PERSONAL NOTES FOR THE ESI MINICOURSE

STATISTICAL PROPERTIES OF INFINITE DIMENSIONAL SYSTEMS

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1. FIRST LECTURE

1.1. **The problem.** We have seen during the first workshop many attempts to bring the theory of dynamical systems to bear on the issue of non-equilibrium statistical mechanics.

In Dolgopyat's lectures we have seen some techniques allowing to show how the complicate behavior of the nonlinearities can give rise to effective noise at the *macroscopic* level.

The main gap to close the circle is to learn how to treat system with *many* (say 10^{25}) components. This is very hard and, at the moment, can be done only in the very simple case of *coupled map lattices*.

1.2. **CML.** A couple map lattice is constructed a follows: given a dynamical system (X,T) we consider the space $\Omega := X^{\mathbb{Z}^d}$ (but more general sets than $\mathbb{Z}^d$ can be also considered) and the product map $F_0(x)_i = T(x_i)$. Next we consider a map $\Phi_\varepsilon : \Omega \to \Omega$ that is ε-close to the identity in a sense to be made precise. The CML that we will consider are then given by $F_\varepsilon := \Phi_\varepsilon \circ F_0$. Interesting cases are:

- T expanding map (either smooth or not)
- T uniformly hyperbolic (either smooth or not)
- T partially hyperbolic (either smooth or not)

The typical approach, going back to Bunimovich-Sinai, is to conjugate F_ε to F_0 and use Markov partitions (see the papers in the references for more details).

A more direct approach, and more dynamical in nature, is desirable (also because in the non-smooth case conjugation fails).

1.3. **Super-brief history of the transfer operator approach.** The possibility to investigate directly the transfer operator for a CML was first investigated by Keller and Künzle [24]. They were able to prove spectral gap in finitely many dimensions and existence of a measure with absolutely continuos marginals in infinite dimensions. Then Fischer, Rugh [8] and Rugh [33] managed to prove space-time decay of correlations in infinite dimensions in the *analytic* case. Then in Baladi, Degli Esposti, Järvenpää, Kupiainen [1] and Baladi, Rugh [2] the spectrum in the analytic case is precisely investigated. Finally, in [27] it was proved the spectral gap for piecewise expanding CML. The latter paper is what I will explain in the following.

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1.4. **Expanding CML.** Consider the case in which $X = [0,1]$ and the map is piecewise $\mathcal{C}^2$ and $|DT| \geq \lambda > 2$. While

$$\Phi_\varepsilon(x)_i = x_i + \varepsilon \sum_{|z|=1} \alpha_z(\tau^i x)(x_{i+z} - x_i),$$

with $\tau^i(x)_j = x_{i+j}$ and $\alpha_z \in \mathcal{C}^1$ with $\partial_{x_j}\alpha_z = 0$ if $|j| \geq 1$. Moreover, we assume

- $\alpha_z \geq 0$. Which, for ε small, insures $x_i \geq 0 \Longrightarrow \Phi_\varepsilon(x)_i \geq 0$.
- $\sum_i \alpha_i = 1$. Which for ε small, insures $x_i \leq 1 \Longrightarrow \Phi_\varepsilon(x)_i \leq 1$.

The goal is show existence and uniqueness of the SRB measure for small ε. For large, but still less than one, ε uniqueness may fail [3].

1.5. **Transfer operator and Lasota-Yorke inequality.** As we want to deal with infinite systems, it is convenient to first define the transfer operator on the set of Borel measures $\mathcal{M}(\Omega)$: for each measurable set A, let $\mathcal{L}\mu(A) := \mu(F_\varepsilon^{-1}(A))$.

Obviously $\mathcal{M}(\Omega)$ is too big to be useful, to restrict it we define two norms:

$$|\mu| := \sup_{|\varphi|_{\mathcal{C}^0} \leq 1} \mu(\varphi)$$

$$\|\mu\| := \sup_{i \in \mathbb{Z}^d} \sup_{\substack{\|\varphi\|_{\mathcal{C}^0} \leq 1 \\ \varphi \in \mathcal{C}^1}} \mu(\partial_{x_i}\varphi).$$

Clearly $|\mu| \leq \|\mu\|$. Let $\mathcal{B} := \{\mu \in \mathcal{M}(\omega) \ : \ \|\mu\| < \infty\}$.

