Global Lipschitz estimates; coupling method and doubling variables

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Mostly Maximum Principle
Roma, 12/09/2012

[joint work with Enrico Priola (Univ. Torino)]

Outlines of the talk

It is well known that the heat equation:

$$u_t - \Delta u = 0 \qquad \text{in } \mathbb{R}^N$$

satisfies the following *global regularizing effect*

$$\|Du(t)\|_\infty \le C\,\frac{\|u_0\|_\infty}{\sqrt{t}} \qquad \forall t > 0\,.$$

Besides the regularization property, the global form of the estimate is important.
Ex: an immediate consequence is the Liouville theorem:

$$u \text{ bounded harmonic un } \mathbb{R}^N \Rightarrow u \text{ is constant}$$

In this talk we discuss:

- Extension of this estimate to variable coefficients, and to fully nonlinear parabolic equations.

  Coefficients are (at least) continuous and possibly unbounded, including models with dissipative drift like Ornstein-Uhlenbeck.

  NB: In these models, coefficients beyond the linear growth may yield Lipschitz solutions, because of the drift-diffusion compensation.

- The methods of proof: we compare the probabilistic approach by coupling method with the viscosity doubling variables method (I. Ishii and P.L. Lions).

- This estimate is embedded into a larger family of global oscillation estimates: global Hölder estimates, data with bounded oscillation

- Applications: Bellman-Isaacs equations, existence results, Liouville theorems ....

Linear case

In the linear setting (semigroup community) there is a huge literature concerning gradient or Lipschitz estimates with variable coefficients. Typically, the results cover:

• bounded and uniformly continuous coefficients
([Stewart '74], via analytic semigroups)

• unbounded coefficients but regular
([Elworthy-X.M. Li '94], [Cerrai '96], [Lunardi '98], [Bertoldi-Fornaro '04], [Kunze-Lorenzi-Lunardi '10],...)

Possible approaches (beyond comparison principles) are:

-classical Bernstein method (but requires regularity..)

-probabilistic methods. Here the solution u of

$$\begin{cases} \partial_t u = \mathrm{tr}\left(q(x)D^2u\right) + b(x)\, Du \\ u(0,\cdot) = u_0, \end{cases}$$

is the *martingale solution*: $u(t,x) = \mathbb{E}_x(u_0(X_t))$ where X_t solves

$$dX_t = b(X_t)dt + \sqrt{2q}(X_t)dB_t\,, \qquad X_0 = x\,.$$

With a probabilistic approach, [Priola-F.Y. Wang '06] prove the following: assume that $\exists\ \lambda > 0$ such that

$$q(x) \geq \lambda I$$

and

$$\|\sigma(x) - \sigma(y)\|^2 + (b(x) - b(y)) \cdot (x - y) \leq \omega(|x - y|) \qquad \text{if } |x - y| \leq 1$$

where $\sigma(x) = \sqrt{q(x) - \lambda I}$, and

$$\int_0^1 \frac{\omega(s)}{s} ds < \infty$$

If the process X_t is nonexplosive, then

$$\|Du(t)\|_\infty \leq C \, \frac{\|u_0\|_\infty}{\sqrt{t} \wedge 1} \qquad \forall t > 0\,.$$

This result is obtained using the so-called coupling method: [Lindvall-Rogers '86], [Chen-Li '89], [Cranston '91,'92],...

The coupling method: a rough description

From the stochastic diffusion

$$dx_t = b(x_t)dt + \sqrt{2q(x_t)}dB_t$$

one is given two processes x_t and y_t starting from x and y, and denotes $\mathbb{P}^x$, $\mathbb{P}^y$ the associated probability measures on $\Omega_N := C([0,\infty);\mathbb{R}^N)$.

A coupling of $\mathbb{P}^x$, $\mathbb{P}^y$ is a probability on Ω_{2N} with marginals $\mathbb{P}^x$ and $\mathbb{P}^y$.

For any coupling $\mathbb{P}^{x,y}$, we have that

$$u(t,x) - u(t,y) = \mathbb{E}^x[u_0(x_t)] - \mathbb{E}^y[u_0(y_t)] = \mathbb{E}^{x,y}[u_0(x_t) - u_0(y_t)]$$

where $\mathbb{E}^{x,y}$ denotes the expectation with respect to $\mathbb{P}^{x,y}$.

