Outlines of the talk

- Brief description of the Mean Field Games model system. Coupling viscous Hamilton-Jacobi & Fokker-Planck.

- “Long time behavior” of Mean Field Games: natural questions and setting. The ergodic problem and expected behavior.

- Main results obtained:
  - (i) long time average convergence: a matter of energy estimates
  - (ii) exponential rate of convergence

- Links with optimal control problems in the long horizon: a general turnpike behavior.

(joint works with P. Cardaliaguet, J-M. Lasry, P-L. Lions)

(joint work E. Zuazua)
The Mean Field Games model was introduced by J.-M. Lasry and P.-L. Lions [CRAS ’06, Cours Collège de France since 2006] and independently by M. Huang-P. Caines-R. Malhamé.

Main goal: describe games with large numbers (a continuum) of agents whose strategies depend on the distribution of the agents.

Typical features of the model:
- players act according to the same principles (they are indistinguishable and have the same optimization criteria).
- players have individually a minor (infinitesimal) influence, but their strategy takes into account the mass of co-players.

Roughly: players are particles but have strategies

Goal: introduce a macroscopic description through a mean field approach as the number of players $N \to \infty$. 

A. Porretta

Long time average of mean field games
The simplest form of the continuum limit is a coupled system of PDEs

\[
\begin{aligned}
(1) &\quad -u_t - \Delta u + H(x, Du) = F(x, m) \quad \text{in } (0, T) \times \Omega \\
(2) &\quad m_t - \Delta m - \text{div}(m H_p(x, Du)) = 0 \quad \text{in } (0, T) \times \Omega,
\end{aligned}
\]

- (1) is the Bellman equation for the agents' value function \( u \).
- (2) is the Kolmogorov-Fokker-Planck equation for the state of the agents. \( m(t) \) is the probability density of the state of players at time \( t \).

Roughly, each agent (infinitesimal) controls the dynamics

\[dX_t = \alpha_t \, dt + \sqrt{2} \, dB_t\]

where \( B_t \) is a \( d \)-dimensional Brownian motion, in order to minimize, among controls \( \alpha_t \), some cost

\[J(\alpha) := \mathbb{E} \left[ \int_0^T [L(X_s, \alpha_s) + F(X_s, m(s, X_s)) \, ds + u_T(X_T) \right]\]

where \( L \) is the Legendre transform of \( H \) and \( u_T \) a final pay-off.
If $u$ solves the Bellman equation it gives the best value:

- $\inf_\alpha J(\alpha) = \int u(x, 0) dm_0(x)$, where $m_0$ is the probability distribution of $X_0$.

- the optimal control is given by the feedback law: $\alpha^*_t = -H_p(X_t, Du(t, X_t))$, $H_p := \frac{\partial H(x, p)}{\partial p}$.

In turn, if $dX_t = \alpha(X_t) dt + \sqrt{2} dB_t$ the probability measure $m(t)$ (distribution law of $X_t$) satisfies

$$m_t - \Delta m + \text{div} (\alpha m) = 0$$

Hence, the evolution of the state of the agents is governed by their optimal decisions $\alpha^*_t$, and $m$ satisfies

$$m_t - \Delta m - \text{div} (m H_p(x, Du)) = 0$$
This is the Mean Field Games system (with horizon $T$):

\begin{align*}
(1) \quad -u_t - \Delta u + H(x, Du) &= F(x, m) \quad \text{in } (0, T) \times \Omega \\
(2) \quad m_t - \Delta m - \text{div}(m H_p(x, Du)) &= 0 \quad \text{in } (0, T) \times \Omega,
\end{align*}

usually complemented with initial-terminal conditions:

- $m(0) = m_0$ (initial distribution of the agents)
- $u(T) = u_T$ (final pay-off)

+ boundary conditions (here for simplicity assume periodic b.c.)

Main novelties are:

- the **backward-forward structure**.

- the interaction in the strategy process: **the coupling** $F(x, m)$

Two coupling regimes are usually considered:

(i) **Nonlocal coupling with smoothing effect** (ex. convolution):
$F : \mathbb{R}^N \times \mathcal{P}_1 \to \mathbb{R}$ is smoothing on the space of probability measures. Ex: $F(x, m) = \Phi(x, k \ast m)$

(ii) **Local coupling**: $F = F(x, m(t, x))$.
(regularity of sol.’s is a big issue)
Pb: What is the behavior of the MFG system when the horizon $T \to \infty$?

\[
\begin{cases}
-u_t^T - \Delta u^T + H(x, Du^T) = F(x, m^T), & \text{in } (0, T) \\
 m_t^T - \Delta m^T - \text{div } (m^T H_p(x, Du^T)) = 0, & \text{in } (0, T) \\
 m^T(x, 0) = m_0(x), & u^T(x, T) = u_T.
\end{cases}
\]

- To fix the ideas, we work in the periodic setting.

