PERSONAL NOTES FOR THE BEIJING MINICOURSE

LIMIT THEOREM FOR HYPERBOLIC SYSTEM. INTRODUCTION: TECHNIQUES TO ESTABLISH FINE STATISTICAL PROPERTIES OF DYNAMICAL SYSTEMS

CARLANGELO LIVERANI

ABSTRACT. This are personal notes, they contain mistakes, the notation is inconsistent and the references are incomplete. Read at your own risk.

1. FIRST TOPIC: EXPANDING MAPS

1.1. **What are we talking about?** The qualitative description of the statistical properties of dynamical systems is very well studies. The possible behavior go from extremely regular motion (for example as in completely integrable dynamical systems), to systems in which the times averages are constant (ergodicity), to systems in which a large class of initial measures converge under the dynamic to the invariant measure (mixing) till systems that are isomorphic to a Bernoulli shift.

This is a very useful classification, widely used in the field of Dynamical Systems and its application. Yet, it turns out that for many applications it is necessary to have a more *quantitative* understanding.

The purpose of the first part of this minicourse is to discuss some of these issue, and show how the relevant properties can be established in various systems from few degree of freedom (mainly Anosov Systems or flows) to infinitely many degree of freedom (Coupled Map Lattices). In the second part of the minicourse D.Dolgopyat will show how such understanding and techniques can be applied to a manifold of interesting problems at the boundary between Dynamical Systems and Probability.

Let me start by being a bit more precise on the type of properties we are interested in

(1) **Decay of correlations** Given a Dynamical System[1] (X, f, Σ) with a class of measures $\mathcal{M}$, the system is mixing with respect to such a class if there exists only one invariant measure μ_* in the weak closure of $\mathcal{M}$, and if $f_*^n\mu$ converges weakly to such a measure for each $\mu \in \mathcal{M}$.[2] The question here is to estimate or compute the speed of convergence. Typically, this question does not make sense (i.e. often the speed can be arbitrarily small depending on the element of $\mathcal{M}$ or the test function). To understand how to restrict the class of measures and obervables, while keeping them as large as possible, is part of the problem.

Date: August 14, 2009.

[1]In the following X will be a Riemannian manifold, possibly with boundaries, f a smooth or piecewise smooth map and Σ the Borel σ-algebra.

[2]Of course $f_*\mu(A) := \mu(f^{-1}A)$ for each $A \in \Sigma$.

(2) **Central limit Theorem** If we consider a mixing system (X, f, Σ, μ_*), then by ergodicity for each L^1 function φ

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ f^k = \mu_*(\varphi) \quad \mu_*\text{-a.s..}$$

This is very nice, yet in reality one cannot perform an infinite time average. Hence it becomes of extreme practical importance to answer the question: if n is large but finite, how will fluctuate the average around the mean? In many cases (*but not all!*) the answer is that the difference will behave like a Gaussian random variable of size $n^{-\frac{1}{2}}$, i.e. it satisfies a CLT. Here also it is necessary to restrict to class of observable in a meaningful way in order to establish such type of results. Actually, in practice, even more precise results are often needed (such as local CLT, Berry-Essen estimates, invariance principles etc.)

(3) **Large deviation Theorems** This is related to the previous situation, only the question is a bit different: is it possible to observe deviations from the average of size one? If the system satisfies the CLT such a possibility must be extremely rare, yet it is well know that rare events can have a catastrophic effect.

(4) **Perturbation Theorems** If we are interested in applications, then the models that we study mathematically are always approximations of the real phenomena. It is then natural to ask what happens of the above properties if one changes a bit the system. This changes can be of many types, here are the most commonly considered
   - Determinist change: one considers a system (X, f_ε) where f_ε is close, in an appropriate sense, to f.
   - Random change: here one considers a random system close to the original one, for example one can iterate maps close to f chosen at random independently at each time, one can add at each time a small random variable, etc.
   - Open systems: one can also consider the possibility to open a small *hole* in the system. That is, consider a small set $A \subset X$ and stop the dynamics when $f^n(x) \in A$. The problem is to study the dynamics conditioned to the fact that it has not been stopped.

(5) **Linear Response** This is similar to the above but the emphasis is on more refined results and explicit formulae: considers a smooth (in some technical sense) one parameter family of systems (X, f_ε) where each f_ε has a unique invariant measure, in some class $\mathcal{M}$, ν_ε. This can be thought as a model of a situation in which one can act on a system by tuning an external parameter (e.g. an electric field). Physicist are interested in the changes of a measurement for small changes of the parameter and have a (mostly heuristic) theory to describe this called *linear response theory.* Mathematically the problem reduces to show that $\frac{d\nu_\varepsilon}{d\varepsilon}$ exists and to find explicit formulae for it.

(6) **Computability** Again for applications is important to be able to actually compute the relevant quantities (e.g. the invariant measure, the rate of decay of correlations, the variance in the CLT, the rate function in the Large deviations, etc.). Since computers use rational numbers, any computation

entails an approximation. Thus the problem of computing is closely related to the problem of perturbing the system, although it comes with its peculiar difficulties.

(7) **Zeta functions** In 1975, Ruelle proposed to study the function

$$\zeta_{Ruelle}(s) = \prod_{\tau}\left(1 - e^{-s\lambda(\tau)}\right)^{-1}$$

where $\lambda(\tau)$ denotes the least period of a closed orbit τ. This definition is analogous to that of the famous Riemann zeta function and its knowledge can be used to estimate the number of closed orbits of the system, in analogy with the well known *prime number theorems*. The study of the ζ function can be carried out by first studying the so called *dynamical determinant*

$$d(s) = \exp\left(-\sum_{\tau}\frac{e^{-s\lambda(\tau)}}{\det(I - D_{\lambda(\tau)}\phi(\tau))}\right)$$

were $D_{\lambda(\tau)}\phi(\tau)$ is the derivative of the map (or of the Poincaré map in the case of flows). Amazingly enough it turns out that the analytic properties of d depend on the topological pressure, and the rates of decay of correlations. We have then that the statistical properties of the systems are closely related to the behavior of the periodic orbits that, naively, seem to have little to do with it. The zeta function is very popular among physicists [19] because it is possible to implement efficient algorithms to compute periodic orbits and thus one can use this theory to effectively estimate interesting properties of the system [65].

In the following I will present two techniques to investigate such problems, the first more robust but less powerful, the second more powerful but also more limited: *coupling* and *spectral methods*. In case people never saw them I will present them first in the easiest setting (expanding maps), sorry for the others.

Remark 1.1. *In this lectures I will always talk about the uniformly hyperbolic (or partially hyperbolic) case. This seems to be at odd with the title of this Conference but it is not really so: the typical strategy to deal with a non-uniform system is to transform it (usually by some inducing procedure) in a uniform one. Also, it is now well understood that when discontinuities in a uniformly hyperbolic system are present, the techniques used to attack the problem are often very similar to the ones used for non uniform systems.*

If one want to know much more about the transfer operator see [1].

1.2. **Smooth expanding maps.** This is the simplest possible case: a dynamical system $(\mathbb{T}^d, f)$ where $f \in \mathcal{C}^2$ and $\|Df^{-1}\|_\infty =: \sigma < 1$.

The basic idea is to study the dynamics on measures: let $\mathcal{M}(\mathbb{T}^d)$ be the space of signed Borel measures on $\mathbb{T}^d$, then, for each Borel set A and measure $\mu \in \mathcal{M}(\mathbb{T}^d)$, define

$$f_*\mu(A) := \mu(f^{-1}(A)).$$

We have now the new dynamical system $(\mathcal{M}(\mathbb{T}^d), f_*)$. This is too big to be studied meaningfully, as already said the idea is to restrict to a smaller class of measures. The class to be chosen depends on the question we want to address, for example: find the SRB measures, or the maximal entropy measure, or find an invariant measure supported on some Cantor set, ...

Here, for simplicity, we will restrict to the first question and we will then chose $\mathcal{M} := \{\mu \in \mathcal{M}(\mathbb{T}^d) \ : \ \mu \ll \text{Lebesgue}\}$. Calling m the Lebesgue measure, an explicit computation (change of variable) shows that if $h \in L^1(\mathbb{T}^d, m)$ and $d\mu := hdm$, then for each $\varphi \in \mathcal{C}^0$,

$$
\begin{aligned}
&f_*\mu(\varphi) = m(\varphi \cdot \mathcal{L}h)\\
&\mathcal{L}h(x) := \sum_{y \in f^{-1}(x)} |\det(D_y f)|^{-1} h(y)
\end{aligned}
$$

in particular $f_*(\mathcal{M}) \subset \mathcal{M}$. The operator $\mathcal{L}$ is called in many way (essentially any arbitrary permutation of a sub-set of the words *Transfer* , *Perron-Frobenious, Ruelle* + operator will do) and is the main object of our discussion. Its key property, that make it possible to study it, is that it has some smoothing properties. To see it assume $h \in \mathcal{C}^1$, then, for each vector $v \in \mathbb{R}^d$,

$$
\begin{aligned}
\langle \nabla \mathcal{L}h, v\rangle &= \sum_{y \in f^{-1}(x)} \langle |\det(D_y f)|^{-1} \cdot \nabla h(y), (D_y f)^{-1} v\rangle\\
&\quad - \sum_{y \in f^{-1}(x)} \langle |\det(D_y f)|^{-2} \nabla |\det(D_y f)| \cdot h(y), (D_y f)^{-1} v\rangle\\
&= \mathcal{L}(\langle \mathcal{D}, v\rangle h) + \mathcal{L}(\langle \nabla h, (D_y f)^{-1} v\rangle),
\end{aligned}
\tag{1.1}
$$

where the quantity $\mathcal{D} = |\det(D_y f)|^{-1}[(D_y f)^{-1}]^t |\nabla[\det(D_y f)]|$ is called *distortion*. Note that for $f \in \mathcal{C}^2$, $\|\mathcal{D}\|_\infty = M < \infty$.

