Gradient estimates for boundary blow-up solutions and applications

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A huge literature has concerned the study of boundary blow-up solutions (also called large-solutions) of elliptic equations like

$$\begin{cases} -\Delta u + g(u) = f(x) & \text{in}\,\Omega\,, \\ u(x) \to +\infty \quad \text{as} \quad d(x) \to 0 & \qquad [d(x) := \mathrm{dist}\,(x, \partial\Omega)] \end{cases}$$

since the works of J. Keller, R. Osserman, who proved that a solution exists if and only if

$$\int^{+\infty} \frac{1}{\sqrt{G(s)}} ds < \infty\,, \qquad G(s) = \int_0^s g(t)dt$$

Keller-Osserman condition

Fundamental problems: existence, asymptotic behavior and uniqueness [Impossible here to recall all contributors, let us mention Loewner, Nirenberg, Bandle, Marcus, Véron, Lazer, McKenna, Lair, Wood, G. Diaz, Letelier, J. López-Gómez, Cirstea, Radulescu, Zhang,...]

New interest was raised recently on qualitative properties of solutions: multiplicity, symmetry, blow-up profile, second order terms, curvature effects
[Del Pino-Letelier, Aftalion-Reichel, Aftalion-Del Pino-Letelier, Du-Guo, Du-Guo-Zhou, ...]

Goal of this talk: show that gradient estimates lead to such qualitative results. Two examples will be discussed

- **Radial symmetry in a ball for semilinear equations** (extension of the Gidas-Ni-Nirenberg result). Joint work with L. Véron
- **Boundary blow-up solutions related to stochastic control problems** (viscous Hamilton-Jacobi equations).
  Joint work with T. Leonori (PHD at Roma Tor Vergata)

Radial symmetry: Gidas-Ni-Nirenberg for large solutions

Recall the celebrated Gidas-Ni-Nirenberg result:

Let g be a locally Lipschitz function. Then any $u \in C^2(\Omega)$ which is a positive solution of

$$\begin{cases} -\Delta u + g(u) = 0 & \textit{in}\, B_R(0)\,, \\ u = 0 & \textit{on}\ \partial B_R(0), \end{cases}$$

is radially symmetric and decreasing.

Remark: Of course the same holds if $u_{|\partial\Omega} = m$ is constant and $u \leq m$ inside Ω.

A natural question is: if g also satisfies the Keller-Osserman condition at infinity, does a similar result holds for boundary blow-up solutions?

(Answer is not trivial: to what extent $u = +\infty$ is constant tangentially?...)

Recall the key points in the GNN approach (as well as in many later symmetry results)

- Hopf boundary lemma
- moving plane method: compare u with its reflection

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Comparing u with its reflection is not easy when solutions blow-up at the boundary:

- how the difference $u - u_\lambda$ behaves near the corner points ?
- how can we replace the information of Hopf lemma ?

With L. Véron, we adopt the following strategy:

(i) we prove that the Gidas-Ni-Nirenberg argument works for boundary blow-up solutions provided one knows that the normal gradient is dominant:

$$\begin{cases} \lim\limits_{|x|\to R} \frac{\partial u}{\partial \nu} = \infty \\ \frac{\partial u}{\partial \tau} = o\left(\frac{\partial u}{\partial \nu}\right) \quad \text{as } |x| \to R, \end{cases} \tag{1}$$

where $\frac{\partial u}{\partial \nu}$ is the normal derivative and $\frac{\partial u}{\partial \tau}$ is *any* tangential derivative of u.

In some sense we use (1) as a version of Hopf lemma for boundary blow-up solutions

(ii) we turn our attention to conditions under which (1) can be proved to hold true.

Theorem (Porretta-Véron, J. Functional Anal. '06)

Let g be a locally Lipschitz continuous function. Assume that

(i) Exists $a > 0$ such that g is positive and convex on $[a, \infty)$

(ii) g satisfies the Keller-Osserman condition at infinity.

Then any $u \in C^2(\Omega)$ solution of

$$\begin{cases} -\Delta u + g(u) = 0 & \text{in } B_R(0)\,, \\ \lim\limits_{|x| \to R} u(x) = +\infty \end{cases}$$

is radially symmetric and increasing.

Rmk: The result allows to characterize all solutions in several situations where uniqueness fails:
Ex: Changing sign g, like $g(u) = u(u-a)(u-1)$ [Aftalion-Reichel, Aftalion-Del Pino-Letelier '03]; $g(u) = u^2$ [Pohozaev '61]

Some comments:

- Partial results were previously proved by McKenna-Reichel-Walter [Nolin. Anal. '97] by using *second order expansion* of solutions. However, that approach requires stronger assumptions on g: indeed,

  proving second order expansion for $u \Rightarrow$ proving first order for ∇u

- We use the assumption that $g(s)$ is "convex at infinity" in order to prove the estimates for derivatives. i.e. $\frac{\partial u}{\partial \tau} = o\left(\frac{\partial u}{\partial \nu}\right)$.