If the coupling satisfies ($\mathbb{P}^{x,y}$-a.s.):

$$x_t = y_t \quad \text{for all } t \geq T_c := \inf\{t \geq 0 \, : \, x_t = y_t\}$$

then one estimate

$$u(t,x) - u(t,y) = \mathbb{E}^{x,y}[u_0(x_t) - u_0(y_t)] \leq 2\|u_0\|_\infty \mathbb{P}^{x,y}(t < T_c)$$

Then, the Lipschitz estimate is reduced to estimate the hitting time of the diagonal (best over all couplings !)

Rmk: this corresponds to an estimate of a Wassernstein distance between the transition probabilities $p(t,x,\cdot)$ and $p(t,y,\cdot)$ in $\mathbb{R}^N$.

If μ, ν are prob. measures in $\mathbb{R}^N$, one set

$$d_W(\mu,\nu) = \inf_{Q\in\pi(\mu,\nu)} \iint \chi(z,w)dQ(z,w)\,, \qquad \chi(z,w) = \begin{cases} 1 & \text{if } z \neq w \\ 0 & \text{if } z = w, \end{cases}$$

where $\pi(\mu,\nu)$ is the set of all couplings of μ,ν on $\mathbb{R}^{2N}$. Then

$$u(t,x) - u(t,y) = \mathbb{E}^{x,y}\left(u_0(x_t) - u_0(y_t)\right) \leq 2\|u_0\|_\infty d_W(p(t,x,\cdot), p(t,y,\cdot))\,.$$

Typically, the constructed coupling is itself associated to a diffusion on the product space:

$$\partial_t - \mathrm{tr}\left(q(x)D_x^2 + q(y)D_y^2 + 2c(x,y)D_{xy}^2\right) - b(x)\,D_x - b(y)\,D_y = 0$$

In order to get the Lipschitz estimate the degrees of freedom are the *coupling diffusions* $c(x,y)$.

Doubling variables

Since many years, there exists a perfect analytic translation: which is, mostly, maximum principle..

the "Theorem of sums" for viscosity solutions (see [Crandall-Ishii-Lions]) implies the following: if u, v are sub/super viscosity sol. of

$$\partial_t u = \mathrm{tr}\left(q(x)D^2 u\right) + b(x)\, Du \qquad \text{in } \mathbb{R}^N$$

then $z(x,y) = u(x) - v(y)$ is a viscosity subsolution of

$$\partial_t z = \mathcal{A}_c(z) \qquad\qquad \text{in } \mathbb{R}^{2N}$$

$$\mathcal{A}_c = \mathrm{tr}\left(q(x)D^2_x + q(y)D^2_y + 2c(x,y)D^2_{xy}\right) - b(x)\, D_x - b(y)\, D_y$$

for every choice of the coupling diffusions $c(x,y)$ (with $\mathcal{A}_c$ elliptic)

Roughly speaking, we have

$$u(t,x) - u(t,y) \leq \inf_{\mathcal{A}_c} \{\psi(t,x,y), \;:\; \partial_t \psi - \mathcal{A}_c(\psi) \geq 0 \;\}.$$

The problem is reduced to the best choice of the coupling matrix $c(x,y)$ and supersolution ψ.

Typically, one chooses a desired

$$\psi(x,y) = K\, f(|x-y|)$$

with f increasing and concave (ex: $|x-y|^\alpha$, $\alpha \le 1$). Then, (here $b=0$)

$$\mathcal{A}_c(\psi) = \frac{f'(|x-y|)}{|x-y|}\,\{\mathrm{tr}(A(x,y)) - A(x,y)\hat p\cdot \hat p\} + f''(|x-y|)A(x,y)\hat p \cdot \hat p$$

where

$$A(x,y) = q(x) + q(y) - 2c(x,y)\,, \qquad \hat p = \frac{x-y}{|x-y|}.$$

Since $\frac{f'(r)}{r}$ is singular near $r=0$, and $\mathrm{tr}(A) - A\hat p\cdot\hat p$ must be positive, one needs to choose $c(x,y)$ so that $\mathrm{tr}(A)$ be compensated by $A\hat p\cdot \hat p$.