The question is: how can we exploit the above smoothing property?

1.3. **Coupling.** One way to use (1.1) is to notice that if $h \in \mathcal{C}^1$ and $ah(x) \geq \|\nabla h(x)\|$, then

$$
\|\nabla \mathcal{L}h(x)\| \leq |\mathcal{D}|_\infty \mathcal{L}h(x) + \sigma(\mathcal{L}\|\nabla h\|)(x) \leq [|\mathcal{D}|_\infty + \sigma a]\mathcal{L}h(x).
$$

In other words, for each $\sigma_1 \in (0, \sigma)$, if we define the cone of function $\mathcal{C}_a := \{h \in \mathcal{C}^1(\mathbb{T}^d, \mathbb{R}) \ : \ ah(x) \geq \|\nabla h(x)\|\}$, with $a > \mathcal{D}(\sigma - \sigma_1)^{-1}$, we have $\mathcal{L}\mathcal{C}_a \subset \mathcal{C}_{\sigma_1 a}$.

The above cone contraction can be used directly by using Hilbert metrics[3] but here we are more interested in the, more flexible, coupling techniques.[4]

Given two probability measures μ_1, μ_2 on $\mathbb{T}^d$ a coupling Q is simply a probability measure on $\mathbb{T}^{2d}$ such that the marginal are exactly μ_1, μ_2, i.e.

$$
\int_{\mathbb{T}^{2d}} g(x) Q(dx, dy) = \mu_1(g) \ ; \quad \int_{\mathbb{T}^{2d}} g(y) Q(dx, dy) = \mu_2(g).
$$

An interesting fact about coupling is that if it has the form

$$
Q(g) = (1 - \delta) \int_{\mathbb{T}^d} g(x, x)\mu(dx) + \delta Q_1(g),
\tag{1.2}
$$

for some probability measures μ, Q, Q_1 and $\delta \in [0, 1]$, then

$$
|\mu_1(g) - \mu_2(g)| = \int |g(x) - g(y)| Q(dx, dy) \leq \delta \int |g(x) - g(y)| Q_1(dx, dy) \leq 2\delta |g|_\infty.
$$

[3]This approach has been introduced in the field of Dynamical Systems by Ferrero [23] and further developed by many people starting with [53].

[4]This approach was present for quite some time in the field of Dynamical System (more precisely abstract ergodic theory) under the name of *joining*, but it has been introduced in this context by Young [77], following the use in probability.

Thus coupling can be used to estimate the distance of two measures (in the present case the *total variation* distance).

Given two different measures $d\mu_i = h_i dm$, the idea is to construct iteratively better and better couplings among the measures with densities $\mathcal{L}^n h_i$. This can be easily done since if one has a coupling of the form (1.2), then, by definition, $\mu_i(g) = (1-\delta)\mu(g) + \delta\nu_i(g)$. On the other hand if the measures have such a form one can always consider the coupling $Q(g) = (1-\delta)\int g(x,x)\mu(dx) + \delta\nu_1 \otimes \nu_2(g)$.

Now, let us assume we have $h \in \mathcal{C}_a$. Note that this implies: (a) $h > 0$;[5] (b) given two points $x, y \in \mathbb{T}^d$, let $\gamma \in \mathcal{C}^1([0,1], \mathbb{T}^d)$ be a reparametrization of the straight segment joining the two points, then $h_i \circ \gamma > 0$ and

$$\frac{h(x)}{h(y)} = e^{\ln h(x) - \ln h(y)} = e^{\int_0^1 \frac{\langle \nabla h \circ \gamma(t), \gamma'(t) \rangle}{h \circ \gamma(t)} dt} \le e^{a|x-y|}.$$

Thus, if h is a smooth probability density, hence its integral is one, it must have value one somewhere, thus $h \ge e^{-a\sqrt{d}}$.

Finally, given two densities of probability measures $h_i \in \mathcal{C}_{\sigma_1 a}$ and setting $r = (1-\sigma_1)e^{-a\sqrt{d}}$ we can write $h_i = r + (1-r)h_{1,i}$, moreover one can easily check that $h_{1,i} \in \mathcal{C}_a$. We are thus in a situation of the above type. Moreover, $\mathcal{L}h_i = r\mathcal{L}1 + (1-r)\mathcal{L}h_{1,i}$, but then $\mathcal{L}h_{1,i} \in \mathcal{C}_{\sigma_1 a}$ and we can keep coupling. After n steps we will have

$$\mathcal{L}^n h_i = \sum_{k=0}^{n-1} r(1-r)^k \mathcal{L}^{n-k} 1 + (1-r)^n h_{n,i}.$$

The above means that the total variation distance between $\mathcal{L}^n h_1$ and $\mathcal{L}^n h_2$ is bounded by $(1-r)^n$, i.e. the sequences $\mathcal{L}^n h$ are Cauchy sequences that converge exponentially fast. In other words there exits a unique $\mu \ll$ Lebesgue such that, for all $f \in \mathcal{C}^0, h \in \mathcal{C}^1$,

$$\left| \int h \cdot \varphi \circ f^n - \mu(\varphi) \int h \right| \le C |\varphi|_\infty |h|_{\mathcal{C}^1} e^{-\alpha n}.$$

1.4. **Spectral methods.** A different method to extract information from (1.1) is to notice that it implies the following inequalities

$$\begin{aligned} &|\mathcal{L}h|_{L^1} = \sup_{|\varphi|_\infty \le 1} m(\varphi \mathcal{L}h) = \sup_{|\varphi|_\infty \le 1} m(\varphi \circ f h) \le |h|_{L^1} \\ &|\nabla \mathcal{L}h|_{L^1} \le \sigma |\nabla h|_{L^1} + M|h|_{L^1}. \end{aligned} \tag{1.3}$$

The above inequalities are the first example of a general class of inequalities commonly called *Lasota-Yorke* or *Doeblin-Fortet* and they will play a fundamental role in our discussion. The reason lies in the following theorem[6]

[5]Indeed, if $h(x) = 0$, then for each y consider a smooth curve γ such that $\gamma(0) = x, \gamma(1) = y$, then

$$\frac{d}{dt} h \circ \gamma = \langle \nabla h \circ \gamma, \gamma' \rangle \le |\gamma'| a h \circ \gamma$$

and then it would follow $h \equiv 0$ by Gronwald inequality and the arbitrariness of y.

[6]This results has a long history starting from Ionescu-Tulcea and Marinescu, Ruelle, Keller and so on. In the form given here is due to Hennion [32]. But see the proof of Lemma 1.10 in the Appendix B for a simple proof of a slightly less general statment.

Theorem 1.2. *Consider two Banach spaces $\mathcal{B} \subset \mathcal{B}_w$, $\|\cdot\| \geq \|\cdot\|_w$, and an operator $\mathcal{L} : \mathcal{B}_w \to \mathcal{B}_w$ such that, $\mathcal{L}(\mathcal{B}) \subset \mathcal{B}$. In addition, assume that for some $L > \theta > 0$, $A, B, C > 0$, and for each $n \in \mathbb{N}$, holds true*

$$\|\mathcal{L}^n f\|_w \leq C L^n \|f\|_w; \quad \|\mathcal{L}^n f\| \leq A\theta^n \|f\| + BL^n \|f\|_w$$

Then the spectral radius of $\mathcal{L}$ is bounded by L. If, in addition, $\mathcal{L}$ is compact as an operator from $\mathcal{B}$ to $\mathcal{B}_w$, then $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ is quasi compact and its essential spectral radius[7] *is bounded by θ.*

Note that considering the Sobolev spaces $\mathcal{B}_w = L^1, \mathcal{B} = W^{1,1}$ and iterating (1.3) we have exactly the above with $C = L = A = 1$, $\theta = \sigma$ and $B = M(1-\sigma)^{-1}$. It follows immediately that there can be at most finitely many invariant measures absolutely continuos with respect to Lebesgue. But we have just seen that there is only one! The simplest way to obtain such a result in the present setting it to remember the Gagliardo-Niremberg-Sobolev inequalities: for each $p \in [1, d)$ and smooth function g

$$\|g\|_{L^{p^*}} \leq C\|g\|_{W^{1,p}},$$

where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$. An Morrey inequality: for each $p > d$ exists $\alpha > 0$ such that

$$\|g\|_{\mathcal{C}^\alpha} \leq C\|g\|_{W^{1,p}}.$$

In addition, by iterating (1.1) we have $\|\nabla \mathcal{L}^n 1(x)\| \leq (1-\sigma)^{-1}|\mathcal{D}|_\infty |\mathcal{L}^n 1(x)|$. Thus $|\mathcal{L}^n 1|_{L^{p^*}} \leq C\|\mathcal{L}^n 1\|_{W^{1,p}} \leq C|\mathcal{L}^n 1|_{L^p}$ and since $p^* = p(1 + \frac{p}{d-p})$, after a finite number of iterations we will be able to apply Morrey inequality, thus $|\mathcal{L}^n 1|_\infty \leq C$. But this implies

$$\int |\mathcal{L}^n h|^p \leq \int |\mathcal{L}^n h|^{p-1} \circ f |h| \leq \left[\int |\mathcal{L}^n h|^p \mathcal{L}^n 1\right]^{\frac{p-1}{p}} |h|_{L^p} \leq |\mathcal{L}^n 1|_\infty^{\frac{p-1}{p}} |\mathcal{L}^n h|_{L^p}^{p-1} |h|_{L^p}.$$

This means that $\|\mathcal{L}^n h\|_{L^p} \leq C\|h\|_{L^p}$ for each $p \geq 1$ and, from (1.1) again, yields the Lasota-Yorke inequality for $L^p, W^{1,p}$. But this means that all absolutely continuos invariant measures must have continuos density.[8] This implies that the support of any invariant measure contains a ball. Since the expansion implies that volume of the image of a ball keeps growing until it covers an invertibility domain for the map the support must be all $\mathbb{T}^d$. But then any two invariant measures would be absolutely continuous with respect to each other, including the ergodic ones, which is impossible, hence there must be only one invariant measure absolutely continuos with respect to Lebesgue.

