Weak solutions to Fokker-Planck equations and Mean Field Games

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Outlines of the talk

- Brief description of the Mean Field Games equilibrium system. Coupling viscous Hamilton-Jacobi & Fokker-Planck.
- Local coupling $\rightarrow$ weak solutions.
- A weak setting for Fokker-Planck (uniqueness, renormalization)
- Uniqueness for mean field games
- The planning problem: an optimal transport for stochastic dynamics
- Vanishing viscosity and first order case

Mean Field Games

The Mean Field Games theory was introduced by Lasry-Lions and Huang-Caines-Malhamé since 2006.

Main goal: *describe dynamics with large numbers (a continuum) of agents whose strategies depend on the distribution law*

Typical features of the model:

- players act according to the same principles (they are indistinguishable and have the same optimization criteria).

- players have individually a minor (infinitesimal) influence, but their strategy takes into account the distribution of co-players.

Idea: introduce a macroscopic description through a mean field approach as the number of players $N \to \infty$.

$\to$ Limit of Nash equilibria of symmetric N-players games will satisfy a system of PDEs coupling individual strategies with the distribution law

The simplest form of the macroscopic model is a coupled system in a time horizon T:

$$\begin{cases} (1) \qquad -u_t - \Delta u + H(t,x,Du) = F(t,x,m) & \text{in } (0,T)\times\Omega \\ (2) \qquad m_t - \Delta m - \mathrm{div}(m\, H_p(t,x,Du)) = 0 & \text{in } (0,T)\times\Omega\,, \end{cases}$$

where H_p stands for $\frac{\partial H(t,x,p)}{\partial p}$.

- (1) is the Bellman equation for the agents' value function u.
- (2) is the Kolmogorov-Fokker-Planck equation for the distribution of agents. $m(t)$ is the probability density of the state of players at time t.

Typically: $p \mapsto H(t,x,p)$ is convex.

Model ex: $H \simeq \gamma(t,x)|Du|^q$.

Roughly, each agent controls the dynamics of a N-d Brownian motion

$$dX_t = \beta_t dt + \sqrt{2} dB_t \, ,$$

in order to minimize, among controls β_t, some cost:

$$\inf J(\beta) := \mathbb{E}\left\{ \int_0^T [L(X_s, \beta_s) + F(X_s, m(s))] ds + G(X_T, m(T)) \right\}$$

where $m(t)$ is the probability measure in $\mathbb{R}^N$ induced by the law of X_t.

The associated Hamilton-Jacobi-Bellman equation is

$$-u_t - \Delta u + H(x, Du) = F(x, m(t))$$

where $H = \sup_\beta[-\beta \cdot p - L(x, \beta)]$. The HJB eq. gives

- the best value $\inf_\beta J(\beta) = \int u(x,0) dm_0(x)$,

where m_0 is the probability distribution of X_0.

- the optimal control through the feedback law: $\beta_t^* = b(t, X_t)$, where $b(t,x) = -H_p(x, Du(t,x))$. , Recall: given a drift-diffusion process

$$dX_t = b(t, X_t)dt + \sqrt{2}dB_t$$

the probability measure $m(t)$ (distribution law of X_t) satisfies

$$m_t - \Delta m + \text{ div } (bm) = 0$$

in a weak sense

$$\int_\Omega \varphi(t,x)m(t,x)dxdt + \int_0^t \int_\Omega m(\tau,x)L^*\varphi \, dxd\tau = \int_\Omega \varphi(0)\, m_0$$

$$\forall \varphi \in C^2 , \ \forall t > 0$$

where $L^* := \partial_t - \Delta - b \cdot D$ and $m(0) =$ initial distribution of X_0.

