On the construction of p-harmonic functions in the cone

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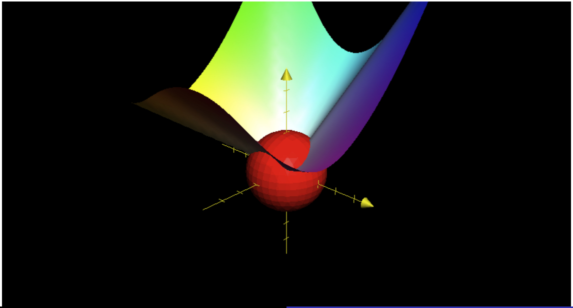
Dedicated to Laurent & Marie-Francoise



Joint work with L. Véron:

A. Porretta & L. Véron: *Separable p-harmonic functions in a cone and related quasilinear equations on manifolds*,
Journal of the European Math. Soc. '09

Motivation and setting of the problem

Let C_S be a cone in $\mathbb{R}^N$ with vertex 0 and opening $S \subset S^{N-1}$, where S is a smooth subdomain on the sphere.

Goal: Construct p harmonic functions in C_S in the form of separable variables

$$u(x) = r^{-\beta}\omega(\sigma) \ : \qquad -\Delta_p u := -\mathrm{div}\left(|Du|^{p-2}Du\right) = 0$$

Motivation: Such functions are fundamental to *describe the precise behaviour near a conical boundary point* of solutions of

$$-\Delta_p v = f(x,v) \qquad \text{in } \Omega,$$

Typically, the (possibly singular) behaviour at those points is described by comparison with explicit solutions in the cone. [Krol, Maz'ya, Tolksdorf, Kichenessamy-Véron,...]

One can check: $u(x) = r^{-\beta}\omega(\sigma)$ is p-harmonic in the cone C_S (and zero on the lateral boundary) if and only if (β, ω) satisfy

$$\begin{cases} -div\left((\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p-2}{2}}\nabla\omega\right) = \\ \qquad\qquad = \beta\left(\beta(p-1) + p - N\right)(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p-2}{2}}\omega \\ \omega = 0 \quad \text{on } \partial S \end{cases} \tag{1}$$

where ∇ and *div* are covariant derivative and divergence operator on S^{N-1}.

Theorem (P. Tolksdorf '83)

There exists a unique $\tilde{\beta} := \tilde{\beta}_S < 0$ *such that the problem* (1) *admits a positive solution* $\omega \in C^1(\bar{S}) \cap C^2(S)$. *Furthermore* ω *is unique up to an homothethy.*

- [L.Véron, Colloquia Mathematica Societatis Jànos Bolyai]: The same proof of Tolksdorf applies when $\beta > 0$ (existence and uniqueness of $\beta_S > 0$: $u = r^{-\beta}\omega(\sigma)$ is p-harmonic in the cone C_S)

  $\Rightarrow$ construction/behaviour of singular solutions

- Recently, new interest in this result has come from the study of the boundary isolated singularities for the equation

$$-\Delta_p u = u^q\,, \qquad q > p-1$$

  [Bidaut Véron-Jazar-Véron, Bidaut Véron-Borghol-Véron, Bidaut Véron-Ponce-Véron ($p = 2$)]

  The construction of positive sol. in the form $u = r^{-\beta}\omega(\sigma)$ would serve as model for the singular behaviour in conical boundary points.

In particular, [Bidaut Verón-Jazar-Véron] prove that

A necessary condition for $\exists$ of sol. $u = r^{-\beta}\omega(\sigma)$ of

$$-\Delta_p u = u^q \qquad \text{in the cone } C_S$$

$$\text{is that} \qquad \beta = \frac{p}{q-(p-1)} < \beta_S$$

Note that this is a condition relating q and S (opening of the cone): $q-(p-1) > \frac{p}{\beta_S}$
(the condition is also sufficient in dimension $N = 2$)

- Unfortunately, the explicit value of β_S is rarely known.

(Ex: $p = 2$, $S = S_+$ half sphere, then $\beta_S = N - 1$)
However, the role of β_S is important as that of an eigenvalue.
(similarly, β_S also appears in Liouville type problems in cones)

$\rightarrow$ Qn: what do we know about β_S?