1.5. **Linear response.** Since the relevant topics are too many to be all discussed here, let us just say two words on *linear response*. Suppose that we have a smooth one parameter family $\{f_\varepsilon\}$ of smooth expanding maps of the type just discussed. Let μ_ε be the associates unique invariant measure associated to f_ε and let h_ε be the relative density. Finally, let $\mathcal{L}_\varepsilon$ be the transfer operator associated to f_ε. Then

$$(\mathcal{L}_0^n - \mathcal{L}_\varepsilon^n)h_\varepsilon = \sum_{k=0}^{n-1} \mathcal{L}_0^k (\mathcal{L}_0 - \mathcal{L}_\varepsilon)\mathcal{L}_\varepsilon^{n-k-1} h_\varepsilon = \sum_{k=0}^{\infty} \mathcal{L}_0^k (\mathcal{L}_0 - \mathcal{L}_\varepsilon) h_\varepsilon - \sum_{k=n}^{\infty} \mathcal{L}_0^k (\mathcal{L}_0 - \mathcal{L}_\varepsilon) h_\varepsilon.$$

---

[7]By *essential spectrum* I mean the complement of the point spectrum with finite multiplicity.

[8]Just approximate the density by a smooth function in the L^1 and iterate by the dynamics: the Lasota-Yorke inequality for $p > d$ implies that one has an equicontinuos sequence, by the L^1 contractions of $\mathcal{L}$ the accumulation points (which exists by Ascoli-Arzelá) are L^1 arbitrarily close to the invariant measure and are still equicontinous, hence the claim.

Since $(\mathcal{L}_0 - \mathcal{L}_\varepsilon)h_\varepsilon$ has zero average, $|h_\varepsilon|_{\mathcal{C}^1}$ is uniformly bounded and the map f_0 enjoys exponential decay of correlations, taking the limit $n \to \infty$ yields

$$h_0 - h_\varepsilon = \sum_{k=0}^{\infty} \mathcal{L}_0^k (\mathcal{L}_0 - \mathcal{L}_\varepsilon) h_\varepsilon.$$

This implies that for all $\varphi \in \mathcal{C}^0$

$$\mu_0(\varphi) - \mu_\varepsilon(\varphi) = \sum_{k=0}^{\infty} \int (\varphi \circ f_0^{k+1} - \varphi \circ f_0^k \circ f_\varepsilon) h_\varepsilon.$$

The above immediately implies weak convergence of μ_ε to μ_0. In fact, more can be proven.

$$|h_0 - h_\varepsilon|_\infty \le C |\mathcal{L}_0 - \mathcal{L}_\varepsilon|_{\mathcal{C}^1 \to \mathcal{C}^0} |h_\varepsilon|_{\mathcal{C}^1}.$$

Which implies that $\lim_{\varepsilon \to 0} h_\varepsilon = h_0$. In fact, stronger results follow immediately from the Lasota-Yorke inequality and the perturbation theory in [42]. Yet, this is far from establishing that μ_ε is differentiable in ε.[9]

To see that let $\varphi \in \mathcal{C}^1$, note that $f_t = t f_0 + (1-t) f_\varepsilon$ is also a map of $\mathbb{T}^d$ to itself, and write

$$\begin{aligned}
\mu_0(\varphi) - \mu_\varepsilon(\varphi) &= \sum_{k=0}^{\infty} \int_0^1 dt \int_{\mathbb{T}^d} \frac{d}{dt} (\varphi \circ f_0^k \circ f_t) h_\varepsilon \\
&= \sum_{k=0}^{\infty} \int_0^1 dt \int_{\mathbb{T}^d} [\nabla(\varphi \circ f_0^k)] \circ f_t \cdot (f_0 - f_\varepsilon) h_\varepsilon \\
&= -\sum_{k=0}^{\infty} \int_0^1 dt \int_{\mathbb{T}^d} \varphi \cdot \mathcal{L}_0^k \mathrm{div}\left[\mathcal{L}_t (f_0 - f_\varepsilon) h_\varepsilon\right].
\end{aligned}$$

Now, by hypothesis there exists $\omega \in \mathcal{C}^2(\mathbb{T}^d, \mathbb{R}^d)$ such that $\|f_0 - f_\varepsilon - \varepsilon\omega\|_{\mathcal{C}^1} \le C\varepsilon^2$, thus

$$\begin{aligned}
&\left| \varepsilon^{-1} \sum_{k=0}^{\infty} \int_0^1 dt \int_{\mathbb{T}^d} \varphi \cdot \mathcal{L}_0^k \mathrm{div}\left[\mathcal{L}_t (f_0 - f_\varepsilon - \varepsilon\omega) h_\varepsilon\right] \right| \\
&\le \left| \varepsilon^{-1} \sum_{k=0}^{c \ln \varepsilon^{-1}} \int_0^1 dt \int_{\mathbb{T}^d} \varphi \cdot \mathcal{L}_0^k \mathrm{div}\left[\mathcal{L}_t (f_0 - f_\varepsilon - \varepsilon\omega) h_\varepsilon\right] \right| + \mathcal{O}(\varepsilon) \\
&\le \varepsilon^{-1} \sum_{k=0}^{c \ln \varepsilon^{-1}} |\varphi|_\infty \|\mathcal{L}_t (f_0 - f_\varepsilon - \varepsilon\omega) h_\varepsilon\|_{\mathcal{C}^1} + \mathcal{O}(\varepsilon) \le C\varepsilon \ln \varepsilon^{-1}.
\end{aligned}$$

Which means that

$$\partial_\varepsilon \mu_\varepsilon(\varphi)|_{\varepsilon=0} = -\lim_{\varepsilon \to 0} \sum_{k=0}^{\infty} \int_0^1 dt \int_{\mathbb{T}^d} \varphi \cdot \mathcal{L}_0^k \mathrm{div}\left[\mathcal{L}_t \omega h_\varepsilon\right].$$

To take the limit we need some uniform converge in the series and to insure that it is need $\mathrm{div}\left[\mathcal{L}_t \omega h_\varepsilon\right] \in \mathcal{C}^\alpha$ for some $\alpha > 0$. In other words we need a uniform bound

[9] Indeed, the general theorem in [42] implies $|h_0 - h_\varepsilon|_\infty \le C\varepsilon \ln \varepsilon^{-1}$ if $|\mathcal{L}_0 - \mathcal{L}_\varepsilon|_{\mathcal{C}^1 \to \mathcal{C}^0} \le \varepsilon$.

on $|h_\varepsilon|_{\mathcal{C}^{1+\alpha}}$.[10] This can easily be obtained by (1.1) by differentiating once more. We have finally the formula

$$\partial_\varepsilon \mu_\varepsilon(\varphi)|_{\varepsilon=0} = -\sum_{k=0}^{\infty} \int_{\mathbb{T}^d} \varphi \cdot \mathcal{L}_0^k \text{div}\,[\mathcal{L}_0 \omega h_0] = \sum_{k=0}^{\infty} \mu_0([\nabla(\varphi \circ f_0^k)] \circ f_0 \cdot \omega). \tag{1.4}$$

Remark 1.3. *In the above argument we have only used a bound of the decay of correlations and some a priori estimates on the regularity of the measures. If we use the full force of the spectral methods much stronger results can be obtained. For example one can prove that the spectral data of $\mathcal{L}_\varepsilon$ are differentiable and this implies, for example, that the rate of decay of correlations is differentiable* [26].

1.6. **Other measures.** Can one use the above methods to construct different invariant measures? It turns out that the spectral method is particularly well suited for that. The idea is to consider the operator

$$\mathcal{L}_g h(x) = \sum_{y \in f^{-1}(x)} g(y) h(y).$$

for some smooth function g. Then one can argue as before and obtain the inequality

$$\|\nabla \mathcal{L}_g^n h\|_\infty \leq \sigma^n \|\mathcal{L}_g^n \nabla h\|_\infty + B\|\mathcal{L}_g^n h\|_\infty.$$

Now the spectral radius of $\mathcal{L}_g : \mathcal{C}^0 \to \mathcal{C}^0$ is $e^{P(g)}$ where $P(g)$ is called *pressure*. Accordingly for each $\rho > e^{P(g)}$ we can apply Lemma 1.2 which implies that $\mathcal{L}_g : \mathcal{C}^1 \to \mathcal{C}^1$ has spectral radius $e^{P(g)}$ and essential spectral radius $\sigma e^{P(g)}$. This means that there exists $h \in \mathcal{C}^1$ such that $\mathcal{L}_g h = e^{P(g)} h$. But what does has this to do with invariant measures?