Hence, the evolution of the state of the agents is governed by their optimal decisions $b_t^* = -H_p(\cdot, Du(\cdot))$:

$$m_t - \Delta m - \text{ div } (m\, H_p(x, Du)) = 0$$

This is the Mean Field Games system (with horizon T):

$$\begin{cases} (1) & -u_t - \Delta u + H(x, Du) = F(x, m) \quad \text{in } (0,T)\times\Omega \\ (2) & m_t - \Delta m - \mathrm{div}(m\, H_p(x, Du)) = 0 \quad \text{in } (0,T)\times\Omega\,, \end{cases}$$

usually complemented with initial-terminal conditions:

-$m(0) = m_0$ (initial distribution of the agents)

-$u(T) = G(x, m(T))$ (final pay-off)

$+$ boundary conditions (here for simplicity assume periodic b.c.)

Main novelties are:

- the backward-forward structure.
- the interaction in the strategy process: the coupling $F(x, m)$

Rmk 1: This is not the most general structure.
Cost criterion $L(X_t, \alpha_t, m(t)) \to H(x, m, Du)$.

Rmk 2: In special cases, the system has a variational structure (so-called *mean field control systems*) $\to$ optimality system

Two coupling regimes are usually considered:

(i) Nonlocal coupling with smoothing effect (ex. convolution): $F, G : \mathbb{R}^N \times \mathcal{P}_1 \to \mathbb{R}$ are smoothing on the space of probability measures. Ex: $F(x,m) = \Phi(x, k \star m)$

$\to$ solutions are smooth & Uniqueness of smooth solutions if $m \mapsto F(x,m)$ increasing, $p \mapsto H(x,p)$ convex [Lasry-Lions].

Proof: Take (u_1, m_1), (u_2, m_2) solutions,

$$-(u_1-u_2)_t - \Delta(u_1-u_2) + [H(Du_1) - H(Du_2)] = F(m_1) - F(m_2) \quad \times (m_1 - m_2)$$

$$(m_1-m_2)_t - \Delta(m_1-m_2) - \operatorname{div}\,[m_1 H_p(Du_1) - m_2 H_p(Du_2)] = 0 \quad \times (u_1 - u_2)$$

Subtract $\to$ second order disappear by duality....:

$$\int_0^T \int_\Omega m_1\,[H(x,Du_2) - H(x,Du_1) - H_p(Du_1)\,D(u_2-u_1)]\,dxdt$$
$$+ \int_0^T \int_\Omega m_2\,[H(x,Du_1) - H(x,Du_2) - H_p(Du_2)\,D(u_1-u_2)]\,dxdt$$
$$+ \int_0^T \int_\Omega [F(x,m_1) - F(x,m_2)]\,[m_1 - m_2]\,dxdt$$
$$= - \int_\Omega [(u_1-u_2)(m_1-m_2)]\,dx\,\Big|_0^T$$
$$\ldots\ldots = 0 \quad \text{from initial-terminal conditions}$$

(ii) Local coupling: $F = F(x, m(t, x))$.

$\to$ regularity of sol.'s is very difficult and mostly unknown.

- Existence of smooth solutions if:

  (i) the Hamiltonian H is globally Lipschitz

  (ii) the coupling $m \mapsto F(x, m)$ has a mild growth or $p \mapsto H(x, p)$ has a mild growth

  ([Lasry-Lions], [Gomes-Pimentel-Sanchez Morgado])

- ([Cardaliaguet-Lasry-Lions-P.]) In the model case (purely quadratic) $H(x, p) = |p|^2$, solutions are smooth for *any* $F(x, m) \geq 0$.

- ([Lasry-Lions]) Existence of weak solutions under much more general growth conditions (ex: $F(x, m) \geq 0$ + any power growth w.r.t. m).

Goal: build a complete theory of weak solutions (existence, uniqueness, stability...)

Motivations:

- Convergence of numerical schemes

  (cfr. [Achdou-Capuzzo Dolcetta], [Achdou-Camilli-Capuzzo Dolcetta])

- Convergence of long time asymptotics

  (cfr. [Cardaliaguet-Lasry-Lions-P.])

  Most results were proved assuming to reach smooth solutions. But any stability argument will naturally get at weak solutions....

- Characterize solutions to the planning problem (prescribed initial and final densities $m(0)$ and $m(T)$):

$$\begin{cases} -u_t - \Delta u + H(x, \nabla u) = F(x, m)\,, \\ m_t - \Delta m - \operatorname{div}\,(m\, H_p(x, \nabla u)) = 0\,, \\ m(0) = m_0\,,\ \ m(T) = m_1 \end{cases}$$

Here, no condition is assumed on u at time T.