Theorem 1.1 (Keller et al.)**.** *For ε small enough there exists $\theta \in (0,1)$ such that, for all $n \in \mathbb{N}$,*

$$\|\mathcal{L}^n\mu\| \leq A\theta^n\|\mu\| + B|\mu|.$$

That is nice but **compactness** is missing. In fact, compactness does not hold, thus we need a way to establish directly the existence of a gap.

2. SECOND LECTURE

2.1. **spectral gap.** To deal with this fix $a \in [0,1]$ and given $x \in \Omega$ let $(x^p)_q = x_q$ for $q \neq p$ and $(x^p)_p = a$. Then define $\Phi_{\varepsilon,p}$ to be the map

$$\Phi_{\varepsilon,p}(x)_q = \begin{cases} \Phi_\varepsilon(x^q)_q & \text{if } q \neq p \\ x_p & \text{if } q = p\,. \end{cases}$$

One can easily verify that

$$|(\mathcal{L} - \mathcal{L}_p)\mu| \leq C\varepsilon\|\mu\|,$$

where $\mathcal{L}_p$ is the operator associated to the coupling $\Phi_{\varepsilon,p}$. Indeed, letting $\Phi_t := (1-t)\Phi_\varepsilon - t\Phi_{\varepsilon,p}$, holds

$$\begin{aligned}
\mu(\varphi \circ \Phi_\varepsilon - \varphi \circ \Phi_{\varepsilon,p}) &= \int_0^1 \mu(\frac{d}{dt}\varphi \circ \Phi_t) = \int_0^1 \sum_{|i-p|\leq 1} \mu(\partial_{x_i}\varphi \cdot [\Phi_\varepsilon - \Phi_{\varepsilon,p}]_i) \\
&= \int_0^1 \sum_{|i-p|\leq 1} \mu(\partial_{x_i}[\varphi(\Phi_\varepsilon - \Phi_{\varepsilon,p})_i]) - \mu(\varphi\partial_{x_i}(\Phi_\varepsilon - \Phi_{\varepsilon,p})_i]) \\
&\leq C\varepsilon\|\mu\| \cdot |\varphi|_\infty.
\end{aligned}$$

Hence

$$|(\mathcal{L}^n - \mathcal{L}_p^n)\mu| \leq \sum_{k=0}^{n-1} |\mathcal{L}^{n-k-1}(\mathcal{L} - \mathcal{L}_p)\mathcal{L}_p^k\mu| \leq C\varepsilon n\|\mu\|.$$

Next, suppose that $\mu(\varphi) = 0$ for each function φ that does not depend on x_p, then

$$\|\mathcal{L}^{n+m}\mu\| \leq A\theta^n\|\mathcal{L}^m\mu\| + B|\mathcal{L}^m\mu| \leq C(\theta^n + m\varepsilon)\|\mu\| + B|\mathcal{L}_p^m\mu|.$$

Then, if h is the invariant density of the single site map,

$$\begin{aligned}
\mathcal{L}_p^m\mu(\varphi) &= \mu(\varphi \circ (\Phi_{\varepsilon,p} \circ F_0)^m) \\
&= \int_\Omega \left[\varphi((\Phi_{\varepsilon,p} \circ F_0)^m(x)) - \int_0^1 dx_p h(x_p)\varphi((\Phi_{\varepsilon,p} \circ F_0)^m(x))\right]\mu(dx) \\
&= \int_\Omega \partial_{x_p}\int_0^{x_p} dx_p \left[\varphi((\Phi_{\varepsilon,p} \circ F_0)^m(x)) - \int_0^1 dx_p h(x_p)\varphi((\Phi_{\varepsilon,p} \circ F_0)^m(x))\right]\mu(dx) \\
&\leq \|\mu\| \sup_{x_{\neq p}} \int_0^1 dy \mathbb{1}_{[0,x_p]}(y)\left[\varphi(x_{\neq p}, T^m y) - \int_0^1 dz h(z)\varphi(x_{\neq p}, z)\right] \\
&\leq C\nu^n\|\mu\| \cdot |\varphi|_\infty,
\end{aligned}$$

where ν is the rate of decay for the single site map. Putting the above estimates together yields

$$\|\mathcal{L}^{n+m}\mu\| \leq C(\theta^n + m\varepsilon + \nu^m)\|\mu\| \leq \sigma^{n+m}\|\mu\|,$$

for some $\sigma \in (0,1)$, provided we choose n, m, ε appropriately.