The point is that one can consider the dual operator $\mathcal{L}_g'$ acting on the distributions of order one. Since $\mathcal{L}$ can be written as a finite rank operator plus a small part, the same hold for the dual operator, hence there exists $\nu \in (\mathcal{C}^1)'$ such that $\mathcal{L}_g' \nu = e^{P(g)} \nu$. The existence of such a conformal distribution does not seem of much interest but, by the Lasota-Yorke inequality

$$\nu(\varphi) = e^{-nP(g)} (\mathcal{L}_g')^n \nu(\varphi) = e^{-nP(g)} (\mathcal{L}_g')^n \nu(\mathcal{L}_g^n \varphi) \leq C\|\nu\| \|\varphi\|_{\mathcal{C}^1} \sigma^n + C\|\nu\| \|\varphi\|_{\mathcal{C}^0}.$$

Taking the limit $n \to \infty$ yields $|\nu(\varphi) \leq C\|\varphi\|_{\mathcal{C}^0}$, i.e. ν is a measure! We can then define the new measure $\mu(\varphi) := \nu(\varphi h)$. Then

$$\mu(\varphi \circ f) = e^{-P(g)} \nu(\mathcal{L}_g(\varphi \circ f h)) = e^{-P(g)} \nu(\varphi \mathcal{L}_g(h)) = \nu(\varphi h) = \mu(\varphi),$$

that is, we have an invariant measure. What is the relation between such an invariant measure and the operator $\mathcal{L}_g$?

$$\nu(\psi \varphi \circ f) = e^{-P(g)} \nu(\varphi \mathcal{L}_g(\psi))$$

So $e^{-P(g)} \mathcal{L}_g$ describes the evolution of the densities with respect to the measure ν exactly like $\mathcal{L}$ describes the evolution with respect tot he Lebesgue measure. Note that one can in this way study a manifold of invariant measures by using exactly the same functional setting used for the measures absolutely continuos with respect

[10] Note that the only property of h_ε used so far is a uniform bound on $|h_\varepsilon|_{\mathcal{C}^1}$, in particular we have not used the decay of correlations for the measure μ_ε. Since it this case it holds true it would have been simpler to use it and interchange the roles of h_ε and h_0 in the formulae above. But it is an interesting fact, pointing to important possible extensions of the above argument, the fact that such an information can be substituted by a uniform bound on some norm of the measure.

to Lebesgue. In particular the above can be applied to the study of the measure of maximal entropy (which corresponds to choosing $g \equiv 1$).

1.7. **Discontinuities.** To conclude I would like to comment on the case of piecewise expanding maps.

More precisely, let $X := [0,1]^d$ together with a finite collection of disjoint open sets $\{\Delta_i\}_{i\in\mathcal{I}\subset\mathbb{N}}$. Assume that $\partial\Delta_i$ consists of the union of finitely many smooth manifolds $\{S_{i,j}\}$ and that

- $\cup_{i\in\mathcal{I}}\overline{\Delta}_i = X$;
- There exists $\varepsilon_0 > 0$, $L \in \mathbb{N}$ such that each ball of radius ε_0 intersects at most L manifolds $\{S_{i,j}\}$.

Next, let $T : X \to X$ be such that, for each $i \in \mathcal{I}$, $T|_{\Delta_j}$ is a $\mathcal{C}^2$ invertible map. Finally, we ask that the map be sufficiently expanding: setting $\|(D_xT)^{-1}\|_\infty = \lambda^{-1}$,[11]

$$
\begin{aligned}
&(L+1)\lambda^{-1} < 1 \quad \text{for all } x \in \cup_j\Delta_j;\\
&|\nabla(D_xT)^{-1}|_\infty < \infty.
\end{aligned}
\tag{1.5}
$$

Here the two strategy (coupling and spectral methods) start to differ a bit: the coupling strives to achieve a local control of the dynamics, on the contrary in the spectral approach one wants more global estimates. We will see that the coupling approach is conceptually simpler, yet the spectral method yield much stronger results.

1.7.1. *Coupling.* The starting idea is to consider absolutely continuous measures supported on a ball of radius $\delta < \varepsilon_0$ with a density as in the smooth case. If we iterate such a measure with the dynamics, we will have that the ball will split in at most L sets W_i on which T is smooth, hence the image measure is supported on the sets TW_i. Such sets are not necessarily disjoint. Clearly we can write the image measure as a convex combination of at most L measures each with density nicely under control. The problem is that now the measures are not supported in a ball anymore, it is then necessary to define a class of sets that can serve as invariant supports for the dynamically generated measures.

Let (W,h) where W is an open set contained in a ball of radius ε_0 and $h \in \mathcal{C}^1(W,\mathbb{R})$, $\|\nabla h\| \le ah(x)$, $\int_W h = 1$, is called a *standard pair*.

A *standard family* $\mathcal{G}$ is given by a (for simplicity say countable) set of indexes A, a probability measure $\nu_\mathcal{G}$ on A and a collection of standard pairs $\{(W_\alpha,h_\alpha)\}_{\alpha\in A}$. Clearly to each family of standard pairs is associated the probability measure

$$\mathbb{E}_\mathcal{G}(\varphi) := \int_A d\nu_\mathcal{G}(\alpha)\int_{W_\alpha}\varphi h_\alpha.$$

What we want to avoid is that the sets W_α have crazy boundaries, so we introduce the following condition: Given a standard pair family $\mathcal{G}$ for each $\varepsilon > 0$ we define $\partial_\varepsilon W_\alpha = \{x, \in W_\alpha \ : \ \mathrm{dist}(x,\partial W_\alpha)\}$. Let us define a measure of the boundary of a standard pair family

$$B_\varepsilon(\mathcal{G}) := \int_A d\nu_\mathcal{G}(\alpha)\int_{\partial_\varepsilon W_\alpha} h_\alpha.$$

[11] If the first condition below is optimal or not it is an issue that remains open to debate, certainly some condition is needed [70] but it in certain cases it can be replace by a smoothness assumption [68, 69, 14].

We then consider the class of measures $\mathcal{M}_{a,B,\varepsilon_0} = \{\mathcal{G} \;:\; B_\varepsilon(\mathcal{G}) \leq B\varepsilon, \;\; \forall \varepsilon \leq \varepsilon_0\}$.

Lemma 1.4. *For a, B large enough and ε_0 small enough, $T_*\mathcal{M}_{a,B,\varepsilon_0} \subset \mathcal{M}_{a,B,\varepsilon_0}$.*

Proof. Let us see how the measures under consideration evolve. Given $\mathcal{G}$ let $W_\alpha \in \mathcal{G}$, then let $\{W_{\alpha,\beta,\gamma}\}$ be the connected components of $W_\alpha \cap \Delta_\beta$ and call f_β the inverse of T restricted to Δ_β, then

$$\int_{W_\alpha} h_\alpha \varphi \circ T = \sum_{\beta,\gamma} \int_{TW_{\alpha,\beta,\gamma}} \varphi h_\alpha \circ f_\beta |\det(DT) \circ f_\beta|^{-1}$$

We can then define the new family $T\mathcal{G} := \{(TW_{\alpha,\beta,\gamma}, h_\alpha \circ f_\beta |\det(DT) \circ f_\beta|^{-1} z_{\alpha,\beta,\gamma}^{-1}\}$ where

$$z_{\alpha,\beta,\gamma} = \int_{TW_{\alpha,\beta,\gamma}} h_\alpha \circ f_\beta |\det(DT) \circ f_\beta|^{-1}.$$

The associated measure is given by $\nu_{T\mathcal{G}}(\{(\alpha,\beta,\gamma)\}) = \nu_{\mathcal{G}}(\{\alpha\}) z_{\alpha,\beta,\gamma}$. It is easy to verify that $\mathbb{E}_{\mathcal{G}}(\varphi \circ T) = \mathbb{E}_{T\mathcal{G}}(\varphi)$. The proof that the new densities satisfy the bound $\|\nabla h_{\alpha,\beta,\gamma}\| \leq a h_{\alpha,\beta,\gamma}$ for a large enough is exactly the same as in the smooth case. So we just need to verify that $T\mathcal{G}$ satisfies $B_\varepsilon(T\mathcal{G}) \leq B\varepsilon$.

First of all notice that if a point is at a distance less than ε form the boundary of $TW_{\alpha,\beta,\gamma}$ then its preimage must be at a distance less than λ^{-1} form the boundary of W_α or from the boundary of Δ_β. Hence

$$\begin{aligned}
B_\varepsilon(T\mathcal{G}) &\leq \int_A d\nu_G(\alpha) \sum_{\beta,\gamma} \int_{\partial_{\lambda^{-1}\varepsilon} W_{\alpha,\beta,\gamma}} h_\alpha \\
&\leq \int_A d\nu_G(\alpha) \left[\int_{\partial_{\lambda^{-1}\varepsilon} W_\alpha} h_\alpha + \sum_\beta \int_{(\partial_{\lambda^{-1}\varepsilon} \Delta_\beta) \cap W_\alpha} h_\alpha \right].
\end{aligned}$$

Since, by hypotheses, the boundaries of the $\{\Delta_i\}$ in a ball of radius ε_0 consist of at most L smooth manifolds $\{S_i\}$, it follows that the last term in the above equation is the ε-neighborhood of at most L smooth manifolds. Also the $d-1$ dimensional manifold of $\Delta_\beta \cap S_i$ must be smaller that $\frac{1+c\varepsilon_0}{2}$ the measure of ∂W_α.[12] Thus the same holds for the measure of an ε neighborhood. Hence

$$B_\varepsilon(T\mathcal{G}) \leq (L+1)(1+c\varepsilon_0) B_{\lambda^{-1}\varepsilon}(\mathcal{G}) \leq \lambda^{-1}(L+1)(1+c\varepsilon_0)B\varepsilon.$$

It seems that we are done but unfortunately there is a last issue to take care of: some of the sets $W_{\alpha,\beta,\gamma}$ could have a diameter larger than ε_0. In this case we further split such sets in sets of diameter smaller then ε_0. Clearly this increase the boundary of the standard pair family. On the other hand if W has radius larger than ε_0, then one can chose a direction in which the projection is larger than ε_0 and cut it by a perpendicular hyperplane Π. Let ℓ be the the volume of $\Pi \cap W$. Since the position of Π is arbitrary, one can chose the minimal volume in an interval $c\varepsilon_0$. But then the volume of W must be larger than $c\varepsilon_0\ell$ while the extra ε-boundary will have volume $2\ell\varepsilon$. Thus the boundary that we add by further refining the family is bounded by

$$C \int d\nu(\alpha) \sum_{i=1}^{cd\lambda} \frac{\ell_i \varepsilon}{|W_\alpha|} \leq C\varepsilon_0^{-1}\varepsilon.$$

[12] The constant c depends on the curvature of S_i.