This is an optimal transport model for the distribution law m of the stochastic flow.

The model case $H(x, Du) = \frac{1}{2}|Du|^2$ was solved by P.L. Lions through a change of unknown using the Hopf-Cole transform. Numerical schemes were studied in [Achdou-Camilli-Dolcetta '12].

In general, solutions (obtained through singular limit of standard MFG systems) can only be proved to be weak.

Main difficulties:

1. The typical setting for well-posedness of

$$(FP) \qquad m_t - \Delta m + \operatorname{div}(m\, b) = 0 \qquad (t,x) \in (0,T)\times\Omega\,, \quad \Omega \subset \mathbb{R}^N$$

is

$$b \in L^\infty(0,T;L^N(\Omega))\,, \qquad \text{or} \quad b \in L^{N+2}((0,T)\times\Omega)$$

or in general $b \in L^r(0,T;L^q(\Omega))$ with $\frac{N}{2q} + \frac{1}{r} \leq \frac{1}{2}$
([Aronson-Serrin] see also [Ladysenskaya-Solonnikov-Uraltseva]).

Under those conditions, plenty of results in the literature (linear and nonlinear operators) & rigorous connections between FP equation and stochastic flow (ex. [Krylov-Röckner '07], [Figalli '08]).

Pb. MFGames: $b = H_p(x, Du) \simeq |Du|^{q-1}$ is in the right class only if q is small or Du highly integrable

2. Uniqueness may fail for unbounded solutions of HJB:

$$\exists \;\; u \in L^2(0,T;H^1_0)\,,\;\; u \neq 0 \text{ sol. of } \qquad \begin{cases} u_t - \Delta u + |Du|^2 = 0 \\ u(0) = 0 \end{cases}$$

Counterexamples are constructed as $u = \log(1+v)$, v solutions to

$$\begin{cases} v_t - \Delta v = \chi \\ v(0) = 0 \end{cases}$$

provided χ is a concentrated measure (ex. $\chi = \delta_{x_0}$)
([Abdellaoui-Dall'Aglio-Peral]).

Back to MFG system:

$$\begin{cases} -u_t - \Delta u + H(x, Du) = F(x, m)\,, \\ m_t - \Delta m - \operatorname{div}\,(m\, H_p(x, Du)) = 0\,, \end{cases}$$

Summary:

(i) HJB has no uniqueness of weak solutions

(ii) The drift in FP is not known to have the right summability

Desperate situation ?......

$$m\,[H_p(x, Du)Du - H(x, Du)] \in L^1(Q_T) \tag{1}$$

which comes from optimization:

$$\int_0^T \int_\Omega L(x, H_p(x, Du))\, mdxdt \;\simeq\; \mathbb{E}\left[\int_0^T L(X_t, H_p(X_t, Du(t, X_t)))dt\right] < \infty$$

Ex: (model case) $H(x, p)$ quadratic $\to H_p(x, Du)Du - H(x, Du) \simeq |Du|^2$

$$(1) \Rightarrow H_p(x, Du) \in L^2(m)\,, \qquad \text{i.e.} \quad m\,|H_p(x, Du)|^2 \in L^1$$

Uniqueness for Fokker-Planck

Key point: we can consider solutions of Fokker-Planck

$$m_t - \Delta m - \operatorname{div}(bm) = 0$$

such that $m \geq 0, \quad m|b|^2 \in L^1$

In this framework, we can prove:

1. Weak (=distributional) solutions of (FP) are unique in this class
2. Weak solutions are renormalized solutions;

   (in the sense of [Di Perna-Lions], extended to second order, see [Boccardo-Diaz-Giachetti-Murat], [Lions-Murat], [Blanchard-Murat])

   Moreover, we show that solutions can be regularized and obtained as limit of smooth solutions.