So, let $\mathcal{B}_p = \{\mu \in \mathcal{B} \;:\; \mu(\varphi) = 0 \text{ for all } \varphi \text{ independent of } p\}$. The situation looks good but there are two problem

(1) in general $\mu \in \mathcal{B}$ does not belong to $\mathcal{B}_p$ for any p.
(2) $\mu \in \mathcal{B}_p \not\Longrightarrow \mathcal{L}\mu \in \mathcal{B}_p$.

No problem: first show that each $\mu \in \mathcal{B}$ can be decomposed as

$$\mu = cm + \sum_{p\in\mathbb{Z}^d} \mu_p$$

where $m \in \mathcal{B}$ is a fixed probability measure and $\mu_p \in \mathcal{B}_p$. Then, for each $\mu_p \in \mathcal{B}_p$, write

$$\mathcal{L}\mu_p = \mathcal{L}_p\mu_p + \varepsilon \sum_{|q-p|\leq 1} \mathcal{L}_{q,p}\mu_p$$

where $\mathcal{L}_p\mathcal{B}_p \subset \mathcal{B}_p$ and $\mathcal{L}_{q,p}\mathcal{B}_p \subset \mathcal{B}_q$ and the operators have all uniformly bounded norm. Only a seemingly catastrophic problem is left: the decomposition sum does not converge in the $|\cdot|$ topology (let alone the $\|\cdot\|$ one).

No problem: let us associate to each measure μ the vector (c, μ_p) given by the terms of its decomposition (this means that one introduces the new super-abstract Banach space $\bar{\mathcal{B}} = \mathbb{C} \times (\times_{p\in\mathbb{Z}^d}\mathcal{B}_p)$ with norm $\|(c, \mu_p)\| := \max\{|c|, \sup_{p\in\mathbb{Z}^d} \|\mu\|_p\})$ and the operator

$$\overline{\mathcal{L}}(c, \mu_p) = (c, \mathcal{L}_p\mu_p + \varepsilon \sum_{|q-p|\leq 1} \mathcal{L}_{p,q}\mu_q + \zeta_p) =: (c, \mathcal{L}_*(\mu_p) + \bar{\zeta}),$$

where ζ_p is the decomposition of $\mathcal{L}m - m$. By applying the previous estimates one has that $\|\mathcal{L}_*\| < 1$. Is that good for something?

Well, $(1, \bar{\mu}) = (1, \mathcal{L}_*\bar{\mu} + \bar{\zeta})$ has the unique solution $\bar{\mu}^* := (\mathbb{1} - \mathcal{L}_*)^{-1}\bar{\zeta}$. Let φ be a local function that depends only the variables in the finite set $\Lambda \subset \mathbb{Z}^d$ and $\mu \in \mathcal{B}$ a probability measure with decomposition $(1, \bar{\mu})$, then

$$\mu(\varphi \circ F_\varepsilon^n) = m(\varphi) + \sum_{p\in\Lambda} \left(\mathcal{L}_*^n\bar{\mu} + \sum_{k=0}^{n-1} \mathcal{L}_*^k\bar{\zeta} \right)_p (\varphi) = \sum_{p\in\Lambda} \mu_p^*(\varphi) + \mathcal{O}(|\Lambda|\|\mathcal{L}_*\|^n).$$

By weak compactness and the Lasota-Yorke inequality we know that $\frac{1}{n}\sum_{k=0}^{n-1} \mathcal{L}^k\mu$ has accumulation points in $\mathcal{B}$, let μ_* be one such accumulation point, then

$$\mu_*(\varphi) = \sum_{p\in\Lambda} \mu_p^*(\varphi)$$

Invariance, uniqueness and spatio-temporal decay of correlation for μ_* readily follow.