The above yields the inequality

$$B_\varepsilon(T\mathcal{G}) \le (L+1)(1+c\varepsilon_0)B_{\lambda^{-1}\varepsilon}(\mathcal{G}) \le \lambda^{-1}(L+1)(1+c\varepsilon_0)B\varepsilon + C\varepsilon,$$

which proves the Lemma when choosing B large enough. □

The above means that if we take a standard pair family then half of the point will belong to an open set of inner radius $\frac{1}{2}B^{-1}$. Hence, given two standard pair families $\mathcal{G}_i$ there are nice open sets on which they can be coupled, provided they coincide. Yet, there is not reason why this must happen, to ensure this we need some topological condition (e.g. topological mixing). Here let us assume that for each ball $\tilde{B}$ of size $\frac{1}{2}b^{-1}$ there exists $m \in \mathbb{N}$ such that $T^m\tilde{B} = X$ almost surely. This mean that $T^M\mathcal{G}_i$ will have manifold that overlap over an open set U of fixed size. On U we can couple the measure exactly as we did in the smooth case. It is then clear that at each time step we can couple a fixed (rather small to be honest) percentage of the mass, hence the exponential decay of correlations follows.

1.7.2. *Spectral approach.* Just not to get too bored let us consider a slightly more general setting (it is an interesting exercise to adapt the previous argument to the present setting). Let $X := [0,1]^d$ together with a (possibly countable) collection of disjoint open sets $\{\Delta_i\}_{i\in\mathcal{I}\subset\mathbb{N}}$ such that

- $\cup_{i\in\mathcal{I}}\overline{\Delta}_i = X$;
- For each orthogonal basis $E := \{e_i\}$ let$L_k(x,j,E)$ be the number of connected components of $\{x + te_k\}_{t\in[-1,1]} \cap \Delta_j$. Then we assume that $L_j = \inf_E \sup_{x\in\Delta_j} \sup_k L_k(x,j,E) < \infty$.

Next, let $T : X \to X$ be such that, for each $i \in \mathcal{I}$, $T|_{\Delta_j}$ is a $\mathcal{C}^2$ invertible map. Finally we ask that the map be expanding and not too singular

$$\begin{aligned} &\|(D_xT)^{-1}\| \le \lambda_j^{-1} < 1 \quad \text{for all } x \in \Delta_j;\\ &|\nabla(D_xT)^{-1}|_{L^d} < \infty. \end{aligned} \tag{1.6}$$

Let us define the following two norms on $\mathcal{M}(X)$:

$$\begin{aligned} |\mu| &:= \sup_{\varphi\in\mathcal{C}^0(X,\mathbb{R})} \frac{\mu(\varphi)}{|\varphi|_\infty}\\ \|\mu\| &:= \sup_{k\in\{1,\dots,d\}} \sup_{\varphi\in\mathcal{C}^1(X,\mathbb{R})} \frac{\mu(\partial_{x_k}\varphi)}{|\varphi|_\infty}. \end{aligned} \tag{1.7}$$

Note that, for each $\varphi \in \mathcal{C}^0(X,\mathbb{R})$ and $\varepsilon > 0$ one can find $\varphi_\varepsilon \in \mathcal{C}^1(X,\mathbb{R})$ such that $|\varphi - \varphi_\varepsilon| \le \varepsilon|\varphi|_\infty$, hence

$$\mu(\varphi) \le |\mu|\varepsilon|\varphi|_\infty + \mu(\varphi_\varepsilon) = |\mu|\varepsilon|\varphi|_\infty + \mu(\partial_{x_1}\int_0^{x_1}\varphi_\varepsilon) \le (|\mu|\varepsilon + \|\mu\|(1+\varepsilon))|\varphi|_\infty.$$

Taking the sup on φ and by the arbitrariness of ε, follows

$$|\mu| \le \|\mu\|. \tag{1.8}$$

The proofs of Lemma 1.5 and 1.7 can be found in Appendix A.

Lemma 1.5. *Let $\mathcal{B} := \{\mu \in \mathcal{M}(X) \ : \ \|\mu\| < \infty\}$. If $\mu \in \mathcal{B}$ then it is absolutely continuos with respect to the Lebesgue measure m. Moreover*

$$\frac{d\mu}{dm} \in L^p(X,m) \quad \textit{for all } p < \frac{d}{d-1}.$$

Remark 1.6. *In fact it follows from the Gagliardo-Nirenberg-Sobolev inequality that the above Lemma holds also for* $p = \frac{d}{d-1}$.

Exercize 1. *Show that, for all* $\mu \in \mathcal{B}$*, setting* $h = \frac{d\mu}{dm}$*, holds* $|\mu| = |h|_{L^1}$ *and* $\|\mu\| = |h|_{BV}$.

The following characterization will be useful in the following: given $h \in L^1(X, m)$ we define

$$\operatorname{Var}^k(h)(x) = \sup_{\varphi \in \mathcal{C}^1([0,1],\mathbb{R})} \frac{\int_0^1 h(x_1, \dots, x_{k-1}, z, x_{k+1}, \dots, x_d)\varphi'(z)dz}{|\varphi|_\infty}.$$

Lemma 1.7. *For each* $\mu \in \mathcal{B}$*, setting* $h = \frac{d\mu}{dm}$*,*

$$\|\mu\| = \sup_{k \in \{1,\dots,n\}} |\operatorname{Var}^k(h)|_{L^1}.$$

Lemma 1.8. $B = \{\mu \in \mathcal{B} \;:\; \|\mu\| \leq 1\}$ *is relatively compact in* $(\mathcal{M}(X), |\cdot|)$.

1.8. **Dynamical inequalities (Lasota-Yorke).** There exists $C > 0$ such that for each $\alpha \in (0,1)$, $\varepsilon > 0$ and $i \in \mathcal{I}$, there are smooth functions ϕ_i^ε supported in a $\alpha^{-i}\lambda_i^{-1}L_i\varepsilon$-neighborhood of Δ_i and such that $|\phi_i^\varepsilon|_\infty = 1$, $|\phi_i^\varepsilon|_{\mathcal{C}^1} \leq C\alpha^i\varepsilon^{-1}\lambda^i L_i^{-1}$ and $\phi_i(x) = 1$ for all $x \in \Delta_i$. Let us define

$$\sigma' := \lim_{\varepsilon \to 0} \left| \sum_{i \in \mathcal{I}} \phi_i^\varepsilon \lambda_j L_j \right|_\infty.$$

Note that, in the simple case in which the partition $\{\Delta_i\}$ is finite and can be chosen (eventually by refining it), such that $L_j = 1$, and if $\lambda = \lambda_i$, then $\sigma' = C_\Delta \lambda^{-1}$ where C_Δ is the complexity of the partition:

$$C_\Delta := \sup_{x \in X} \#\{i \in \mathcal{I} \;:\; x \in \overline{\Delta}_i\}.$$

Lemma 1.9 (Lasota-Yorke inequality)**.** *For each* $\sigma \in (\sigma', 1)$ *there exists a constant* $B > 0$ *such that, for each* $\mu \in \mathcal{B}$*, holds*

$$\begin{aligned} &|T'\mu| \leq |\mu| \\ &\|T'\mu\| \leq \sigma\|\mu\| + B|\mu|. \end{aligned}$$

Proof. First of all notice that, if $\mu \in \mathcal{B}$, then (Remembering Lemma 1.5 and Exercise 1)

$$|T'\mu| = \sup_{|\varphi|_{\mathcal{C}^0} \leq 1} \mu(\varphi \circ T) \leq |\mu|.$$

Next, for all $\varphi \in \mathcal{C}^1$, $|\varphi|_\infty \leq 1$ and $k \in \{1, \dots, d\}$ we have

$$\begin{aligned} T'\mu(\partial_{x_k}\varphi) &= \sum_{i \in \mathcal{I}} \mu(\mathbb{1}_{\Delta_i}(\partial_{x_k}\varphi) \circ T) \\ &= \sum_{i \in \mathcal{I}} \sum_{j=1}^d \mu(\mathbb{1}_{\Delta_i}\partial_{x_j}((DT)^{-1}_{kj}\varphi \circ T)) - \sum_{i \in \mathcal{I}} \sum_{j=1}^d \mu(\mathbb{1}_{\Delta_i}\varphi \circ T\partial_{x_j}((DT)^{-1}_{kj})). \end{aligned}$$

Setting $h = \frac{d\mu}{dm}$ and $\psi_{kj} = (DT)^{-1}_{kj}\varphi \circ T$, note that $\sum_j |\psi_{kj}|_\infty \leq \lambda_i^{-1}$, moreover we can rotate the coordinates as is most convenient (by redefining ψ_{kj} as well)

$$\begin{aligned}\mu(\mathbb{1}_{\Delta_i}\partial_{x_j}\psi_{kj}) &= \mu(\phi_i^\varepsilon \mathbb{1}_{\Delta_i}\partial_{x_j}\psi_{kj}) \\ &\leq \int h(x)\partial_{x_j}\left[\phi_i^\varepsilon \int_0^{x_j} [\mathbb{1}_{\Delta_i}\partial_{x_j}\psi_{kj}](x_1,\dots,x_{j-1},z,x_{j+1},\dots,x_d)dz\right] \\ &\quad + \lambda_i^{-1}L_i|\mu||\phi_i|_{\mathcal{C}^1}.\end{aligned}$$

Hence, remembering the hypotheses on T,

$$\begin{aligned}T'\mu(\partial_{x_k}\varphi) &= \int \operatorname{Var}^k h \left|\sum_{i\in\mathcal{I}} \phi_i^\varepsilon \lambda_i^{-1} L_i\right|_\infty + \sum_{i\in\mathcal{I}} \lambda_i^{-1}L_i|\mu||\phi_i|_{\mathcal{C}^1} + C\mu(\|\nabla(DT)^{-1}\|) \\ &\leq \|\mu\|\sigma + B|\mu| + (\sigma - \sigma')\|\mu\|.\end{aligned}$$

□

Lemma 1.10. *The operator T' has spectral radius equal one and essential spectral radius smaller than σ.*

This follows from Lemma 1.2, but see Appendix B for a direct proof based on the Analitic Fredholm alternative.

