi$.
- If there exists no solution φ, then necessarily $\min u_\lambda \to -\infty$.

  We prove that minimum points of u_λ remain sufficiently far from the boundary. Hence

  $$u_\lambda + \|u_\lambda\|_\infty \sim u_\lambda(x) - u_\lambda(x_\lambda) \lesssim |\nabla u_\lambda||x - x_\lambda|\,,$$

  which is bounded if $\{x_\lambda\}$ remains in compact sets. Then $u_\lambda + \|u_\lambda\|_\infty$ is bounded in $W^{1,\infty}_{loc}$.

- Local compactness yields

$$u_\lambda + \|u_\lambda\|_\infty \to v \quad \text{locally uniformly,}$$

where v solves

$$-\Delta v + |\nabla v|^q + c = f(x) \qquad \text{and } \min_\Omega v = 0.$$

and on the boundary $v(x) \to +\infty$ as $x \to \partial\Omega$.

- The result of [Lasry-Lions] implies that $c = c_0$, v_0 is unique $\to$ convergence for the whole sequence.

Large time behavior

[G.Barles-A.P.-T. Tabet Tchamba '10]

$$\begin{cases} u_t - \Delta u + |\nabla u|^q = f(x) & \text{in } \Omega\,, \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), & u(0) = u_0\,. \end{cases}$$

where $1 < q \leq 2$, $f \in W^{1,\infty}(\Omega)$ and $u_0 \in L^\infty(\Omega)$.

Theorem 1

(i) If $c_0 > 0$ then $u(t)$ converges uniformly to the stationary solution

(ii) If $c_0 \leq 0$, then

$$\begin{cases} u(t) \to -\infty & \text{for every } x \in \Omega, \\ \frac{u(t)}{t} \to c_0 & \text{locally uniformly} \end{cases}$$

Rmk: When $c_0 > 0$, the convergence to the stationary state holds at exponential rate (see [P-Zuazua '12], [Benachour-Dabuleanu Hapca -Laurencot '07] if $f = 0$)

Concerning the blow-up profile and blow-up rate, our main result is the following:

Theorem 2

(i) If $c_0 < 0$ and $\frac{3}{2} < q \leq 2$, then

$$u(x,t) - c_0 t \to v \quad \textit{locally uniformly}$$

where v is a solution of the ergodic problem.

(ii) If $c_0 < 0$ and $1 < q \leq \frac{3}{2}$ or if $c_0 = 0$, it may happen that

$$u(x,t) - c_0 t \to -\infty$$

- Recall: in the periodic case (as well as in the case $q > 2$) it is always true that $u(t) - c_0 t \to v$, where v is a sol. of the ergodic problem. A new threshold $q = \frac{3}{2}$ appears here; the blow-up rate is influenced by the profile of blow-up solutions .
- Case (ii) happens e.g. in star-shaped domains whenever $c_0(f) + \|f\|_{W^{1,\infty}} < 0$

Idea: compare $u - c_0 t$ with a blow-up sol. v of the ergodic problem

$$\begin{cases} -\Delta v + |\nabla v|^q + c_0 = f(x) & \text{in } \Omega, \\ \lim\limits_{x \to \partial\Omega} v(x) = +\infty\,, \end{cases}$$

Indeed, set $\tilde{u} = u - c_0 t$, it solves the equation

$$\begin{cases} \tilde{u}_t - \Delta \tilde{u} + |\nabla \tilde{u}|^q = f - c_0 \\ \tilde{u}_{|_{\partial\Omega \times (0,T)}} = -c_0\, t \end{cases}$$

and one expects $u - c_0 t \simeq v(x) + \ldots.$ (error terms).

Main steps towards Theorem 2:

- (L^∞-contraction) $\sup\limits_{\Omega}(u(t) - c_0 t - v)$ is decreasing in t.
- local uniform bounds on $|u(t) - c_0 t - v|$
- compactness (after time rescaling) + strong maximum principle (and Hopf lemma).

The key point is the error estimate of the blow-up rate (bounds are locally uniform):

- Case $c_0 < 0$:

$$u - c_0 t = \begin{cases} O(1) & \text{when } \frac{3}{2} < q \le 2 \\ O(\log t) & \text{when } q = \frac{3}{2} \\ O(t^{\frac{3-2q}{2-q}}) & \text{when } 1 < q < \frac{3}{2} \end{cases}$$

- Case $c_0 = 0$:

$$u - c_0 t = \begin{cases} O(\log t) & \text{when } q = 2 \\ O(t^{2-q}) & \text{when } 1 < q < 2 \end{cases}$$

NB: The bound from above is trivial: $u - c_0 t \le v + \|u_0\|_\infty$.