Rmk: The importance of the class $\{m : b \in L^2(m)\}$ was also stressed in [Bogachev-Da Prato-Röckner '11], [Bogachev-Krylov-Röckner]

The typical statement is the following (adapted to Dirichlet, Neumann, or to entire space $\mathbb{R}^N$ under suitable modifications)

Theorem (P. '14)

Let $b \in L^2(Q_T)^N$ and $m_0 \in L^1(\Omega)$. Then the problem

$$\begin{cases} m_t - \Delta m - \mathrm{div}(m\, b) = 0\,, & \text{in } (0,T)\times\Omega, \\ m(0) = m_0 & \text{in } \Omega. \\ + BC & \end{cases} \tag{2}$$

admits at most one weak sol. $m \in L^1(Q_T)_+$: $m|b|^2 \in L^1(Q_T)$.

Moreover, in this case any weak solution is a renormalized solution, belongs to $C^0([0,T];L^1)$ and satisfies (for a suitable truncation $T_k(\cdot)$):

$$(T_k(m))_t - \Delta T_k(m) - \mathrm{div}(T_k'(m) m\, b) = \omega_k\,, \qquad \text{in } Q_T \tag{3}$$

where $\omega_k \in L^1(Q_T)$, and $\omega_k \overset{k\to\infty}{\to} 0$ in $L^1(Q_T)$.

A nonlinear look at a linear equation

- The equivalence *weak=renormalized* follows from a nonlinear argument.

(i) If $m|b|^2 \in L^1$, then

$$m = \lim_\varepsilon m^\varepsilon \,,$$

$$\begin{cases} m^\varepsilon_t - \Delta m^\varepsilon - \mathrm{div}(\sqrt{m^\varepsilon}\, B^\varepsilon) = 0\,, & \text{in } (0,T)\times\Omega, \\ m^\varepsilon(0) = m_0\,, \ + \text{BC} \end{cases}$$

provided

$$B^\varepsilon \xrightarrow{L^2} \sqrt{m}\, b$$

(ii) The sequence m^ε converges in $C^0([0,T];L^1)$ and produces a renormalized solution

- This is a very general principle for convection-diffusion problems (e.g. nonlinear $Am = -\operatorname{div}(a(x, m, Dm))$)

$$\begin{cases} m^\varepsilon_t + Am^\varepsilon = \operatorname{div}(\phi(t,x,m^\varepsilon)) & \text{in } Q_T \\ m^\varepsilon(0) = m^\varepsilon_0\,, \ +\text{BC} \end{cases}$$

we have: if

$$|\phi(t,x,m)| \le c(1+\sqrt{m})\,k(t,x)\,, \qquad k \in L^2(Q_T) \tag{4}$$

then

$$m^\varepsilon_0 \xrightarrow{L^1} m_0 \quad \Rightarrow \begin{cases} m^\varepsilon \to m & \text{in } C^0([0,T];L^1) \\ T_k(m^\varepsilon) \to T_k(m) & \text{in } L^2([0,T];H^1) \end{cases}$$

and m is renormalized solution relative to m_0.

- One can apply this idea even in the Di Perna-Lions approach, regularizing m by convolution:

$$m_t - \Delta m - \mathrm{div}(m\, b) = 0 \qquad \star \rho_\varepsilon$$

$$\Rightarrow m^\varepsilon := m \star \rho_\varepsilon \quad \text{solves}$$

$$m^\varepsilon_t - \Delta m_\varepsilon - \mathrm{div}((mb) \star \rho_\varepsilon) = 0$$

where Schwartz's inequality $+\ m \geq 0$ imply

$$|(mb) \star \rho_\varepsilon| \leq \underbrace{(m \star \rho_\varepsilon)^{\frac{1}{2}}}_{\sqrt{m^\varepsilon}} \underbrace{\left((m|b|^2) \star \rho_\varepsilon\right)^{\frac{1}{2}}}_{B^\varepsilon}$$

with B^ε converging in $L^2(Q_T)$.

$\rightarrow$ for purely second order operators, no need of commutators !