In fact, one can have a much more deep understanding of the structure of the spectrum and of its dynamical meaning, see Appendix C.

2. Second Topic: Hyperbolic Systems

The issue is to extend the previous results to the case of hyperbolic systems: i.e. when a contracting direction is present. Given a Riemannian manifold M we will consider $f \in \text{Diff}^2(M, M)$, f Anosov.

The first problem is to consider a class of measures that take the place of the ones absolutely continuos with respect to Lebesgue, clearly a to restrictive class given the the SRB measure may not belong to it. The idea is a follows: Consider a manifold W close to the unstable direction and $\varphi \in \mathcal{C}^\infty(W, \mathbb{T})$ and for each function $g \in \mathcal{C}^0(M, \mathbb{R})$ define the measure

$$\mu_{W,\varphi}(g) = \int_W gf$$

where the integral is made with respect to the restriction of the Riemannian measure to W. The above are again standard pairs. The dynamics restricted to manifolds close to the unstable one is expanding, hence all what we have said about family of standard pairs for discontinuous expanding can be used here almost verbatim. Then $\mathcal{M}$ is the set of of measures obtained by families of standard pairs.

It looks quite good but there is a problem: if we want to imitate what we did for expanding maps then given two standard pairs we want to couple the measure when they are at the same place, but the dynamics is invertible now, **different points will never coincide in the future**! So we need to modify our coupling scheme. The basic idea is to notice the following: suppose we have two near by manifolds W, W' and the map $\Psi : W \to W'$ defined by $\{\Psi(x)\} = W' \cap W^s_\delta(x)$.[13]

[13]By $W^s_\delta(x)$ we mean the stable manifold at x of inner size δ.

and define the measure on M^2 by

$$\mu(\varphi) := \int_W \varphi(x, \Psi(x))h(x). \tag{2.1}$$

Then ν is a coupling of the measures defined by two standard pairs: (W, h) and $(W', h \circ \Psi^{-1} J\Psi)$, where $J\Psi$ is the Jacobian of the Holonomy. Also calling ν_i the measures associated to the two standard pairs, we have, for each $\varphi \in$
$cC^1(M)$,

$$|\nu_1(\varphi)-\nu_2(\varphi)| \leq \int_{M^2} |\varphi(x)-\varphi(y)|\mu(dx, dy) = \int_W |\varphi(x)-\varphi(\Psi(x))|h(x) \leq |\varphi|_{\mathcal{C}^1} d(W, W').$$

Even more, it is an easy exercise to prove that

$$|f_*^n \nu_1(\varphi) - f_*^n \nu_2(\varphi)| \leq |\varphi|_{\mathcal{C}^1} d(f^n W, f^n W') \leq C\lambda^{-n} |\varphi|_{\mathcal{C}^1}.$$

This means that the iterates of the two measures get exponentially close when viewed as distributions of order one. The idea is then to take a two standard families and wait that some manifold of the first gets close to a manifold of the second and then couple a little part of the mass as in (2.1) and, by the same reasoning as in the previous discussions it exponential decay of correlations follows, namely, there exists a unique measure μ such that for each $h, \varphi \in \mathcal{C}^1$,

$$\left|\int_M h \cdot \varphi \circ f^n - \int_M h\mu(\varphi)\right| \leq C|\varphi|_{\mathcal{C}^1} |h|_{\mathcal{C}^1} e^{-cn}. \tag{2.2}$$

Remark 2.1. *In fact, a moment of thought shows that, calling $|h|_u = |h|_\infty + |D^u h|_\infty$, where D^u is the derivative in the unstable direction and the analogous for $|h|_s$,*

$$\left|\int_M h \cdot \varphi \circ f^n - \int_M h\mu(\varphi)\right| \leq C|\varphi|_s |h|_u e^{-cn}.$$

This is not an irrelevant observation because the form of the constant in from of the exponential plays a crucial role in many applications. In particular the above formula immediately imply decay of multiple correlations which does non follows from (2.2).

The above strategy can be applies to many different systems such as some partially hyperbolic systems [21] and billiards [16].

2.0.1. *Spectral approach.* It is not immediately clear how to apply the spectral method to the present situation. Indeed the realization that this is possible has been very recent [10] and since than a considerable amount of work has been carried out in order to extend this approach to an ever large class of systems [2, 26, 6, 57, 59, 7, 27, 20] and references therein. This work is not completed, indeed the applicability of this method to partially hyperbolic systems is still not complete, in spite of considerable progresses [58, 71] not is the application to billiard (but see [8]).

The logical obstacle to develop this theory was that one has to leave the space of measures and consider distributions. Indeed, if at first sight strange in reality it is very natural: we have already seen that the convergence of measures takes place only in the topology of distributions (i.e. limited to test function with some smoothness). Moreover the distributions must be regular in the unstable direction and wild in stable.

One this concept is clear one can realize “good” Banach spaces in many different ways, here I'll present briefly one closely connected with the present point of view.

The basic idea is to consider the set of standard pairs Ω for the map f^{-1} (this are just constructed with manifolds close to the stable one rather than to the unstable one). Let Ω_r be the standard pairs where W is a $\mathcal{C}^r$ manifold (with uniformly bounded derivatives) and $|\varphi|_{\mathcal{C}^r} \leq C$. Then define

$$\|h\|_{p,q} := \sup_{(W,\varphi)\in\Omega_{p+q}} \int \varphi \partial^p h$$

Let $\mathcal{B}^{p,q}$ be the closure of $\mathcal{C}^\infty$ with respect to the above norms. Then one can prove

- $\mathcal{L}$ is bounded in each $\mathcal{B}^{p,q}$.
- $\mathcal{L}$ satisfies a Lasota-Yorke inequality with respect to the spaces $\mathcal{B}^{p,q}, \mathcal{B}^{p-1,q+1}$.
- The unit ball of $\mathcal{B}^{p,q}$ is compact in $\mathcal{B}^{p-1,q+1}$.

The above ingredients imply that $\mathcal{L}$ is quasi-compact on each $\mathcal{B}^{p,q}$, $p \leq 1$, that the spectral radius is one and that the essential spectral radius is $r_{p,q}$ tending to zero when $p, q \to \infty$.

One interesting consequence of this facts is that, for $\mathcal{C}^\infty$ Anosov diffeos the zeta function is meromorphic on all $\mathbb{C}$, [57].

3. Third Topic: Coupled Map lattices

3.1. **CML.** A couple map lattice is constructed a follows: given a dynamical system (X, T) we consider the space $\Omega := X^{\mathbb{Z}^d}$ (but more general sets than $\mathbb{Z}^d$ can be also considered) and the product map $F_0(x)_i = T(x_i)$. Next we consider a map $\Phi_\varepsilon : \Omega \to \Omega$ that is ε-close to the identity in a sense to be made precise. The CML that we will consider are then given by $F_\varepsilon := \Phi_\varepsilon \circ F_0$. Interesting cases are:

- T expanding map (either smooth or not)
- T uniformly hyperbolic (either smooth or not)
- T partially hyperbolic (either smooth or not)

The typical approach, going back to Bunimovich-Sinai, is to conjugate F_ε to F_0 and use Markov partitions (see the papers in the references for more details).

A more direct approach, and more dynamical in nature, is desirable (also because in the non-smooth case conjugation fails).

3.2. **Super-brief history of the transfer operator approach.** The possibility to investigate directly the transfer operator for a CML was first investigated by Keller and Künzle [46]. They were able to prove spectral gap in finitely many dimensions and existence of a measure with absolutely continuos marginals in infinite dimensions. Then Fischer, Rugh [24] and Rugh [64] managed to prove space-time decay of correlations in infinite dimensions in the *analytic* case. Then in Baladi, Degli Esposti, Järvenpää, Kupiainen [3] and Baladi, Rugh [4] the spectrum in the analytic case is precisely investigated. Finally, in [49] it was proved the spectral gap for piecewise expanding CML. The latter paper is what I will explain in the following.

3.3. **Expanding CML.** Consider the case in which $X = [0, 1]$ and the map is piecewise $\mathcal{C}^2$ and $|DT| \geq \lambda > 2$. While

$$\Phi_\varepsilon(x)_i = x_i + \varepsilon \sum_{|z|=1} \alpha_z(\tau^i x)(x_{i+z} - x_i),$$

with $\tau^i(x)_j = x_{i+j}$ and $\alpha_z \in \mathcal{C}^1$ with $\partial_{x_j}\alpha_z = 0$ if $|j| \geq 1$. Moreover, we assume

- $\alpha_z \geq 0$. Which, for ε small, insures $x_i \geq 0 \Longrightarrow \Phi_\varepsilon(x)_i \geq 0$.
- $\sum_i \alpha_i = 1$. Which for ε small, insures $x_i \leq 1 \Longrightarrow \Phi_\varepsilon(x)_i \leq 1$.