• In the approach of P. Tolksdorf, there is no appearing of an eigenvalue problem. His theorem is a consequence of the results for solutions in the cone.

The existence of (β, ω) is deduced by constructing a self-similar sol. in the unit cone ($u(R\,x) = R^{\beta}\, u(x)$) and defining $\omega(\sigma) := \frac{u(R\sigma)}{R^{\beta}}$ (uniqueness of β, ω is proved next using Harnack inequalities in the infinite cone)

Pb: Is there an intrinsic construction of (β, ω) ? Does this problem have an independent meaning on S^{N-1}? Note that problem

$$\begin{cases} -div\left((\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p-2}{2}}\nabla\omega\right) = \\ \qquad\qquad = \beta\,(\beta(p-1) + p - N)\,(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p-2}{2}}\,\omega \\ \omega = 0 \quad \text{on } \partial S \end{cases}$$

is a kind of "nonlinear eigenvalue problem" (invariant by dilations of ω) - but it is not variational !

When $p = 2$, the equation

$$-div\left((\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p-2}{2}}\nabla\omega\right) = \\ = \beta(\beta(p-1) + p - N)(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p-2}{2}}\omega$$

is exactly an eigenvalue problem

$$-\Delta_g\omega = \beta(\beta + 2 - N)\,\omega \qquad \text{in } S \subset S^{N-1} \tag{2}$$

where Δ_g is the Laplace-Beltrami operator.

$$\beta(\beta + 2 - N) = \lambda_{1,S}$$

when $\lambda_{1,S}$ is the first eigenvalue on S.

Note in the case $p = 2$:

- ω is precisely an eigenfunction
- β is not precisely an eigenvalue, but is obtained in terms of λ_1

(β solves an equation $F(\beta, \lambda_1) = 0$)

$\exists$ of sol. $u(x) = r^{-\beta}\omega(\sigma) \longrightarrow$ eigenvalue-type problems in S^{N-1}.
What if $p \neq 2$? Key point: set

$$v = -\frac{1}{\beta} \ln \omega$$

Then the equation

$$-div \left((\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p-2}{2}} \nabla\omega \right) = \\ = \beta(\beta(p-1) + p - N)(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p-2}{2}} \omega$$

is transformed into

$$-div \left(\left(1 + |\nabla v|^2\right)^{\frac{p-2}{2}} \nabla v \right) + \beta(p-1) \left(1 + |\nabla v|^2\right)^{\frac{p-2}{2}} |\nabla v|^2 \\ = -\left(\beta(p-1) + p - N\right) \left(1 + |\nabla v|^2\right)^{\frac{p-2}{2}} \quad \text{in } S$$

Divide by $\left(1 + |\nabla v|^2\right)^{\frac{p-2}{2}}$ .....

We see that $v = -\frac{1}{\beta} \ln \omega$ solves

$$-\Delta_g v - (p-2)\frac{D^2 v \nabla v \cdot \nabla v}{1+|\nabla v|^2} + \beta(p-1)|\nabla v|^2 = -\left(\beta(p-1) + p - N\right)$$

We immediately remark:

- In the equation of v, the case $p = 2$ and $p \neq 2$ are very similar
- the principal part is an elliptic operator independent of β
- The number $(\beta(p-1) + p - N)$ has a role of "ergodic constant":
  given any $\beta > 0$, is there a unique λ_β such that the equation

  $$-\Delta_g v - (p-2)\frac{D^2 v \nabla v \cdot \nabla v}{1+|\nabla v|^2} + \beta(p-1)|\nabla v|^2 = -\lambda_\beta$$

  has a solution v ?
  Important: with the boundary behaviour $v \to +\infty$ on ∂S !

When $p = 2$, the problem

$$\begin{cases} -\Delta_g v + \beta |\nabla v|^2 = -\lambda_\beta \\ v(\sigma) \to +\infty \qquad \text{as } \sigma \to \partial S \end{cases}$$

is related to a state constraint problem for the Brownian motion (see [J.M. Lasry-P.L.Lions '89]).