2.2. **Coupled Anosov maps.**

Let $X = \mathbb{T}^m$ and T be a smooth Anosov map.

$$\Phi_\varepsilon(x)_i = x_i + \varepsilon \sum_{|z|=1} \alpha_z(\tau^i x) \sin(2\pi(x_{i+z} - x_i)) \mod 1,$$

note that, contrary to the previous case Φ_ε is a diffeomorphism of Ω. The transfer operator is defined as before on $\mathcal{M}(\Omega)$. To restrict the space we again have to introduce norms. This are in the spirit of Baladi's lectures. Yet there is an abundance of spaces that can be used, I would like a space that reduces to BV in the expanding case, this is what I am going to describe. I will describe it in the case of one Anosov map, its application to the coupled system is *work in progress*.

2.3. **Norms.**

Definition 1. *A $\mathcal{C}^k$ t-dimensional foliation W is a collection $\{W_\alpha\}_{\alpha\in A}$, for some set A, such that the W_α are pairwise disjoint, $\cup_{\alpha\in A}W_\alpha = M$ and for each $\xi \in W_\alpha$ there exists a neighborhood $B(\xi)$ such that the connected component $W(\xi)$ of $W_\alpha \cap B(\xi)$ containing ξ is a $\mathcal{C}^k$ t-dimensional open submanifold of M. We will call $\mathcal{F}^k$ the set of $\mathcal{C}^k$ d_s-dimensional foliations.*

Definition 2. *A foliation W is adapted to the cone filed $\mathcal{C}$ if, for each $\xi \in M$, holds $T_\xi W(\xi) \subset C(\xi)$. We will call $\mathcal{F}^k_{\mathcal{C}}$ the set of $\mathcal{C}^k$ d_s-dimensional foliations adapted to $\mathcal{C}$.*

Given a d_s-foliation adapted to $\mathcal{C}$ we can associate to it a system of local coordinates as follows. For each point $\xi \in M$ choose a chart (V_i, ϕ_i), $\xi \in \phi_i(V_i)$, and a neighborhood $U_\xi \subset V_i$ of $(x_\xi, y_\xi) := \phi_i^{-1}(\xi)$. Next, for each $z \in U_\xi$, let $W(z)$ be the connected component of $\phi_i^{-1}(W)$ containing z. Finally, let $U^0_\xi := \{z \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_s} \ : \ z + (x_\xi, y_\xi) \in U_\xi\}$ and define the function $F_\xi : U^0_\xi \to \mathbb{R}^{d_u}$ by $\{F_\xi(x,y)\} = \{(w, y + y_\xi)\}_{w\in\mathbb{R}^{d_u}} \cap W(x + x_\xi, y_\xi)$. In other words the graph of the function $F_\xi(x,\cdot)$ is exactly $W(x + x_\xi, y_\xi)$, moreover $F(0,0) = (x_\xi, y_\xi)$.

Note that this construction defines the local triangular coordinates $\mathbf{F}_\xi(x,y) = (F_\xi(x,y), y)$ which describe locally the foliation. Also let r_ξ such that $D_{r_\xi} = \{(x,y) \in \mathbb{R}^d \ : \ \|(x,y)\| \le r_\xi\} \subset U^0_\xi$.

Definition 3. *For each $\xi \in M$ and $r \le r_\xi$, $\|\cdot\|_{\mathcal{C}^p_s(D_r,\mathbb{R}^l)}$ is a semi-norm defined as follows: For any function $\psi \in \mathcal{C}^p(D_r,\mathbb{R}^l)$, consider the function $\hat\psi(y) = \psi \circ \mathbf{F}_\xi(0,y)$ and let $D^s_r = \{y \in \mathbb{R}^{d_s} \ : \ \|y\| < r\}$ be its domain of definition, then*

$$\|\psi\|_{\mathcal{C}^p_s(D_r,\mathbb{R}^l)} := \|\hat\psi\|_{\mathcal{C}^p(D^s_r,\mathbb{R}^l)}.$$