Summary on FP:

- the class of weak solutions m such that $m|b|^2 \in L^1$ gives: uniqueness, renormalized formulation, solutions obtained by regularization, estimates. Ex:

$$m_0 > 0\,, \log m_0 \in L^1_{loc}(\Omega) \Rightarrow \log m(t) \in L^1_{loc}(\Omega),$$

  hence $m(t) > 0$ a.e.

- the class $m|b|^2 \in L^1$ is consistent with the stochastic flow:

  we are considering only trajectories X_t along which the drift is L^2-integrable:

$$\int_0^T [\mathbb{E}|b(X_t)|^2]\, dt < \infty$$

  Possible development: one should prove uniqueness in law for (SDE) under this condition and establish rigorously the connection with a stochastic flow

Mean Field Games

$$\begin{cases} -u_t - \Delta u + H(x, \nabla u) = F(x, m)\,, \\ m_t - \Delta m - \text{ div } (m\, H_p(x, \nabla u)) = 0\,, \\ u(T) = G(x, m(T))\,,\ \ m(0) = m_0 \end{cases}$$

- $F, G \in C^0(\overline{\Omega} \times \mathbb{R})$
- $p \mapsto H(x,p)$ is convex and satisfies structure conditions

Ex: $H \simeq \gamma(t,x)|\nabla u|^q$, $q \le 2$.

Def. of weak solutions:

- $u, m \in C^0([0,T]; L^1(\Omega))$, $m\,|Du|^q \in L^1$

-$G(x, m(T)) \in L^1(\Omega)$, $H(x, Du) \in L^1$, $F(x,m) \in L^1$,

- the equations hold in the sense of distributions.

Theorem (case $q = 2$)

Assume that $m \mapsto G(x, m)$ is nondecreasing, and let $m_0 \in L^\infty(\Omega)_+$.

(i) If F, G are bounded below, then there exists a weak solution.

(ii) If in addition $m \mapsto F(x, m)$ is nondecreasing, $p \mapsto H(x, p)$ is strictly convex (at infinity), and $\log m_0 \in L^1_{loc}(\Omega)$, then there is a unique weak solution.

Rmk: The coupling functions F, G have no growth restriction from above

- The case $F = F(x)$ is included !! New viewpoint for

$$\begin{cases} u_t - \Delta u + H(x, Du) = F(x) \\ u_{\partial\Omega} = 0\,, \;\; u(0) = u_0 \end{cases}$$

Uniqueness $\iff$ $m_t - \Delta m -$ div $(H_p(x, Du)\, m) = 0$ admits a sol. m with $H_p(Du) \in L^2(m)$.

$\to$ new uniqueness results even with $F, u_0 \in L^1$.

Proof requires previous results for Fokker-Planck and the following crucial lemma

Lemma (crossed integrability)

Given any two weak solutions (u_1, m_1) *and* (u_2, m_2)*, we have*

$$F(m_i)m_j \in L^1(Q_T), \qquad m_i |Du_j|^2 \in L^1(Q_T), \qquad \forall i,j = 1,2. \tag{5}$$

Uniqueness is then proved with all the ingredients:

- m is a weak solution to FP with drift $H_p(x, Du) \in L^2(m)$
  $\to$ m is unique and is a renormalized sol.
  In addition, $m > 0$ a.e. provided $\log m_0 \in L^1_{loc}(\Omega)$.
- Since $u_t - \Delta u \in L^1(Q_T)$, a weak solution u is also renormalized (L^1 theory...)
- Apply the Lasry-Lions argument to the renormalized system
- Pass to the limit thanks to the crossed integrability lemma

Similar result holds for $q < 2$ with minor variations.

Theorem (case $q < 2$)

Let $m_0 \in L^\infty(\Omega)_+$.

(i) If F, G are bounded below, there exists a weak solution.

(ii) Assume in addition that $m \mapsto F(x,m), G(x,m)$ are nondecreasing and

$$F(x,m) \simeq f(m)\,, \qquad G(x,m) \simeq g(m) \quad \textit{as } m \to \infty$$

where $f(s)s$ and $g(s)s$ are convex.
If $p \mapsto H(x,p)$ is strictly convex (at infinity), and $\log m_0 \in L^1_{loc}(\Omega)$, then there is a unique weak solution which is bounded below.