  This assumption is satisfied by any "reasonable" example of function enjoying the Keller-Osserman condition
  (recall that K-O condition $\Rightarrow$ superlinearity at infinity).
  However, the most general result (assuming only K-O condition) is open.

This is a special case of a general problem: in a smooth domain Ω, prove that boundary blow-up solutions of

$$\begin{cases} -\Delta u + H(u, \nabla u) = f(x) & \text{in}\, \Omega\,, \\ u(x) \to +\infty \quad \text{as} \quad d(x) \to 0 \end{cases}$$

satisfy $\frac{\partial u}{\partial \tau} = o\left(\frac{\partial u}{\partial \nu}\right)$.

Many situations can be dealt with using asymptotic estimates and blow-up arguments [Bandle-Essen, Bandle-Marcus, Porretta-Véron]

If one can prove that $u(x) \sim \psi(d(x))$ where ψ satisfies the associated ODE

$$\begin{cases} \psi'' = H(\psi, \psi')\,, \\ \psi(0) = +\infty \end{cases}$$

then the strategy is:

scaling and blow–up near a point $x_0 \in \partial\Omega$: $\quad u_\delta = \psi(\delta)\, u(x_0 + \delta\xi)$

elliptic $W^{2,p}$–estimates on u_δ $\qquad \Rightarrow C^1$–compactness

$$\Rightarrow \nabla u \sim \psi'(d(x))\nabla d(x) = -\psi'(d)\,\nu$$

(Related topics: symmetry/uniqueness results in half spaces)

Boundary blow-up in viscous Hamilton-Jacobi equations

We consider now the problem

$$\begin{cases} -\Delta u + u + |\nabla u|^q = f(x) \quad \text{in } \Omega\,, \\ u(x) \to +\infty \quad \text{as} \quad d(x) \to 0 \end{cases} \tag{2}$$

- Ω is a bounded smooth subset in $\mathbb{R}^N$, f is (at least) bounded
- $1 < q \leq 2$
  (this range is necessary: no such solutions if $q > 2$ or $q \leq 1$)

Motivation & origin of this model is a state constraint problem for the Brownian motion

J-M Lasry, P.L. Lions, Math. Ann. **283** (1989):

"constraining a Brownian motion in a given domain by controlling its drift"

Given a Brownian motion B_t and the SDE

$$\begin{cases} dX_t = a(X_t)dt + \sqrt{2}\, dB_t\,, \\ X_0 = x \in \Omega\,, \end{cases}$$

find an optimal feedback control $a \in C(\Omega)$ such that X_t does never leave the domain Ω. Admissible controls:

$$a \in \mathcal{A} = \{a \in C(\Omega) : X_t \in \Omega\,, \forall t > 0\, a.s.\}$$

Rmk: in order to constrain a diffusion one needs vector fields $a(x)$ which blow-up at $\partial\Omega$.

Given the cost functional

$$J(x,a) = E\int_0^\infty \left\{ f(X_t) + \gamma_q\, |a(X_t)|^{q'} \right\} e^{-t} dt$$

where $q' = \dfrac{q}{q-1}$, then the value function

$$u(x) = \inf_{a\in\mathcal{A}} J(x,a)\,,$$

is a solution of (2) if $1 < q \leq 2$ (dynamic programming principle).

Theorem (JM. Lasry-PL. Lions)

Let $1 < q \leq 2$. Then the value function u is the unique solution (in $W^{2,p}_{\mathrm{loc}}(\Omega)$ for every $p < \infty$) of

$$\begin{cases} -\Delta u + u + |\nabla u|^q = f(x) \quad \textit{in } \Omega\,, \\ u(x) \to +\infty \quad \textit{as} \quad d(x) \to 0 \end{cases}$$

and

$$a(x) = -q|\nabla u(x)|^{q-2}\nabla u(x)$$

is the unique optimal control law.

Moreover u satisfies, as $d(x) \to 0$,

$$\begin{cases} u(x) \sim C_q d(x)^{-\frac{2-q}{q-1}} & \textit{if } 1 < q < 2, \\ u(x) \sim -\log(d(x)) & \textit{if } q = 2, \end{cases}$$

where C_q is a universal constant, ($C_q = \frac{(q-1)^{-\frac{2-q}{q-1}}}{2-q}$).