This is a classical connection (through logarithmic tranform) between the first eigenvalue and the ergodic constant of stochastic control problems

$$\begin{cases} -\Delta u = \lambda_1 \, u \qquad \text{in } \Omega \\ u = 0 \qquad \text{on } \partial\Omega \end{cases} \quad \overset{v=-\ln u}{\longleftrightarrow} \quad \begin{cases} -\Delta v + |\nabla v|^2 = -\lambda_1 \quad \text{in } \Omega \\ v \to +\infty \qquad \text{on } \partial\Omega \end{cases}$$

So-called stochastic control interpretation of the first eigenvalue [C.J. Holland '77]

(see also Donsker-Varadhan, J.M.Lasry-P.L.Lions '89, W.H. Fleming-McEneaney '95, W. H. Fleming-S.J. Sheu '97, ...)

The heart of our approach is the following

Theorem (P-V)

Let $S \subset S^{N-1}$ be a smooth bounded open subdomain. Then for any $\beta > 0$ there exists a unique $\lambda_\beta > 0$ such that the problem

$$\begin{cases} -\Delta_g v - (p-2)\frac{D^2 v \nabla v \cdot \nabla v}{1+|\nabla v|^2} + \beta(p-1)|\nabla v|^2 = -\lambda_\beta \\ v(\sigma) \to +\infty \qquad \text{as } \sigma \to \partial S \end{cases}$$

admits a solution $v \in C^2(S)$.

Furthermore, v is unique up to an additive constant.

This result has an intrinsic independent interest:

- Our proof applies replacing S^{N-1} with a general $N-1$-dimensional Riemannian manifold (M, g).
- This result extends [J.M.Lasry-P.L.Lions '89] (where $p = 2$ and $S \subset \mathbb{R}^N$). It seems new when $p \neq 2$ even in the euclidean case.

The proof of this Theorem stands on the following steps:

- As is typical for ergodic-type problems, we start from solutions of

$$\begin{cases} \varepsilon\, v_\varepsilon - \Delta_g v_\varepsilon - (p-2)\frac{D^2 v_\varepsilon \nabla v_\varepsilon \cdot \nabla v_\varepsilon}{1+|\nabla v_\varepsilon|^2} + \beta(p-1)|\nabla v_\varepsilon|^2 = 0 \\ v_\varepsilon(\sigma) \to +\infty \qquad \text{as } \sigma \to \partial S \end{cases}$$

and then we let $\varepsilon \to 0$.

What happens in such models is that
- v_ε has a complete blow-up as $\varepsilon \to 0$

On the other hand,

-$\varepsilon\, v_\varepsilon$ remains bounded (locally)

-$|\nabla v_\varepsilon|$ remains locally bounded due to the barrier effect of the absorption term.

Therefore we have

(i) $\|\varepsilon v_\varepsilon\|_\infty \leq C$

(ii) $\varepsilon \nabla v_\varepsilon \to 0$ locally uniformly, hence, up to subsequences,

$$\varepsilon\, v_\varepsilon \text{ converges to a constant } \lambda_\beta$$

If we fix $\sigma_0 \in S$, then, locally uniformly,

$$v_\varepsilon(\cdot) - v_\varepsilon(\sigma_0) \quad \text{converges to a function } v$$

and v solves

$$\lambda_\beta - \Delta_g v - (p-2)\frac{D^2 v \nabla v \cdot \nabla v}{1 + |\nabla v|^2} + \beta(p-1)|\nabla v|^2 = 0$$

with the boundary behaviour $v \to +\infty$ on ∂S

Key point: Compactness relies on interior gradient estimates:

For every compact subset $S' \subset\subset S$, we have

$$\|\nabla v_\varepsilon\|_{L^\infty(S')} \le \frac{K}{\mathrm{dist}(S', S)}$$

To get the gradient bound, we use the (intrinsic) Weitzenböck formula

$$\frac{1}{2}\Delta_g |\nabla v|^2 = \|D^2 v\|^2 + \nabla(\Delta_g v)\cdot \nabla v + Ricc_g(\nabla v, \nabla v)$$

and the classical Bernstein's method . Since

$$\|D^2 v\|^2 \ge \tfrac{1}{N-1}|\Delta_g v|^2$$

$$\Delta_g v = (p-2)\frac{\overbrace{D^2 v \nabla v}^{\nabla|\nabla v|^2}\cdot\nabla v}{1+|\nabla v|^2} + \beta(p-1)|\nabla v|^2$$

we find an equation satisfied by $|\nabla v|^2$ where we apply max. principle.