- We also have robust stability results on the nonlinearities

  (i) $F^\varepsilon(x,s) \to F(x,s)$, $G^\varepsilon \to G(x,s)$, $H^\varepsilon(x,p) \to H(x,p)$ (under uniform structure assumptions)

  (ii) $m_{0\varepsilon} \to m_0$

  Then $(u^\varepsilon, m^\varepsilon) \to (u,m)$ weak solutions.

- [joint work with Y. Achdou] Existence of weak solutions can be proved from the convergence of numerical schemes.

We use finite differences approximations of the mean field games system as in [Achdou-Capuzzo Dolcetta]:

$$\begin{cases} \frac{u_{i,j}^{k+1}-u_{i,j}^{k}}{\Delta t} - (\Delta_h u^k)_{i,j} + g(x_{i,j}, \left[\nabla_h u^k\right]_{i,j}) = F(m_{i,j}^{k+1}), \\ \frac{m_{i,j}^{k+1}-m_{i,j}^{k}}{\Delta t} - (\Delta_h m^{k+1})_{i,j} + \mathcal{T}_{i,j}(u^k, m^{k+1}) = 0, \end{cases}$$

where g is a monotone approximation of the Hamiltonian H as in upwind schemes:
Ex (1-d): $g = g\left(\frac{u_{i+1}-u_i}{h}, \frac{u_i-u_{i-1}}{h}\right)$ with $g(p_1, p_2)$ increasing in p_2 and decreasing in p_1, $g(q,q) = H(q)$.

while $\mathcal{T}$ is the discrete adjoint of the associated linearized transport:

$$\mathcal{T}(v, m) \cdot w = m\, g_p([\nabla_h v]) \cdot [\nabla_h w]$$

Similar structure allows to have discrete estimates and compactness as in the continuous model.

The planning problem

Further application to Mean Field Games: the "(stochastic) optimal transport problem":

$$\begin{cases} -u_t - \Delta u + H(x, \nabla u) = F(x, m)\,, \\ m_t - \Delta m - \operatorname{div}\,(m\, H_p(x, \nabla u)) = 0\,, \\ m(0) = m_0\,,\;\; m(T) = m_1 \end{cases}$$

Here, no condition is assumed on u at time T.

This is an optimal transport model for the distribution law m of the stochastic flow.

Ex: (model case $H = \frac{1}{2}|Du|^2$, $F = F(m)$):

$$\min_{\alpha \in L^2(m\,dxdt)} \int_0^T \int_\Omega \frac{1}{2}\,|\alpha|^2\, m\; dxdt + \int_0^T \int_\Omega \Phi(m) dxdt\,, \qquad [\Phi = \int_0^s F(r)dr]$$

$$\begin{cases} m_t - \Delta m - \operatorname{div}\,(\alpha\, m) = 0 \\ m(0) = m_0\,, m(T) = m_1 \end{cases}$$

(compare with deterministic case, $F \equiv 0$ [Benamou-Brenier])

Theorem (P. '13)

Under the above assumptions on F, H. Let $m_0, m_1 \in C^1(\overline{\Omega})$, $m_0, m_1 > 0$, $\int_\Omega m_0\,dx = \int_\Omega m_1\,dx = 1$. Then, there exists a weak solution (u, m) of the planning problem.

If in addition $H(x, \cdot)$ is strictly convex, then the weak solution (u, m) is uniquely characterized: m is unique, u is unique up to a constant.

- Uniqueness follows from the same method as before.

- Smoothness of solutions is open.

- Existence is not easy: this is an exact controllability result (bilinear control) with representation of the optimal control. Ex. is obtained from penalized MFG systems:

$$\begin{cases} -(u_\varepsilon)_t - \Delta u_\varepsilon + H(x, Du_\varepsilon) = F(x, m_\varepsilon) & \text{in } Q_T \\ (m_\varepsilon)_t - \Delta m_\varepsilon - \mathrm{div}(m_\varepsilon\, H_p(x, Du_\varepsilon)) = 0 & \text{in } Q_T \\ m_\varepsilon(0) = m_0\,, \quad u_\varepsilon(T) = \frac{m_\varepsilon(T) - m_1}{\varepsilon} & \varepsilon \to 0 \end{cases}$$

Here we use the variational structure..