Ex: in the Laplace case, the best choice is the "coupling by reflection":

$$c(x,y) = I - 2\frac{x-y}{|x-y|}\otimes\frac{x-y}{|x-y|}$$

This corresponds exactly to a crucial estimate in [Ishii-Lions '90].
NB: a good choice is also $c(x,y) = I - t\frac{x-y}{|x-y|}\otimes\frac{x-y}{|x-y|}$ for any $t\in[0,2]$; $t=1$ is the coupling by projection.

In [Priola-P.]:

• we give purely analytic proofs of the results obtained in the linear case with the coupling method

• we extend the previous results to a nonlinear setting

$$u_t = F(t, x, Du, D^2 u) \qquad \text{in } (0, T) \times \mathbb{R}^N$$

• we distinguish a two steps procedure:

- a general global estimate in terms of the oscillation of u (only depends on short term interaction: conditions when $|x - y|$ is small)

- an estimate on the oscillation of u (under an additional condition on the long term interaction: some condition when $|x - y|$ is large)

Our main structure condition is (simplified form for the model example):

Hypothesis (F) There exists $\lambda > 0$:

$$F(t, x, \mu(x-y), X) - F(t, y, \mu(x-y), Y)$$
$$\geq -\lambda \mathrm{tr}\,(X - Y) - \mu\, \omega(|x-y|) - \underbrace{[....]}_{l.o.t+r.h.s.}$$

for every $\mu > 0$, $X, Y \in \mathcal{S}_N$:

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \mu \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

The assumptions on the x-dependence is in the function $\omega(|x-y|)$:

- $\frac{\omega(s)}{s} \in L^1(0,1) \quad \Rightarrow$ Lipschitz estimates
- $\omega(s) \overset{s\to 0}{\to} 0 \quad \Rightarrow \quad C^{0,\alpha}-$ estimates for any $\alpha \in (0,1)$
- $\limsup_{s\to 0} \omega(s) < 4\lambda \quad \Rightarrow \quad C^{0,\alpha}-$ estimates for some $\alpha \in (0,1)$

  [Cordes type condition, ex: $\lambda I \leq q(x) \leq \Lambda I$, with $\frac{\Lambda}{\lambda} < 1 + \frac{4}{N}$.]

In order to take care of infinity, for the estimates to be global, we also assume the existence of a Lyapunov function:

Hypothesis (L) For any $L > 0$, $\exists\, \varphi = \varphi_L \in C^{1,2}(\bar{Q}_T)$:

$$\begin{cases} \varphi(t,x) \to +\infty \quad \text{as } |x| \to \infty, \text{ uniformly for } t \in [0,T]. \\ \varepsilon \partial_t \varphi + F(t,x,p+\varepsilon D\varphi, X+\varepsilon D^2\varphi) - F(t,x,p,X) \geq 0 \end{cases}$$

for every $|p| \leq L + \varepsilon |D\varphi(t,x)|$, $X \in \mathcal{S}_N$, and for ε small.

Rmk: In case $F = \mathrm{tr}\left(q(t,x)D^2 u\right) + b(t,x)\, Du$ we recover [Priola-Wang]:

- (F) is equivalent to

$$\|\sigma(x) - \sigma(y)\|^2 + (b(x) - b(y)) \cdot (x-y) \leq \omega(|x-y|)$$

where $\sigma(x) = \sqrt{q(x) - \lambda I}$.

Ex: $\omega \to 0 \sim$ uniform continuity; $\frac{\omega(s)}{s} \in L^1(0,1)$ weaker than Dini condition.

- (L) ensures that the associated stochastic process is non explosive.