Definition 4. *For each $D > 0$, $k \in \mathbb{N}$, we define*

$$\overline{\mathcal{F}}^k_{\mathcal{C}} := \left\{W \in \mathcal{F}^k_{\mathcal{C}} \ : \ \mathbf{F}_\xi \in \mathcal{C}^k(U^0_\xi, \mathbb{R}^d) \quad \forall\, \xi \in M\right\}$$

$$\mathcal{W}^k_D := \left\{ W \in \overline{\mathcal{F}}^k_{\mathcal{C}} \ : \ \sup_{\xi\in M}\left[\|F_\xi(0,\cdot)\|_{\mathcal{C}^k(D^s_{r_\xi},\mathbb{R}^{d_u})} + \|H^F_\xi\|_{\mathcal{C}^{k-1}_s(D_{r_\xi},\mathbb{R}^{d_s})}\right] \le D \right\},$$

where $H^F_\xi(x,y) = \sum_{j=1}^{d_u} \partial_{x_j}\left([\partial_y(F_\xi)_j] \circ \mathbf{F}_\xi^{-1}\right)(x,y)$.

$$\Omega_{D,q,l} = \left\{(W,\varphi) \in \mathcal{W}_D \times \mathcal{C}^q(M,\mathbb{R}^l) \ : \ \|\varphi\|^W_q \le 1\right\} \tag{2.1}$$

It is now time to define the norms. Given a function $h \in \mathcal{C}^1(M,\mathbb{R})$ we define

$$\begin{aligned} \|h\|_{0,q} &:= \sup_{(W,\varphi)\in\Omega_{D,q,1}} \int_M h\,\varphi \\ \|h\|_{1,q} &:= \|h\|_{0,q} + \sup_{(W,\varphi)\in\Omega_{D,q+1,d}} \int_M h\,\mathrm{div}\varphi \end{aligned} \tag{2.2}$$

Let us call $\mathcal{B}^{0,q}, \mathcal{B}^{1,q}$, respectively, the Banach spaces obtained by completing $\mathcal{C}^1(M,\mathbb{R})$ in the above norms.

Lemma 2.1. *For all $h \in \mathcal{C}^1(M,\mathbb{R})$ holds true*

$$\begin{aligned} &\|\mathcal{L}h\|_{0,q} \le \|h\|_{0,q} \\ &\|\mathcal{L}^n h\|_{1,q} \le A\sigma^n\|h\|_{1,q} + B\|h\|_{0,q}. \end{aligned}$$

Moreover the unit ball of $\mathcal{B}^{1,q}$ *is relatively compact in* $\mathcal{B}^{0,q+1}$.

Lemma 2.2. *For* $q \in \mathbb{R}_+$ *large enough, the essential spectrum of* $\mathcal{L}$ *on* $\mathcal{B}^{1,q}$ *is contained in the disc* $\{z \in \mathcal{C} \,:\, |z| \leq \lambda^{-1}\}$.

The above proves exponential mixing for any smooth Anosov map in any dimension but gives no information on how the spectral gap depends on the dimension. In general the dependence can be arbitrarily bad, to get some meaningful estimates we need to use the product structure as we did in the expanding case. This is the issue under scrutiny at this moment.

3. THIRD LECTURE

3.1. **Partially hyperbolic systems.** Let $X = [0,1]^2$, and T be a piecewise expanding map, then

$$F_\varepsilon(x,E)_i = \begin{pmatrix} Tx_i + \varepsilon \sum_{|z|=1} \alpha_z(\tau^i(x,E))(x_{i+z} - x_i) \\ E_i + \varepsilon \sum_{|z|=1} \pi_z(\tau^i(x,E))(E_{i+z} - E_i) \end{pmatrix} \tag{3.1}$$

where α_z, π_z are smooth and $\alpha_z, \pi_z \geq 0$ and $\sum_z \alpha_z = \sum_z \pi_z = 1$. As explained by Dolgopyat this is a very hard case even if one has only two maps. So we need some simplifying assumptions. Let us start with a very drastic one that has recently been treated in [7].