To simplify the presentation, I will consider a reference case: $H(x, Du) = \frac{1}{2} |Du|^2$, initial data $m_0$, $u_T$ smooth, $m_0 > 0$.

Long time behavior is a very natural question in the viewpoint of SDE. In the long horizon, agents are expected to behave in a way to minimize the average (ergodic) cost, regardless of the initial distribution.
Recall the case of a single equation (with no coupling):
(see e.g. [Bensoussan-Frehse], [Namah-Roquejoffre], [Barles-Souganidis])

\[
\begin{cases}
-u_t - \Delta u + \frac{1}{2}|Du|^2 = F(x) \quad \text{in } (0, T) \\
u(T, x) = G(x)
\end{cases}
\]

(i) \( \frac{u(x, 0)}{T} \) converges uniformly to a constant \( \bar{\lambda} \in \mathbb{R} \), which is the ergodic (minimal) cost

\[
\bar{\lambda} = \inf_{\alpha} \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left\{ \int_0^T \frac{1}{2} |\alpha(X_s)|^2 + F(X_s)|ds \right\}
\]

(ii) \( u(x, 0) - \bar{\lambda}T \to \bar{u} \), periodic solution of the "ergodic problem"

\[
\bar{\lambda} - \Delta \bar{u} + \frac{1}{2}|D\bar{u}|^2 = F(x).
\]

• If \( Du \to D\bar{u} \), then \( m^T \) converges to the unique invariant measure \( \bar{m} \) associated to the process

\[
dX_t = -D\bar{u}(X_t)dt + \sqrt{2}dB_t
\]
What happens for Mean Field Games?

Good news: if the coupling $F(x, \cdot)$ is monotone, then the ergodic problem is well posed ([Lasry-Lions '07]).

There exists a unique couple $(\bar{u}, \bar{m})$ and a unique constant $\bar{\lambda}$ which solve

$$
\begin{cases}
\bar{\lambda} - \Delta \bar{u} + \frac{1}{2} |D\bar{u}|^2 = F(x, \bar{m}), & \int_{\Omega} \bar{u} = 0 \\
-\Delta \bar{m} - \text{div} (\bar{m}D\bar{u}) = 0, & \int_{\Omega} \bar{m} = 1
\end{cases}
$$

Moreover, $\bar{u}$, $\bar{m}$ are smooth, $\bar{m} > 0$

Expected long time behavior: $u^T / T \to \bar{\lambda}$ and $m^T \to \bar{m}$.

However:

- one can not use the arguments of the single equation: there are no standard/simple comparison arguments, gradient estimates, etc...

- forward-backward conditions: there is not just an evolution forward in time! Some boundary layer could appear at $t = 0$ or $t = T$.

$\to$ stability will appear in a large transient time $[\delta T, (1 - \delta) T]$
The kind of results which we prove [Cardaliaguet-Lasry-Lions-P.]

- (ergodic behavior) \( \frac{u^T(x,0)}{T} \to \bar{\lambda} \)
- \((Du^T, m^T)\) is close in average to \((D\bar{u}, \bar{m})\):

\[
\frac{1}{T} \int_0^T \int_\Omega |Du - D\bar{u}|^2 + (F(x, m) - F(x, \bar{m}))(m - \bar{m}) \, dx \to 0
\]

Eventually, under some stronger assumption we also get:

- \((Du^T, m^T)\) are exponentially close to \((D\bar{u}, \bar{m})\) in the transient time:

\[
\|Du^T(t) - D\bar{u}\| + \|m^T(t) - \bar{m}\| \leq C \left( e^{-\kappa(T-t)} + e^{-\kappa t} \right) \quad \forall t \in (a, T-a),
\]

where \(a, C\) may depend on initial-terminal conditions.

The norms of the above convergences may vary according to local/nonlocal coupling.

Rmk: This is a turnpike result, in the terminology of math. economics, since the work of Nobel Price P. Samuelson in 1949: an efficient expanding economy should for most of the time be nearly an equilibrium path.
Main ingredient: energy equality [Lasry-Lions] → uniqueness, stability of the system when $F(x, \cdot)$ monotone.