Ex: Bellman-Isaacs equations

$$u_t = \inf_{\beta\in\mathcal{B}} \sup_{\alpha\in\mathcal{A}} \left\{-\mathrm{tr}\left(q_{\alpha,\beta}(t,x)D^2u\right) - b_{\alpha,\beta}(t,x)\cdot Du - f_{\alpha,\beta}(t,x)\right\}$$

Previous conditions are satisfied if we have, uniformly in $\alpha\in\mathcal{A}$, $\beta\in\mathcal{B}$:

(i) $q_{\alpha,\beta}(t,x) = \lambda I + \sigma_{\alpha,\beta}(t,x)^2$, for some $\lambda > 0$, where

$$\|\sigma_{\alpha,\beta}(x) - \sigma_{\alpha,\beta}(y)\|^2 + (b_{\alpha,\beta}(x) - b_{\alpha,\beta}(y))\cdot(x-y) \leq \omega(|x-y|)$$

(ii) $f_{\alpha,\beta}$ have bounded oscillation on $\bar{Q}_T$

(iii) $\exists\ \varphi \in C^{1,2}(\bar{Q}_T)$:

$$\begin{cases} \partial_t\varphi + \left\{-\mathrm{tr}\left(q_{\alpha,\beta}(t,x)D^2\varphi\right) - b_{\alpha,\beta}(t,x)\cdot D\varphi\right\} \geq 0 & \forall\alpha\in\mathcal{A},\beta\in\mathcal{B} \\ \varphi(t,x)\to+\infty \quad \text{as } |x|\to\infty, \text{ uniformly in } t\in[0,T]. \end{cases}$$

ex: OK if $\mathrm{tr}\left(q_{\alpha,\beta}(t,x)\right) + b_{\alpha,\beta}(t,x)\cdot x \leq C(1+|x|^2) \quad (\longrightarrow \varphi \simeq |x|^2)$.

Theorem ([Priola-P.], Lipschitz case)

Assume (L) and (F) with $\frac{\omega(s)}{s} \in L^1(0,1)$.

(i) Any viscosity solution u with bounded oscillation satisfies

$$\|Du(t)\|_\infty \le \frac{C}{\sqrt{t \wedge 1}},$$

where $C = C\left(osc_{(\frac{t}{2}\,,\,T\wedge\frac{3}{2}t)}(u), \omega, \dots\right)$.

(ii) If $\omega = O(s^2)$ *as* $s \to \infty$, *and if* u_0 *satisfies*

$$|u_0(x) - u_0(y)| \le k_0 + k_\alpha |x-y|^\alpha + k_1|x-y|, \quad x, y \in \mathbb{R}^N,$$

any u such that $u = o(\varphi)$ *in* $\bar{Q}_T$ *has bounded oscillation and*

$$\|Du(t)\|_\infty \le c_T \left\{ \frac{k_0}{\sqrt{t \wedge 1}} + \frac{k_\alpha}{(t\wedge 1)^{\frac{1-\alpha}{2}}} + k_1 \right\}$$

for some $c_T(T, \lambda, g, \alpha)$.

Remarks:

- (F) does not imply any growth restriction on the coefficients. Ex:

$$A_t = A = (1 + |x|^4)\triangle u - 4N|x|^2 x \cdot Du.$$

- Is the assumption $\frac{\omega(s)}{s} \in L^1(0,1)$ sharp? .... It is certainly optimal for this technique: we construct a supersol. $\psi = f(|x-y|)$ from the ODE

$$4\lambda f'' + \frac{\omega(r)}{r} f' = -1 \qquad \text{with } r \in (0,\delta)$$

and $\frac{\omega(s)}{s} \in L^1(0,1)$ is necessary in order that f be Lipschitz.

- In the model case, if $u_0 \in C_b(\mathbb{R}^N)$ we get

$$\|Du(t)\|_\infty \leq \frac{C\|u_0\|_\infty}{\sqrt{t \wedge 1}}, \qquad C \lesssim \frac{1+2\lambda}{\lambda} e^{\frac{1}{4\lambda}\int_0^1 \frac{\omega(s)}{s}\, ds}$$

An estimate global in time can also be obtained (see later...)

Theorem ([Priola-P.], Hölder case)

Assume (L) and (F) with* $\omega(s) \overset{s\to 0}{\to} 0$.

(i) Any viscosity solution u with bounded oscillation satisfies

$$|u(t,x)-u(t,y)| \le \frac{C}{(t\wedge 1)^{\frac{\alpha}{2}}}|x-y|^{\alpha},$$

where $C = C\left(osc_{(\frac{t}{2}\,,\,T\wedge\frac{3}{2}t)}(u), \omega, \dots\right)$.