The problem is the bound from below since $u \to -\infty$.

Idea: construct a subsolution using the graph $v(x)$ as a propagating front.

Example of our construction in the case Ω star-shaped: we take

$$\tilde{v}(x,t) = r(t)^{\frac{2-q}{q-1}} v(r(t)x)$$

$$\text{with } r(t) < 1, \;\; r(t) \uparrow 1 \text{ as } t \to \infty$$

This corresponds to a translation of the profile:

(i) $\tilde{v}$ is defined on $\frac{\Omega}{r(t)} \supset \Omega$

(ii) The graph of v moves with velocity $1 - r(t)$

(iii) The velocity $r(t)$ is chosen in a way that $\tilde{v}$ is comparable to $u - c_0 t$ on the boundary: $r(t)^{\frac{2-q}{q-1}} v(r(t)x) \simeq -c_0 t$ on $\partial\Omega \times (0,t)$.

Important: since we know the blow-up rate of v near the boundary, this will fix the velocity $r(t)$

indeed: $v(x) \simeq d(x)^{-\alpha}$ implies $\tilde{v}(x,t) \simeq (1 - r(t))^{-\alpha}$ $(\alpha = \frac{2-q}{q-1})$.

$$\tilde{v}(x,t) \simeq -c_0 t \quad \Rightarrow \quad 1 - r(t) \simeq t^{-\frac{1}{\alpha}}$$

Computing the equation for $\tilde{v}(x,t) = r(t)^{\frac{2-q}{q-1}} v(r(t)x)$ we find

$$\tilde{v}_t - \Delta\tilde{v} + |\nabla\tilde{v}|^q = r(t)^{\frac{q}{q-1}}(f - c_0) + r'(t)....$$
$$= f - c_0 - (1 - r(t))(f - c_0) + \underbrace{r'(t)....}_{\text{negligeable}}$$

hence

$$\tilde{v}_t - \Delta\tilde{v} + |\nabla\tilde{v}|^q \simeq f - c_0 \pm K(1 - r(t))$$

hence

$$u - c_0 t \sim \tilde{v}(x,t) + O\left(H(t)\right), \quad H \simeq \int_0^t (1 - r(s))ds.$$

Since the velocity $1 - r(t)$ is fixed by the boundary rate: $1 - r(t) \simeq t^{-\frac{1}{\alpha}}$, we get

$$u - c_0 t \sim v(r(t)x) + O(t^{1-\frac{1}{\alpha}})$$

Then $u - c_0 t$ is (locally) bounded $\iff \alpha < 1$

(since $\alpha = \frac{2-q}{q-1}$ this means $q > \frac{3}{2}$)

Conclusion, comments, work in progress

- The existence of solutions to elliptic problems with superlinear first order terms can be characterized in terms of a constant c_0 which plays the role of first eigenvalue
- Such constant is associated to boundary blow-up solutions (and to stochastic control problems)
- In case the stationary problem does not have solutions, c_0 determines the blow-up rate of the evolution problem.
- The profile of the long time behavior may be detected from the associated boundary blow-up solution (new thresholds in blow-up rate appear related to singularity of the boundary profile).

  It is essential here a precise knowledge of the qualitative behavior of large solutions.

- The blow-up profile in case $1 < q \leq \frac{3}{2}$ (whenever $u(t) - c_0 t \to -\infty$) is not clear.. suggests to look closer at the radial case

- As I suggested, c_0 plays the role of eigenvalue.... Indeed, a Faber-Khran inequality has been recently proved ([Ferone-Giarrusso-Messano-Posteraro])

- The case $q > 2$ is similar but different [T. Tchamba '11].

  The blow-up rate is still determined by the ergodic constant c_0 related to maximal solutions (state constraint problems). But maximal solutions are bounded ! Hence $u(t) - c_0 t$ always converges.

- (work in progress) Most results (e.g. the blow-up behavior $\frac{u(t)}{t} \to c_0$ when stationary solutions are missing) can be extended to p-Laplace operator

Thanks for the attention!