Any couple of solutions $(u_1, m_1)$ and $(u_2, m_2)$ satisfy

$$-rac{d}{dt} \int_{\Omega} (u_1 - u_2)(m_1 - m_2) dx =$$

$$\int_{\Omega} \frac{(m_1 + m_2)}{2} |Du_1 - Du_2|^2 + (F(x, m_1) - F(x, m_2))(m_1 - m_2) \, dx$$

Apply the energy equality to $(u, m)$ and $(\bar{u}, \bar{m})$ between 0 and $T$:

$$\int_0^T \int_{\Omega} \frac{(m + \bar{m})}{2} |Du - D\bar{u}|^2 + (F(x, m) - F(x, \bar{m}))(m - \bar{m}) \, dx$$

$$= - \left[ \int_{\Omega} (u - \bar{u})(m - \bar{m}) dx \right]_0^T \lesssim C$$

Bounds at $t = 0$ and $t = T \Rightarrow$ convergence in average.

Main point: obtain an estimate $\|Du^T(0)\|$ independent of $T$.
Typically, \( u^T(0) \sim C T \). However, if we set \( \langle u \rangle := \int u \, dx \), we have

\[
\int_{\Omega} u^T(0)(m_0 - \bar{m})dx = \int_{\Omega} (u^T(0) - \langle u^T(0) \rangle)(m_0 - \bar{m})dx \leq c \| Du^T(0) \|_{L^2}
\]

\[ \Rightarrow \] it is enough to bound \( \| Du^T(0) \| \) independently of \( T \).

We get estimates differently according to local or nonlocal coupling.

(i) **Smoothing coupling** \( F(x, m) \): we use a (uniform in time) semiconcavity estimate \( \Rightarrow \) Lipschitz bound for \( u^T \).

(ii) **Local coupling** \( F(x, m) \): we use the property that the system has an Hamiltonian structure \( \Rightarrow \) there exists an invariant (constant in time)

\[
\mathcal{E}(u, m) = \int_{\Omega} \left[ \frac{1}{2} m |Du|^2 + Du \cdot Dm - F(x, m) \right] dx
\]

Thanks to this fact, we obtain a bound on \( \| Du^T(0) \|_{L^2} \).
The exponential rate of convergence may come from two possible ingredients:

- stronger coercivity of the coupling $F(x, \cdot)$
- stability of the linearized pb. (if sol.'s are smooth !)

1. **Local coupling $F(x, m)$.**

We strengthen the monotonicity condition

$$
(F(x, m_1) - F(x, m_2))(m_1 - m_2) \geq \gamma (m_1 - m_2)^2
$$

(1)

**Theorem**

*Under assumption (1), there is some $\kappa > 0$ (independent of $T$) such that (we denote $\tilde{u} = u - \langle u \rangle$)*

$$
\|\tilde{u}(t) - \bar{u}\|_{L^1} + \|m(t) - \bar{m}\|_{L^1} \leq C \left( e^{-\kappa(T-t)} + e^{-\kappa t} \right) \quad \forall t \in (1, T-1)
$$
2. Nonlocal smoothing coupling.
We strengthen the regularizing property of the coupling term

\[ \| F(x, m_1) - F(x, m_2) \|_{C^{1+\alpha}} \leq \tilde{C} \| m_1 - m_2 \|_{H^{-1}} \quad \forall m_1, m_2 \quad (2) \]

for some \( \alpha > 0 \).

**Theorem**

*Under assumption (2), there exists \( \kappa > 0 \) (independent of \( T \)) such that*

\[ \| \tilde{u}(t) - \bar{u} \|_{C^{3,\alpha}} \leq C \left( e^{-\kappa(T-t)} + e^{-\kappa t} \right) \quad \forall t \in (a, T-a) , \]

\[ \| m(t) - \bar{m} \|_{C^{2,\alpha}} \leq C \left( e^{-\kappa(T-t)} + e^{-\kappa t} \right) \quad \forall t \in (a, T-a) \]

\((C, a \ depend \ on \ initial-terminal \ conditions)\).
Look at the linearized system around \((\bar{m}, \bar{u})\):

\[
\begin{aligned}
\begin{cases}
-A v \\
-v_t \\
-\Delta v + D\bar{u}Dv
\end{cases} &= F'(\bar{m})\mu \\
\begin{cases}
\mu_t \\
-\Delta \mu - \text{div} (\mu D\bar{u})
\end{cases} &= \text{div} (\bar{m} Dv)
\end{aligned}
\]