(ii) If $\omega = O(s^2)$ *as* $s\to\infty$, *and if* u_0 *satisfies*

$$|u_0(x)-u_0(y)| \le k_0 + k_\alpha|x-y|^{\alpha} + k_1|x-y|, \;\; x,y\in\mathbb{R}^N,$$

any u such that $u = o(\varphi)$ *in* $\bar{Q}_T$ *has bounded oscillation and*

$$|u(t,x)-u(t,y)| \le c_T\left\{\frac{k_0}{(t\wedge 1)^{\frac{\alpha}{2}}} + \max(k_\alpha,k_1)\right\}|x-y|^{\alpha} + k_1|x-y|$$

for some $c_T(T,\lambda,g,\alpha)$.

Rmk: (L)* means the Lyapunov condition uniform w.r.t. gradient

Liouville property

Taking care of the constants c_T one can obtain global (in time) estimates, hence some Liouville-type results when F is independent of time:

$$F(x, Du, D^2u) = 0 \qquad \text{in } \mathbb{R}^N. \tag{1}$$

Theorem

Assume Hypothesis (L) and (F), with $\frac{\omega(s)}{s} \in L^1(0,1)$. Assume in addition that

$$\int_0^{+\infty} e^{-\frac{1}{4\lambda}\int_0^r \frac{\omega(s)}{s}ds} dr = +\infty. \tag{2}$$

Then any bounded viscosity solution of (1) is constant.

(ii) Let $\alpha \in (0,1)$, and assume in addition that

$$\limsup_{s\to\infty} \omega(s) < 4\lambda(1-\alpha).$$

Then, any viscosity solution u such that $|u| \le k_0(1+|x|^\alpha)$ is constant.

Rmk: if $F(x,0,0)$ is not identically zero, then $\nexists$ solution.

Remarks, comments, open questions

- The method admits a natural extensions to a general weighted case ($\lambda = \lambda(t,x) \to \infty$): if

$$q(t,x)\xi \cdot \xi \geq \lambda_{t,x} |\xi|^2,$$

with $\lambda_{t,x} \geq \lambda > 0$ and

$$\begin{aligned} \|\sigma(t,x,y) - \sigma(t,y,x)\|^2 + (b(t,x) - b(t,y))\,(x-y) \\ \leq (\lambda_{t,x} \wedge \lambda_{t,y})\omega(|x-y|) \qquad \text{if } |x-y| \leq 1 \end{aligned}$$

where $\sigma(t,x,y) = \sqrt{q(t,x) - (\lambda_{t,x} \wedge \lambda_{t,y})I}$

- Extension to quasilinear operators is also natural; although the assumption concerning the Lyapunov function may become involved in this case

- These are regularity results. But they can be used as a priori estimates in order to provide the existence of globally Lipschitz (or Hölder) solutions.

  Note: Perron's method can not be applied, we are beyond the assumptions under which comparison principle is known.

- Advertising: what about uniqueness ?

  -Uniqueness for Lipschitz solutions is known if $\omega(s) = O(s^{\frac{1}{2}})$ as $s \to 0$ [Trudinger, Ishii-Lions]

  But solutions are Lipschitz up to $\frac{\omega(s)}{s} \in L^1(0,1)$...

  - Uniqueness can be proved for more general ω if solutions are known to be $C^{1,\gamma}$ (see e.g. [Crandall-Ishii-Lions]).

  But counterexamples exist to $C^{1,\gamma}$ regularity for any $\gamma < 1$ if the operator is not concave [Nadirashvili].

- Coupling method Vs Doubling variables.

  Technically, our estimates are nothing but a global version of the method developed by I. Ishii and P.L. Lions (see also [Barles '91,'92]).

  On the other hand, establishing a bridge with the coupling method used in the probabilistic approach we hope to provide a deeper insight into those questions. Some hints:

  - the Liouville property is known to be roughly equivalent to the existence of a successful coupling. Assumptions on the long range interaction seem more natural from this viewpoint.

  - the coupling method is currently used to get estimates for Levy processes too, or more general processes with jumps. It seems interesting to have a similar comparison with recent estimates for nonlocal operators (see e.g. [Barles-Chasseigne-Imbert])