- Uniqueness of (λ_β, v)

[Rmk: Uniqueness of (λ_β, v) implies that the convergence holds for the whole sequence v_ε]

- The uniqueness of λ_β is a consequence of the strong maximum principle.

Rmk: $A(v) := -\Delta_g v - (p-2)\dfrac{D^2 v \nabla v \cdot \nabla v}{1 + |\nabla v|^2}$ is nondegenerate

$$A(v_1) - A(v_2) + \beta\left[|\nabla v_1|^2 - |\nabla v_2|^2\right] = -(\lambda_\beta^1 - \lambda_\beta^2)$$

$$\lambda_\beta^1 \neq \lambda_\beta^2 \quad \Rightarrow \quad v_1 - v_2 \equiv \text{const}.$$

-Uniqueness of v can be proved with a typical argument for "large solutions " [Bandle, Marcus, Véron]

Detailed estimates on the boundary behaviour of v, ∇v allow to compare v_1 with $(1+\varepsilon)v_2$
$\Rightarrow$ uniqueness (up to an additive constant) of the boundary blow-up solution v.

NB: the operator is quasilinear. To handle the ε-perturbation we need precise gradient estimates: $\frac{\gamma_1}{\mathrm{dist}(\sigma,\partial\Sigma)} \leq |\nabla v(\sigma)| \leq \frac{\gamma_2}{\mathrm{dist}(\sigma,\partial\Sigma)}$

(here we use $C^{1,\alpha}$ estimates up to the boundary for p-Laplace type equations)

Back to Tolksdorf's result. Recall that

$$v = -\frac{1}{\beta}\ln\omega \quad \leftrightarrow \quad \omega = e^{-\beta\, v}$$

We proved that, for any given $\beta > 0$, there exists a unique $\lambda_\beta > 0$:

$$\begin{cases} -div\left((\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p-2}{2}}\nabla\omega\right) = \beta\,\lambda_\beta(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p-2}{2}}\omega \\ \omega = 0 \quad \text{on } \partial S \end{cases}$$

So we rephrase Tolksdorf's problem as:

$u(x) = r^{-\beta}\omega(\sigma)$ is p-harmonic in the cone

if and only if $\lambda_\beta = \beta(p-1) + p - N$

Theorem (P-V)

There exists a unique $\beta > 0$ such that

$$\lambda_\beta = \beta(p-1) + p - N \tag{3}$$

As a consequence, for any subdomain S there exists a unique $\beta_S > 0$ and a unique (up to dilation) positive $\omega \in C^1(\overline{S}) \cap C^2(S)$: $u(x) = r^{-\beta}\omega(\sigma)$ is p-harmonic in the cone C_S.

Remarks:

- As in the case $p = 2$: ω is an eigenfunction, β is not exactly an eigenvalue but a solution of an equation $F(\beta, \lambda_\beta) = 0$ where λ_β is an eigenvalue.
- When $p = 2$ we have $\lambda_\beta = \frac{\lambda_1}{\beta}$ and (3) is the algebraic equation $\beta(\beta + 2 - N) = \lambda_{1,S}$.

Proof of Theorem 3 is simple using the ergodic problem

$$-\Delta_g v - (p-2)\frac{D^2 v \nabla v \cdot \nabla v}{1+|\nabla v|^2} + \beta(p-1)|\nabla v|^2 = -\lambda_\beta$$

to study the mapping $\beta \mapsto \lambda_\beta$.