After [LL], one knows that the constrained dynamics

$$\begin{cases} dX_t = a(X_t)dt + \sqrt{2}\, dB_t\,, \\ X_0 = x \in \Omega\,, \end{cases}$$

is determined by the unique optimal control

$$a(X_t) = -q|\nabla u(X_t)|^{q-2}\nabla u(X_t)$$

where u is the boundary blow-up solution of the viscous Hamilton-Jacobi equation

$$\begin{cases} -\Delta u + u + |\nabla u|^q = f(x) \quad \text{in}\,\Omega\,, \\ u(x) \to +\infty \quad \text{as} \quad d(x) \to 0 \end{cases}$$

Next goal: study the qualitative behavior (near the boundary) of ∇u to understand the control mechanism

First order asymptotics of the gradient

As a particular case of results in [Porretta-Véron, Adv. Nonlin. Stud. '06] we have:

$$\lim_{x\to\partial\Omega} \quad d(x)^{\frac{1}{q-1}}\nabla u(x) = \tilde{c}_q\, \nu(x)$$

where $\nu(x)$ is the outward unit normal on $\partial\Omega$, and $\tilde{c}_q = {\scriptstyle (q-1)^{-\frac{1}{q-1}}}$.
In particular this implies:

$$\frac{\partial u}{\partial \nu} \sim \frac{\tilde{c}_q}{d(x)^{\frac{1}{q-1}}} \quad \text{and} \quad \frac{\partial u}{\partial \tau} = o\left(\frac{\partial u}{\partial \nu}\right) .$$

As before, this is the scaling of the asymptotics of u: set $\alpha = \frac{2-q}{q-1}$

$$\begin{cases} \text{if} \quad 1 < q < 2\,, \quad u \sim C_q d(x)^{-\alpha} \quad \to \quad \nabla u \sim -\overbrace{C_q\alpha}^{\tilde{c}_q}\, d(x)^{-(\alpha+1)}\nabla d(x) \\ \text{if} \quad q = 2\,, \quad u \sim -\log(d(x)) \quad \to \quad \nabla u \sim -\frac{1}{d(x)}\nabla d(x) \end{cases}$$

(note that $\alpha + 1 = \frac{1}{q-1}$, $\tilde{c}_q = C_q\,\frac{2-q}{q-1}$ and $\nabla d(x) = -\nu$)

We recover the typical result: the first order behavior of u and ∇u is independent of Ω and is described by the associated ODE

$$\psi'' = |\psi'|^q + \psi$$

Recently, for the equation $\Delta u = u^p$, [Del Pino-Letelier '02], [Bandle-Marcus '05] showed that the influence of the domain in the blow–up appears in second order terms (with curvature effects). Proof is through sub–super solutions which provide a detailed (second order) expansion of u.

Natural question for our model is: how the feedback control process depends on the geometry of domain ?

To get an answer:

- Give a precise description of the blow–up of ∇u
  (role of normal and tangential components, second order terms...)

Second order terms: curvature effects

Theorem (Leonori–Porretta SIAM J.Math. Anal. '07)

Let Ω be a smooth bounded open subset of $\mathbb{R}^N$, and let u be the unique solution of (2).
Set $\overline{x}$ the projection of x onto $\partial\Omega$ and by $H(\overline{x})$ the mean curvature of $\partial\Omega$ computed at $\overline{x}$.
Being ν and τ the normal and tangent vectors, we have, as $d(x) \to 0$,

$$\frac{\partial u}{\partial \nu} = \frac{\tilde{c}_q}{d(x)^{\frac{1}{q-1}}} \left[1 + \frac{(N-1)}{2} H(\overline{x})\, d(x) + o(d(x))\right], \quad \forall 1 < q \leq 2,$$

and

$$\begin{cases} \dfrac{\partial u}{\partial \tau} \in L^\infty(\Omega) & \text{if } \frac{3}{2} < q \leq 2, \\ \dfrac{\partial u}{\partial \tau} = O\left(|\log d|\right) & \text{if } q = \frac{3}{2}, \\ \dfrac{\partial u}{\partial \tau} = O\left(\dfrac{1}{d^{\frac{3-2q}{q-1}}}\right) & \text{if } 1 < q < \frac{3}{2}. \end{cases}$$

Corollary (Representation of the optimal control)

Let $a(x) = -q|\nabla u(x)|^{q-2}\nabla u(x)$ *be the optimal control for the state contraint problem.*

As $d(x) \to 0$, *we have: for any* $1 < q < 2$

$$a(x) = -\left[\frac{q'}{d(x)} + \frac{q'(N-1)}{2}\ H(\overline{x})\right] \nu(x) + o(1)$$

For $q = 2$ *we have*

$$a(x) = -\left[\frac{2}{d(x)} + {\scriptstyle (N-1)}\ H(\overline{x}) + o(1)\right] \nu(x) + \psi(x)\,\tau(x)$$

where $\psi \in L^\infty(\Omega)$.

Note in particular:

(i) The control tangentially is zero on $\partial\Omega$ if $q \neq 2$, bounded if $q = 2$.