3.2. **Cocycles and random walks.** Assume that $\alpha_z = 0$ (in fact, the case $\partial_E \alpha_z = 0$ can be treated in the same way) and $\partial_E \pi_z = 0$. Then the above system can be treated as a random walk in random environment. Indeed the if we take an initial condition such that $\sum_i E_i = 1$, such a condition will be preserved by the dynamics. But then we can interpret the E_i as the probability of an imaginary particle (a ghost particle) to be at site i. Looking at the dynamics we see that if we interpret the π_z as environment dependent transition probability, then the dynamics specify exactly the evolution of the probability distribution of the ghost particle if such a particle performs a random walk with transition probabilities π_z. Let $X_n \in \mathbb{Z}^d$ be the position of the ghost particle at time n.

3.3. **The egocentric point of view.** Consider the process $\boldsymbol{\omega} =: (\omega^n)_{n\in\mathbb{N}} \in \Omega^{\mathbb{N}}$ described by the action of the Markov operator $S : L^\infty(\Omega) \to L^\infty(\Omega)$ defined by

$$Sf(\omega) := \sum_{z\in\Lambda} \pi_z(\omega) f \circ F(\tau^z \omega) =: \sum_{z\in\Lambda} S_z f. \tag{3.2}$$

Remark 3.1. *It is easy to verify that the process $\boldsymbol{\omega}$, $\omega^0 = x$, has the same distribution as the process $(\tau^{X_n} T^n x)_{n\in\mathbb{N}}$.*

We can use the same techniques used to study CML to study the operator S and prove that it has a unique invariant measure $\mu_* \in \mathcal{B}$. We can then consider the measure $\mathbb{P}$ on $\Omega^{\mathbb{N}}$ of the associated Markov process started with a measure μ_*.

3.4. **Annealed statistical properties.**

Lemma 3.2. *There exists a vector $v \in \mathbb{R}^d$ and a matrix $\Sigma \geq 0$ such that, for each probability measure $\nu \in \mathcal{B}$ we have*

$$\frac{1}{N}\mathbb{E}(X_N) \to v$$
$$\frac{X_N - vN}{\sqrt{N}} \Rightarrow \mathcal{N}(0,\Sigma) \quad \textit{under } \mathbf{P}_\nu.$$

Note that if $v = 0$ (which can be insured by a symmetry assumption) and $\varphi \in \mathcal{C}^0(\mathbb{R}^d)$, then we have (essentially)

$$\lim_{N\to\infty} \sum_{q\in\mathbb{Z}^d} \mu_*(\varphi(N^{-\frac{1}{2}} q) E_q(Nt)) = \lim_{N\to\infty} \sum_{q\in\mathbb{Z}^d} \mathbb{E}(\varphi(N^{-\frac{1}{2}} X_{Nt})) = \int_{\mathbb{R}^d} \Psi(y,t)\varphi(x)dy$$

where

$$\partial_t \Psi = \sum_{ij} \Sigma^2_{ij} \partial_{y_i y_j} \Psi.$$

In other words, the *local average* of the E is described by the function Ψ which satisfy the heat equation. Since

$$\mathbb{E}\left(e^{\frac{i}{\sqrt{N}}\langle t,\Delta_k - v\rangle} \mid \mathcal{F}_k\right) = \sum_{z\in\Lambda} \pi_z(\tau^{X_k}\theta^k)e^{\frac{i}{\sqrt{N}}\langle t,z-v\rangle},$$

it is natural to introduce the operators, for all $t \in \mathbb{C}^d$,

$$\mathcal{M}_t h(\theta) := \sum_{z\in\Lambda} \pi_z(\theta)e^{\langle t,z-v\rangle} h(\tau^z F(\theta)) = \sum_{z\in\Lambda} e^{\langle t,z-v\rangle} S_z h. \tag{3.3}$$

Then,

$$\mathbb{E}\left(e^{\frac{i}{\sqrt{N}}\langle t,\tilde{X}_N\rangle}\right) = \nu(\mathcal{M}^N_{it/\sqrt{N}}1). \tag{3.4}$$

The operator $\mathcal{M}'_t$ acting on the space $\mathcal{B}$ is an analytic perturbation of the operator $S' = \mathcal{M}'_0$. Unfortunately, S' does not have a nice spectrum on $\mathcal{B}$, but if we lift it to our covering space $\overline{\mathcal{B}}$ then it has a simple maximal eigenvalue and we can apply standard perturbation theory to prove that the maximal eigenvalue is of the form $1 - N^{-1}\langle t, \Sigma^2 t\rangle$ which implies the result.