Indeed, it is (intuitively, and rigorously !!) true that

- $\beta \mapsto \lambda_\beta$ is decreasing (since $\lambda_\beta = \lim_{\varepsilon \to 0} \varepsilon\, v_\varepsilon$)
- $\beta \mapsto \lambda_\beta$ is continuous (stability of the ergodic constant constant is consequence of its uniqueness)
- we have $\lambda_\beta \to +\infty$ when $\beta \to 0$.

Therefore, the mapping

$$\varphi(\beta) := \lambda_\beta - \beta(p-1)$$

is continuous, decreasing and such that $\varphi(0) = +\infty$, $\varphi(+\infty) = -\infty$.

By continuity, the equation

$$\lambda_\beta - \beta(p-1) = Y$$

has a unique sol. for every Y.

When $Y = p - N$ we get the unique $\beta > 0$ which makes $u = r^{-\beta}\omega$ p-harmonic in the cone.

Remarks:

- In the same way one can prove that $\exists\,!\ \beta < 0$ such that $u(x) = r^{-\beta}\omega(\sigma)$ is p-harmonic in the cone (regular case, i.e. Tolksdorf's result)

  Changing β into $-\beta$ is equivalent to change $Y = p - N$ into $Y^* = N - p$

- The monotonicity of the map $\beta \mapsto \lambda_\beta$ gives a typical monotonicity property of eigenvalues (also found in Tolksdorf's paper):

$$\text{if } S,\, S' \subset S^{N-1}, \qquad S \subset S' \Rightarrow \beta_S \geq \beta_{S'}$$

Comments and work in progress

- Our proof of Tolksdorf's result is not easier
- However we provide an intrinsic interpretation of the unique couple (β_S, ω_S) such that

$$u(r,\sigma) = r^{-\beta}\omega(\sigma)$$

  is p-harmonic in the cone C_S, and a new construction of (β, ω) (valid in general manifolds).

- Our approach suggests that in some cases it can be useful *to embed eigenvalue problems into the family of ergodic problems*
- Our construction can be useful to understand the role of β_S in the Lane-Emden-Fowler problem

Back to the source problem in the cone C_S:

$$-\Delta_p u = u^q\,, \quad \text{in the cone } C_S, \text{ with } q > p-1.$$

A positive solution $u = r^{-\beta}\omega(\sigma)$ exists if and only if (β,ω) satisfy

$$-div_g\left((\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\nabla\omega\right) =$$
$$= \beta(\beta(p-1)+p-N)(\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\omega + \omega^q$$

Recall the necessary conditions [Bidaut Véron-Jazar-Véron]:

$$\beta = \frac{p}{q-(p-1)} \qquad \text{and} \quad \beta < \beta_S$$

The construction of β_S now implies:

$$\beta < \beta_S \quad \Longleftrightarrow \quad (\beta(p-1)+p-N) < \lambda_\beta$$

where λ_β is the unique "eigenvalue":

$$-div_g\left((\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\nabla\omega\right) = \beta\lambda_\beta(\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\omega$$

Now it is clear the analogy with the euclidean case:

$$\exists \text{ pos. sol. of } \quad -\Delta_p u = \lambda\, u^{p-1} + u^q \quad \Rightarrow \quad \lambda < \lambda_1(-\Delta_p, \Omega)$$

This suggests how to go further [P-V, work in progress]:

- - $\beta = \frac{p}{q-(p-1)} < \beta_S$ (we are below the "eigenvalue" λ_β)

  - $q < \frac{N(p-1)+1}{N-1-p}$

  (Sobolev critical exponent in dim. $N-1$, if $p < N-1$)

  then $\exists$ a sol. of

  $$-div_g\left((\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\nabla\omega\right) =$$
  $$= \beta(\beta(p-1)+p-N)(\beta^2\omega^2 + |\nabla\omega|^2)^{p/2-1}\omega + \omega^q$$

  (hence $\exists$ a separable sol. of $-\Delta_p u = u^q$).

- If S is "star shaped with respect to the North pole", then there is no solution when $q = \frac{N(p-1)+1}{N-1-p}$
  [Pohozaev type identity on the sphere]
  (using ideas similar to [Bidaut Véron-Ponce-Véron])

Thanks for the attention !