The goal is show existence and uniqueness of the SRB measure for small ε. For large, but still less than one, ε uniqueness may fail [9].

3.4. **Transfer operator and Lasota-Yorke inequality.** As we want to deal with infinite systems, it is convenient to first define the transfer operator on the set of Borel measures $\mathcal{M}(\Omega)$: for each measurable set A, let $\mathcal{L}\mu(A) := \mu(F_\varepsilon^{-1}(A))$.

Obviously $\mathcal{M}(\Omega)$ is too big to be useful, to restrict it we define two norms:

$$
\begin{aligned}
|\mu| &:= \sup_{|\varphi|_{\mathcal{C}^0}\leq 1} \mu(\varphi)\\
\|\mu\| &:= \sup_{i\in\mathbb{Z}^d} \sup_{\substack{\|\varphi\|_{\mathcal{C}^0}\leq 1\\ \varphi\in\mathcal{C}^1}} \mu(\partial_{x_i}\varphi).
\end{aligned}
$$

Clearly $|\mu| \leq \|\mu\|$. Let $\mathcal{B} := \{\mu \in \mathcal{M}(\omega) \;:\; \|\mu\| < \infty\}$.

Theorem 3.1 (Keller et al.)**.** *For ε small enough there exists $\theta \in (0,1)$ such that, for all $n \in \mathbb{N}$,*

$$
\|\mathcal{L}^n\mu\| \leq A\theta^n\|\mu\| + B|\mu|.
$$

That is nice but **compactness** is missing. In fact, compactness does not hold, thus we need a way to establish directly the existence of a gap.

3.5. **spectral gap.** To deal with this fix $a \in [0,1]$ and given $x \in \Omega$ let $(x^p)_q = x_q$ for $q \neq p$ and $(x^p)_p = a$. Then define $\Phi_{\varepsilon,p}$ to be the map

$$
\Phi_{\varepsilon,p}(x)_q = \begin{cases} \Phi_\varepsilon(x^q)_q & \text{if } q \neq p\\ x_p & \text{if } q = p\,. \end{cases}
$$

One can easily verify that

$$
|(\mathcal{L} - \mathcal{L}_p)\mu| \leq C\varepsilon\|\mu\|,
$$

where $\mathcal{L}_p$ is the operator associated to the coupling $\Phi_{\varepsilon,p}$. Indeed, letting $\Phi_t := (1-t)\Phi_\varepsilon - t\Phi_{\varepsilon,p}$, holds

$$
\begin{aligned}
\mu(\varphi\circ\Phi_\varepsilon - \varphi\circ\Phi_{\varepsilon,p}) &= \int_0^1 \mu(\frac{d}{dt}\varphi\circ\Phi_t) = \int_0^1 \sum_{|i-p|\leq 1} \mu(\partial_{x_i}\varphi\cdot[\Phi_\varepsilon - \Phi_{\varepsilon,p}]_i)\\
&= \int_0^1 \sum_{|i-p|\leq 1} \mu(\partial_{x_i}[\varphi(\Phi_\varepsilon - \Phi_{\varepsilon,p})_i]) - \mu(\varphi\partial_{x_i}(\Phi_\varepsilon - \Phi_{\varepsilon,p})_i])\\
&\leq C\varepsilon\|\mu\|\cdot|\varphi|_\infty.
\end{aligned}
$$

Hence

$$
|(\mathcal{L}^n - \mathcal{L}_p^n)\mu| \leq \sum_{k=0}^{n-1} |\mathcal{L}^{n-k-1}(\mathcal{L} - \mathcal{L}_p)\mathcal{L}_p^k\mu| \leq C\varepsilon n\|\mu\|.
$$

Next, suppose that $\mu(\varphi) = 0$ for each function φ that does not depend on x_p, then

$$
\|\mathcal{L}^{n+m}\mu\| \leq A\theta^n\|\mathcal{L}^m\mu\| + B|\mathcal{L}^m\mu| \leq C(\theta^n + m\varepsilon)\|\mu\| + B|\mathcal{L}_p^m\mu|.
$$

Then, if h is the invariant density of the single site map,

$$\begin{aligned}
\mathcal{L}_p^m\mu(\varphi) &= \mu(\varphi\circ(\Phi_{\varepsilon,p}\circ F_0)^m)\\
&= \int_\Omega \left[\varphi((\Phi_{\varepsilon,p}\circ F_0)^m(x)) - \int_0^1 dx_p h(x_p)\varphi((\Phi_{\varepsilon,p}\circ F_0)^m(x))\right]\mu(dx)\\
&= \int_\Omega \partial_{x_p}\int_0^{x_p} dx_p \left[\varphi((\Phi_{\varepsilon,p}\circ F_0)^m(x)) - \int_0^1 dx_p h(x_p)\varphi((\Phi_{\varepsilon,p}\circ F_0)^m(x))\right]\mu(dx)\\
&\leq \|\mu\|\sup_{x_{\neq p}}\int_0^1 dy\mathbb{1}_{[0,x_p]}(y)\left[\varphi(x_{\neq p},T^m y) - \int_0^1 dz h(z)\varphi(x_{\neq p},z)\right]\\
&\leq C\nu^n\|\mu\|\cdot|\varphi|_\infty,
\end{aligned}$$

where ν is the rate of decay for the single site map. Putting the above estimates together yields

$$\|\mathcal{L}^{n+m}\mu\| \leq C(\theta^n + m\varepsilon + \nu^m)\|\mu\| \leq \sigma^{n+m}\|\mu\|,$$

for some $\sigma\in(0,1)$, provided we choose n, m, ε appropriately.

So, let $\mathcal{B}_p = \{\mu\in\mathcal{B} \;:\; \mu(\varphi) = 0 \text{ for all } \varphi \text{ independent of } p\}$. The situation looks good but there are two problem

1. in general $\mu\in\mathcal{B}$ does not belong to $\mathcal{B}_p$ for any p.
2. $\mu\in\mathcal{B}_p \not\Longrightarrow \mathcal{L}\mu\in\mathcal{B}_p$.

No problem: first show that each $\mu\in\mathcal{B}$ can be decomposed as

$$\mu = cm + \sum_{p\in\mathbb{Z}^d}\mu_p$$

where $m\in\mathcal{B}$ is a fixed probability measure and $\mu_p\in\mathcal{B}_p$. Then, for each $\mu_p\in\mathcal{B}_p$, write

$$\mathcal{L}\mu_p = \mathcal{L}_p\mu_p + \varepsilon\sum_{|q-p|\leq 1}\mathcal{L}_{q,p}\mu_p$$

where $\mathcal{L}_p\mathcal{B}_p\subset\mathcal{B}_p$ and $\mathcal{L}_{q,p}\mathcal{B}_p\subset\mathcal{B}_q$ and the operators have all uniformly bounded norm. Only a seemingly catastrophic problem is left: the decomposition sum does not converge in the $|\cdot|$ topology (let alone the $\|\cdot\|$ one).

No problem: let us associate to each measure μ the vector (c,μ_p) given by the terms of its decomposition (this means that one introduces the new super-abstract Banach space $\bar{\mathcal{B}} = \mathbb{C}\times(\times_{p\in\mathbb{Z}^d}\mathcal{B}_p)$ with norm $\|(c,\mu_p)\| := \max\{|c|, \sup_{p\in\mathbb{Z}^d}\|\mu\|_p\}$) and the operator

$$\overline{\mathcal{L}}(c,\mu_p) = (c, \mathcal{L}_p\mu_p + \varepsilon\sum_{|q-p|\leq 1}\mathcal{L}_{p,q}\mu_q + \zeta_p) =: (c, \mathcal{L}_*(\mu_p) + \bar\zeta),$$

where ζ_p is the decomposition of $\mathcal{L}m - m$. By applying the previous estimates one has that $\|\mathcal{L}_*\| < 1$. Is that good for something?

Well, $(1,\bar\mu) = (1, \mathcal{L}_*\bar\mu + \bar\zeta)$ has the unique solution $\bar\mu^* := (\mathbb{1} - \mathcal{L}_*)^{-1}\bar\zeta$. Let φ be a local function that depends only the variables in the finite set $\Lambda\subset\mathbb{Z}^d$ and $\mu\in\mathcal{B}$ a probability measure with decomposition $(1,\bar\mu)$, then

$$\mu(\varphi\circ F_\varepsilon^n) = m(\varphi) + \sum_{p\in\Lambda}\left(\mathcal{L}_*^n\bar\mu + \sum_{k=0}^{n-1}\mathcal{L}_*^k\bar\zeta\right)_p(\varphi) = \sum_{p\in\Lambda}\mu_p^*(\varphi) + \mathcal{O}(|\Lambda|\|\mathcal{L}_*\|^n).$$

By weak compactness and the Lasota-Yorke inequality we know that $\frac{1}{n}\sum_{k=0}^{n-1}\mathcal{L}^k\mu$ has accumulation points in $\mathcal{B}$, let μ_* be one such accumulation point, then

$$\mu_*(\varphi) = \sum_{p\in\Lambda}\mu_p^*(\varphi)$$

Invariance, uniqueness and spatio-temporal decay of correlation for μ_* readily follow.