3.5. **Kinetic limit.** The idea is to consider the system described by (3.1) and look at $E(\varepsilon^{-2}t)$ in the limit $\varepsilon \to 0$. This is very similar to the work of Gaspard-Gilbert and Bricmont-Kupiainen that we heard in the previous weeks. The goal is to show that, in the limit, we have a limiting stochastic process $e(t)$ that satisfy the SDE

$$de_i = \sum_{|i-k|=1} \alpha(e_i, e_k)dt + \sum_{|i-k|=1} \sigma\gamma(e_i, e_k)dB_{\{i,k\}} \tag{3.5}$$

with $B_{\{i,k\}} = -B_{\{k,i\}}$ independent standard Brownian motions. This is an open problem at the moment but similar results have been obtained for different model by Liverani-Olla (in preparation).

The above equation looks very similar to the non-gradient Ginzburg-Landau equation studied by Varadhan in [35]. So it is conceivable that it may be possible to take diffusive scaling limit (like in the previous example) and obtain a non-linear heat-equation.

This is the current research plan of several people.

References

[1] V. Baladi, M. Degli Eposti, S. Isola, E. Järvenpää, A. Kupiainen, *The spectrum of weakly coupled map lattices,* J. Math. Pures Appl. **77** (1998), 539–584.

[2] V. Baladi, H.H Rugh, *Floquet spectrum of weakly coupled map lattices,* Commun. Math. Phys.**220** (2001), 561–582.

[3] Jean-Baptiste Bardet, G. Keller, *Phase transitions in a piecewise expanding coupled map lattice with linear nearest neighbour coupling*, Nonlinearity **19** (2006), 2193-2210.

[4] J. Bricmont, A. Kupiainen, *Coupled analytic maps*, Nonlinearity **8** (1995), no. 3, 379–396.

[5] J. Bricmont, A. Kupiainen, *High temperature expansions and dynamical systems*, Comm. Math. Phys. **178** (1996), no. 3, 703–732.

[6] L.A. Bunimovich, Ya.G. Sinai, *Space-time chaos in coupled map lattices*, Nonlinearity **1** (1988), 491–516.

[7] D.Dolgopyat, C.Liverani, *Random Walk in Deterministically Changing Environment*, ALEA **4**, 89-116 (2008).

[8] T. Fischer, H.H. Rugh, *Transfer operators for coupled analytic maps,* Ergod. Th.& Dynam. Sys. **20** (2000), 109–143.

[9] G. Gielis, R.S. MacKay, *Coupled map lattices with phase transition,* Nonlinearity **13** (2000), 867–888

[10] V.M. Gundlach, D.A. Rand, *Spatio-temporal chaos: 1. Hyperbolicity, structural stability, spatio-temporal shadowing and symbolic dynamics,* Nonlinearity **6** (1991), 165–200.

[11] V.M. Gundlach, D.A. Rand, *Spatio-temporal chaos: 2. Unique Gibbs states for higher-dimensional symbolic systems,* Nonlinearity **6** (1993), 201–214.

[12] V.M. Gundlach, D.A. Rand, *Spatio-temporal chaos: 3. Natural spatio-temporal measures for coupled circle map lattices,* Nonlinearity **6** (1993), 215–230.

[13] V.M. Gundlach, D.A. Rand, *Spatio-temporal chaos (Corrigendum),* Nonlinearity **9** (1996), 605–606.

[14] E. Järvenpää, *A note on weakly coupled expanding maps on compact manifolds,* Annales Academiæ Scientiarum Fennicæ Mathematica **24** (1999), 511-517.

[15] E. Järvenpää, M. Järvenpää, *On the definition of SRB-measures for coupled map lattices*, Comm. Math. Phys. **220** (2001), no. 1, 1–12.

[16] M. Jiang, *Equilibrium states for lattice models of hyperbolic type*, Nonlinearity **8** (1995), no. 5, 631–659.