1. The structure of Hamiltonian system gives a kind of observability inequality: any solution (u, m) satisfies

$$\int_\Omega |Du(0)|^2\, dx \le C \left\{ \int_0^T\!\!\int_\Omega m\, |Du|^2\, dxdt + 1 \right\} \tag{6}$$

where $C = C(T, H, m_0)$.

2. Coupling the energy estimates of the system with the observability inequality, we end up with a uniform bound

$$\|u_\varepsilon(t)\|_{L^2(\Omega)} \quad \text{bounded, uniformly in } [0, T]$$

and in particular

$$\|u_\varepsilon(T)\|_{L^2(\Omega)} = \frac{1}{\varepsilon}\|m_\varepsilon(T) - m_1\|_{L^2(\Omega)} \le C$$

so

$$m_\varepsilon(T) \xrightarrow{\varepsilon \to 0} m_1$$

and the target will be achieved !!

Vanishing viscosity and first order case

[joint work with Cardaliaguet, Graber & Tonon]

The vanishing viscosity limit is possible at least for $F(x, m) \simeq m^\gamma$ and leads to weak solutions of the first order system in the sense:

(i) u is a distributional subsolution:

$$-u_t + H(x, \nabla u) \leq F(x, m)$$

(ii) m is a distributional solution for the continuity equation

$$m_t - \text{ div } (m\, H_p(x, \nabla u)) = 0$$

(iii) $mH(x, Du), F(x, m)m \in L^1$ and the energy equality holds

$$\int_0^T\!\!\int_\Omega m\, F(x, m) dxdt + \int_0^T\!\!\int_\Omega m \left\{ H_p(x, Du)Du - H(x, Du) \right\} dxdt = \int_\Omega m_0\, u(0) - \int_\Omega u_T\, m(T)$$

Theorem (CGPT '15)

Assume that

(i) $p \mapsto H(x,p)$ *is strictly convex and* $H(x,p) \simeq |p|^q$ *at infinity,* $q > 1$.

(ii) $m \mapsto F(x,m)$ *is increasing and* $F(x,m) \simeq m^\gamma$ *at infinity.*

Then, for smooth initial-terminal data m_0, u_T *the first order system*

$$\begin{cases} -u_t + H(x, \nabla u) = F(x,m)\,, \\ m_t - \ div\,(m\, H_p(x, \nabla u)) = 0\,, \\ m(0) = m_0\,,\ u(T) = u_T \end{cases}$$

admits a unique weak solution (u,m) *in the sense that m is unique and u is unique in* $\{m > 0\}$.

Moreover, the solution can be obtained from the vanishing viscosity limit.

Ingredients of proof:

- Integral estimates for sub solutions of HJ (possibly degenerate)
- Characterization of the weak solution (u,m) as the unique minimizer of optimal control problem (as in [Cardaliaguet], [Cardaliaguet-Graber]).

Further directions of research

- Use the PDE results on weak solutions to prove uniqueness in law for the associated trajectories of the SDE:

$$dX_t = b(X_t)dt + \sqrt{2}dB_t$$

- More general MFG models, e.x. congestion models: $H = \frac{|Du|^2}{m^\alpha}$
Multi-populations interactions, more general stochastic dynamics (Levy processes,...), etc...

- Optimal transport: a bridge with the deterministic case ?

$$\begin{cases} -u_t + H(x, Du) = F(x, m) \\ m_t - \operatorname{div}(m\, H_p(x, Du)) = 0 \\ m(0) = m_0\,, \quad m(T) = m_1 \end{cases}$$

-$F = 0 \rightarrow$ optimal transport ([Benamou-Brenier], [Villani],...).

-$F = F(m)$ increasing $\rightarrow$ results by P.L. Lions (totally different method).

General results ? Is there some unifying framework ?

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