(ii) On the hypersurfaces parallel to $\partial\Omega$, the control is maximum where the domain has a maximal mean curvature

The “constrained dynamics”

Near the boundary, the dynamics looks like

$$\begin{cases} dX_t = \left[\frac{q'}{d(X_t)} + \frac{q'(N-1)}{2}\, H(\overline{X_t})\right] \nabla d(X_t) dt + \sqrt{2}\, dB_t \,, \\ X_0 = x \in \Omega \,, \end{cases}$$

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The control (i.e. the drift) has to be ”stronger” where the domain is more curved.

Method of proof: asymptotic expansion of the gradient

Remarks (with respect to first order asymptotics):

- The second order expansion of the gradient cannot be obtained here just using sub–super solutions nor "rescaling from the expansion of u".

(it may happen that $u - C_q\, d(x)^{-\alpha}$ has a non trivial trace on $\partial\Omega$, the second order behavior of u cannot be determined)

Our approach relies instead on a regularity result, and we obtain the previous statements by proving a complete asymptotic expansion for ∇u with respect to $d(x)$:

- introduce a formal asymptotic expansion S
- prove *directly* that $u - S$ is Lipschitz
  (without knowing the boundary value of $u - S$): this is possible thanks to a priori estimates and approximation with Neumann-type boundary condition

It sounds similar to a *corrector result*:

Let here $q < 2$: we already know that

$$u(x) \sim C_q\, d(x)^{-\alpha} \qquad \alpha = \frac{2-q}{q-1}$$

Then we introduce as a *corrector*

$$S = d(x)^{-\alpha} \sum_{k=0}^{m_\alpha} \sigma_k(x) d(x)^k$$

and look for a result of the type

$$u - S \quad \text{is Lipschitz in } \Omega.$$

Of course one has that $\sigma_0 = C_q$ is known, and σ_k, $k = 1, \ldots, m$ are smooth functions to be determined.

Indeed, we will prove that there exists a unique choice of the functions σ_k such that

$$u - S \qquad \text{is Lipschitz}$$

where $S = d(x)^{-\alpha} \sum_{k=0}^{m_\alpha} \sigma_k(x) d(x)^k$.

The coefficients σ_k can be explicitly computed, hence we deduce all singular terms of the expansion, since

$$\nabla u - \nabla S \in L^\infty$$

In particular, the computation of σ_k gives

$$\sigma_1(x) = \frac{\tilde{c}_q}{1-\alpha} \frac{\Delta d(x)}{2} \,,$$

hence the mean curvature in second order terms $\left(\Delta d(x)\big|_{\partial\Omega} = -(N-1)H(x)\right)$

Key point: Lipschitz estimates on the reduced ("linearized") equation.

(a) Take $S = d(x)^{-\alpha} \sum_{k=0}^{m} \sigma_k(x) d(x)^k$ and look at the equation satisfied by $z = u - S$

Using the first order behavior $[\frac{\partial u}{\partial \tau} = o\left(\frac{\partial u}{\partial \nu}\right) \ldots]$ and an asymptotic development near the boundary the equation for z looks like

$$-\Delta z + z - \frac{\alpha+2}{d(x)} \nabla z \nabla d(x) + O(d^{\alpha} |\nabla z|^2) = f(x) + g(x),$$

$$g = \Delta S - S - |\nabla S|^q$$

(b) Using Bernstein's method we get estimates for $|\nabla z|^2$ depending on the regularity of f and g. Next two ingredients:
(i) Choose the coefficients $\sigma_k(x)$ of S in a way that g is smooth (this gives a unique choice of the corrector S)

(ii) In order to get global Lipschitz estimates in Ω, we approximate $u - S$ with solutions satisfying Neumann boundary conditions.

Comments, extensions, work in progress

- The result extends to inhomogeneous diffusions

$$\begin{cases} dX_t = a(X_t)dt + \sqrt{2}\,\sigma(X_t)dB_t\,, \\ X_0 = x \in \Omega\,, \end{cases}$$

with associated HJB equation

$$-\mathrm{tr}\left(A(x)D^2u\right) + \lambda u + |\nabla u|^q = f(x)$$

where $A(x) = \sigma(x)\sigma^T(x)$.
Assuming $A(x)$ elliptic and smooth, one can use the same approach replacing the distance function $d(x)$ with the solution of the first order equation

$$\begin{cases} A(x)\nabla\rho\nabla\rho = \gamma\,|\nabla\rho|^q \qquad \text{in } \Omega \\ \rho > 0\,, \\ \rho = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Things to be done (or in progress)...

- Existence/blow–up of explosive solutions in singular domains (link with Wiener criteria for the Brownian motion)

- general diffusions, possibly non smooth and/or degenerate. $\Rightarrow$ approach by viscosity solutions (cfr. degenerate state constraint problems [Katsoulakis], [Ishii–Loreti], [Barles-Burdeau, Barles-Rouy, B-R-Souganidis]...)