[17] M. Jiang, *Equilibrium measures for coupled map lattices: existence, uniqueness and finite-dimensional approximations,* commun. Math. Phys. **193** (1998), 675-712.

[18] M. Jiang, *Sinai-Ruelle-Bowen measures for lattice dynamical systems*, J. Statist. Phys. **111** (2003), no. 3-4, 863–902.

[19] M. Jiang, A.E. Mazel, *Uniqueness and exponential decay of correlations for some two-dimensional spin lattice systems*, J. Statist. Phys. **82** (1996), no. 3-4, 797–821.

[20] M. Jiang, Ya. Pesin, *Equilibrium measures for coupled map lattices: existence, uniqueness and finite-dimensional approximations*, Comm. Math. Phys. **193** (1998), no. 3, 675–711.

[21] G. Keller, *Coupled map lattice via transfer operators on functions of bounded variation*, Stochastic and spatial structures of dynamical systems (Amsterdam, 1995), 71–80, Konink. Nederl. Akad. Wetensch. Verh. Afd. Natuurk. Eerste Reeks **45** North-Holland, Amsterdam, 1996.

[22] G. Keller, *Mixing for finite systems of coupled tent maps*, Tr. Mat. Inst. Steklova **216** (1997), Din. Sist. i Smezhnye Vopr., 320–326; translation in Proc. Steklov Inst. Math. 1997, no. 1 (216), 315–32

[23] G. Keller, *An ergodic theoretic approach to mean field coupled maps*, Fractal geometry and stochastics, II (Greifswald/Koserow, 1998), 183–208, Progr. Probab., **46**, Birkhäuser, Basel, 2000.

[24] G. Keller, M.Künzle, *Transfer operators for coupled map lattices*, Ergodic Theory Dynam. Systems **12** (1992), no. 2, 297–318.

[25] G. Keller, C. Liverani, *Coupled map lattices without cluster expansion*, Discrete and Continuous Dynamical Systems, 11, n.2,3, 325–335 (2004).

[26] G.Keller, C.Liverani, *A spectral gap for a one-dimensional lattice of coupled piecewise expanding interval maps*, to appear in Lecture Notes in Physics, Springer.

[27] G.Keller, C.Liverani, *Uniqueness of the SRB measure for piecewise expanding weakly coupled map lattices in any dimension*, Communications in Mathematical Physics, **262**, 1, 33–50, (2006).

[28] G. Keller, R. Zweimüller, *Unidirectionally coupled interval maps: between dynamics and statistical mechanics*, Nonlinearity **15** (2002), no. 1, 1–24.

[29] M. Künzle: Invariante Maße für gekoppelte Abbildungsgitter, Dissertation, Universität Erlangen (1993)

[30] Ch. Maes, A. van Moffaert, *Stochastic stability of weakly coupled map lattices,* Nonlinearity **10** (1997), 715–730.

[31] J. Miller, D.A. Huse, *Macroscopic equilibrium from microscopic irreversibility in a chaotic coupled-map lattice*, Pys. Rev. E, **48**, 2528–2535 (1993).

[32] Ya.B. Pesin, Ya.G. Sinai, *Space-time chaos in chains of weakly interacting hyperbolic mappings*, adv.Sov.Math. **3** (1991), 165-198.

[33] H.H. Rugh, *Coupled maps and analytic function spaces*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 4, 489–535.

[34] M. Schmitt: BV-spectral theory for coupled map lattices, Dissertation, Universität Erlangen (2003). See also: Nonlinearity **17**, 671–690 (2004).

[35] Varadhan, S. R. S. *Nonlinear diffusion limit for a system with nearest neighbor interactions. II.* Asymptotic problems in probability theory: stochastic models and diffusions on fractals (Sanda/Kyoto, 1990), 75–128, Pitman Res. Notes Math. Ser., 283, Longman Sci. Tech., Harlow, 1993.

[36] D.L. Volevich, *Kinetics of coupled map lattices,* Nonlinearity **4** (1991), 37-45.

[37] D.L. Volevich, *Construction of an analogue of Bowen-Sinai measure for a multidimensional lattice of interacting hyperbolic mappings*, Russ. Acad. Math. Sbornink **79**, 347–363 (1994).

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