We show that \(w := K^{-\frac{1}{2}} \mu\) satisfies

\[
\frac{d^2}{dt^2} \|w(t)\|^2 \geq \omega_0^2 \|w(t)\|^2
\]

for some \(\omega_0 > 0\).

\[
\|w(t)\|^2 \leq \max\{\|w(0)\|^2, \|w(T)\|^2\} \left(e^{-\omega_0 t} + e^{-\omega_0 (T-t)}\right).
\]

\[
\Rightarrow \|\mu(t)\|_{H^{-1}} \lesssim \|w(t)\|^2 \leq \max\{\|m_0\|^2, \|m(T)\|^2\} \left(e^{-\omega_0 t} + e^{-\omega_0 (T-t)}\right)
\]
Through a fixed point argument, we can preserve such property for the nonlinear problem:

$$\|m(t) - \bar{m}\|_{H^{-1}}^2 \lesssim C \left( e^{-\omega_0 t} + e^{-\omega_0 (T-t)} \right)$$

Using

$$\|F(x, m(t)) - F(x, \bar{m})\|_{C^{1+\alpha}} \leq \bar{C} \|m(t) - \bar{m}\|_{H^{-1}}$$

we bootstrap the estimates between the two equations, using the exponential decay of the operators.
MFG as optimality system (for a bilinear control problem).

Ex: Optimize in terms of the field $\alpha$

$$\inf_{\alpha} \int_0^T \left[ \int_{\Omega} \frac{1}{2} m |\alpha|^2 + \mathcal{F}(m(s)) \right] ds$$

State eq.

$$m_t - \Delta m - \text{div} (\alpha m) = 0, \quad m(0) = m_0$$

Optimality gives:

$$\alpha_{opt} = Du(t, x)$$

$$-u_t - \Delta u + \frac{1}{2} \alpha_{opt} \cdot Du = F(m)$$

We proved: Controls $[Du^T]$ and trajectories $[m^T]$ which are optimal in $[0, T]$ are close to the corresponding steady-state ones.
The convergence holds in average and exponentially in the transient time.

Is this a general issue of optimality systems?
It turns out that similar exponential estimates hold for a wide class of optimal control problems in the long horizon (joint work with E. Zuazua).

**Ex:** linear case $\iff$ minimize a quadratic cost

$$J(u) = \frac{1}{2} \int_0^T \left[ \|Cx - z\|^2 + \|u\|^2 \right] dt$$

over the dynamics

$$\begin{cases} x_t + Ax = Bu \\ x(0) = x_0. \end{cases}$$

where $A, B, C \in M_N, z \in \mathbb{R}^N$ is some target observation.

The optimality system reads as

$$\begin{cases} x_t + Ax = -BB^*p \\ -p_t - A^*p = C^*Cx - C^*z \end{cases}$$
Theorem

If \((A, B)\) is controllable and \((A, C)\) is observable, then there exist \(\kappa > 0\) and \(K\):

\[
|u^T(t) - \bar{u}| + |x^T(t) - \bar{x}| \leq K(e^{-\kappa t} + e^{-\kappa(T-t)}) \quad \forall t \in [0, T],
\]

where \((u^T, x^T)\) and \((\bar{u}, \bar{x})\) are the evolution and the stationary optimal control and state.

- Actually, \(\kappa\) is characterized as the exponential rate of the dynamics stabilized through the solution of algebraic Riccati equation.

- We extend the same approach to infinite dimensional setting (at least for a large class of examples, ex. heat and wave equations). Stabilization and observability estimates play a crucial role.
  The linearized MFG system (around the ergodic solution) is an example of this kind.

- Properties of the linearized systems + fixed point arguments are a possible approach for nonlinear systems, as in MFG.
Conclusions

- Under mild monotonicity conditions, we have shown, as $T \to \infty$
  
  (i) the convergence of $\frac{u(t)}{T}$ to $\bar{\lambda}(T - t)$
  
  (ii) the convergence of $u(t) - \int_\Omega u(t, y)dy$ to $\bar{u}$
  
  (iii) the convergence of $m(t)$ to $\bar{m}$

expressed in different norms or scales.

- Under either stronger monotonicity in the local case or stronger continuity in the nonlocal case we have shown that the convergence has exponential rate in the transient time.

- The results obtained are consistent with a general behavior of optimality systems in the long horizon. The structure of the linearized system explains the exponential stability and suggests more general viewpoints.
Thanks for the attention!