Appendix A. Measure with 'densities' of bounded variation

Proof of Lemma 1.5. Let $\varphi \in \mathcal{C}^0(X,\mathbb{R})$, then for each $\varepsilon \in (0,1)$ there exists $\varphi_\varepsilon \in \mathcal{C}^1(\mathbb{R}^d,\mathbb{R})$, supported in $[-\varepsilon, 1+\varepsilon]^d$, such that $|\varphi-\varphi_\varepsilon|_{\mathcal{C}^0(X,\mathbb{R})} \le \varepsilon$, $|\varphi_\varepsilon|_\infty \le |\varphi|_\infty(1+\varepsilon)$. In addition, if we define

$$\Gamma(\xi) := \begin{cases} -\frac{1}{2}\|\xi\| & \text{if } d=1 \\ -\frac{1}{2\pi}\ln\|\xi\| & \text{if } d=2 \\ \frac{1}{d(d-2)\alpha_d\|\xi\|^{d-2}} & \text{if } d\ge 3, \end{cases} \tag{A.1}$$

where α_d is the d-dimensional volume of the unit ball in $\mathbb{R}^d$, we can define the Newtonian potential $w_\varepsilon(x) = \int_{\mathbb{R}^d}\Gamma(x-z)\varphi_\varepsilon(z)dz$. It is then well know from potential theory that $\Delta w_\varepsilon = \varphi_\varepsilon$, thus

$$\begin{aligned} \mu(\varphi) &\le \mu(\varphi_\varepsilon) + |\mu|\varepsilon = \sum_{k=1}^d \mu(\partial_{x_k}\partial_{x_k}w_\varepsilon) + |\mu|\varepsilon \\ &\le \sum_{k=1}^d \|\mu\| \sup_{x\in X}\int |\partial_{x_k}\Gamma(x-z)\varphi_\varepsilon(z)dz| + |\mu|\varepsilon \\ &\le C\sum_{k=1}^d \|\mu\|\,|\varphi_\varepsilon|_{L^q}\left[\int_{[-1,2]^d}\frac{|x_k-z_k|^p}{\|x-z\|^{dp}}dz\right]^{\frac{1}{p}} + |\mu|\varepsilon, \end{aligned}$$

where $q^{-1}+p^{-1}=1$. Since the integral in square brackets is finite for $p < \frac{d}{d-1}$, we have, be the arbitrariness of ε,

$$\mu(\varphi) \le C(\|\mu\|+|\mu|)|\varphi|_{L^q}.$$

This means that the linear functional $\mu : \mathcal{C}^0 \to \mathbb{R}$ can be extended to a bounded functional on L^q. Since the dual of L^q is L^p it follows that there exists $h \in L^p$ such that $\mu(\varphi) = \int_X h(x)\varphi(x)dx$. □

Proof of Lemma 1.7. First,

$$\|\mu\| \le \sup_k \sup_{|\varphi|_\infty\le 1}\int h\partial_{x_k}\varphi = \sup_k \sup_{|\varphi|_\infty \le 1}\int \operatorname{Var}^k h \sup_{x_k}|\varphi| \le \sup_k |\operatorname{Var}^k(h)|_{L^1}.$$

For the opposite inequality one need a bit of preparation.

For each $n\in\mathbb{N}$ and function $\eta \in \mathcal{C}^2_0([-1,1]^n,\mathbb{R}_+)$, $\int\eta = 1$, let us define $\eta_\varepsilon(x) = \varepsilon^{-n}\eta(\varepsilon^{-1}x)$. Then, for each $h \in L^1([0,1]^n, m)$ and $\varphi \in \mathcal{C}^1_0(\mathbb{R}^n,\mathbb{R})$ let $h_\varepsilon(x) = \int dz\, h(z)\,\eta_\varepsilon(x-z)$,

$$\begin{aligned} \int \partial_{x_k}h_\varepsilon(x)\cdot\varphi(x) &= \int h(z)\partial_{x_k}\eta_\varepsilon(x-z)\cdot\varphi(x) \\ &= -\int h(z)\partial_{z_k}\eta_\varepsilon(x-z)\cdot\varphi(x) \le |h|_{BV}|\varphi|_\infty. \end{aligned} \tag{A.2}$$

That is $\sup_k|\partial_{x_k}h_\varepsilon|_{L^1} \le |h|_{BV}$. On the other hand, for each $\delta > 0$ and $k \in \{1,\dots,d\}$ there exists $\phi\in\mathcal{C}^1$, $|\phi|_\infty = 1$, such that $|h|_{BV} \le \int h\partial_{x_k}\phi + \delta$. Next, consider a compact support extension $\tilde\phi \in \mathcal{C}^1_0$ of ϕ on all $\mathbb{R}^n$ such that $|\tilde\phi|_\infty \le 1+\delta$ and choose $\varepsilon_0 > 0$ such that, for all $\varepsilon < \varepsilon_0$,

$$\sup_{x\in[0,1]^n}\left|\partial_{x_k}\phi(x) - \int_{\mathbb{R}^n}\eta_\varepsilon(x-z)\partial_{z_k}\tilde\phi(z)dz\right| \le \delta|\mu|^{-1}.$$

Hence,

$$|h|_{BV} \le \int h_\varepsilon \partial_{x_k} \tilde\phi + 2\delta = -\int \partial_{x_k} h_\varepsilon \tilde\phi + 2\delta \le |\partial_{x_k} h_\varepsilon|_{L^1}(1+\delta) + 2\delta.$$

Thus, by the arbitrariness of δ,

$$\liminf_{\varepsilon\to 0} \sup_k |\partial_{x_k} h_\varepsilon|_{L^1} = |h|_{BV}. \tag{A.3}$$

Finally, let $\tilde\eta : \mathbb{R} \to \mathbb{R}_+$ and $\eta_\varepsilon(x) = \varepsilon^{-1}\tilde\eta(\varepsilon^{-1}x_k)$, using first (A.3) for $n = 1$, then Fatu and finally arguing as in (A.2),

$$\begin{aligned} |\operatorname{Var}^k(h)|_{L^1} &= \int dx_1 \cdots dx_{k-1} dx_{k+1} \cdots dx_d \operatorname{Var}^k h(x) \\ &= \int dx_1 \cdots dx_{k-1} dx_{k+1} \cdots dx_n \liminf_{\varepsilon\to 0} \int dx_k |\partial_{x_k} h_\varepsilon(x)| \\ &\le \liminf_{\varepsilon\to 0} |\partial_{x_k} h_\varepsilon|_{L^1} \le \liminf_{\varepsilon\to 0} \sup_{\substack{\varphi\in\mathcal{C}^1 \\ |\varphi|_\infty \le 1}} \int h(x)\partial_{x_k}\varphi_\varepsilon(x) \le |h|_{BV}. \end{aligned}$$

□

Proof of Lemma 1.8. For each $t \in \mathbb{N}$, let us consider a partition $\{A_j\}$ of $[0,1]$ in intervals of size t^{-1} and, for each $k \in \{1,\dots,d\}$, define

$$\begin{aligned} &P_{t,k}\varphi(x) = t\sum_j \mathbb{1}_{A_j}(x_k) \int_{A_j} dz \varphi(x_1,\dots,x_{k-1},z,x_{k+1},\dots,x_d) \\ &P_t\varphi = P_{t,1}\cdots P_{t,d}\varphi. \end{aligned} \tag{A.4}$$

First of all note that

$$P'_{t,k}\mu(\varphi) = \mu(P_{t,k}\varphi) = \int hP_{t,k}\varphi = \int P_{t,k}h\cdot\varphi.$$

Next, if $j \ne k$

$$P'_{t,k}\mu(\partial_{x_j}\varphi) = \int hP_{t,k}\partial_{x_j}\varphi = \int h\partial_{x_j}P_{t,k}\varphi \le \|\mu\|.$$

and

$$P'_{t,k}\mu(\partial_{x_k}\varphi) = \int hP_{t,k}\partial_{x_k}\varphi = \|\mu\| \left| \int_0^{x_k} dx_k P_{t,k}\partial_{x_k}\varphi \right|_\infty \le 4\|\mu\|.$$

In addition,

$$\mu(P_{i,k}\varphi - \varphi) = \|\mu\| \left| \int_0^{x_k} dx_k(P_{t,k}\varphi - \varphi) \right|_\infty.$$

If $x_k \in A_j = [jt^{-1},(j+1)t^{-1}]$, then

$$\int_0^{x_k} dx_k(P_{t,k}\varphi - \varphi) = \int_{jt^{-1}}^{x_k} \varphi \le |\varphi|_\infty t^{-1}.$$

Accordingly, $\|P'_t\mu\| \le 4^d\|\mu\|$ and $|P'_t\mu - \mu| \le 4^{d+1}t^{-1}$. In addition, notice that $P'_t\mu = t^d\sum_{i_1,\dots,i_d}\mu(\mathbb{1}_{A_{i_1}}\cdots\mathbb{1}_{A_{i_d}})m_{A_1\times\cdots\times A_{i_d}}$, where $t^{-d}m_{A_1\times\cdots\times A_{i_d}}$ is the Lebesgue measure restricted to the set $A_1\times\cdots\times A_{i_d}$. In other words the range of P'_t is a finite dimensional space. This implies that if $\{\mu_j\} \subset B$, then $\{P'_t\mu_j\}$ lives in a finite dimensional bounded set, hence it is compact. Thus there exists μ_t and n_j such that $\lim_{j\to\infty}\|P'_t\mu_{n_j} - \mu_t\| = 0$. In addition, for $t' \ge t$,

$$|\mu_t - \mu_{t'}| \le |\mu_t - P'_t\mu_{n_j}| + |\mu_t - P'_{t'}\mu_{n_j}| + |P'_t\mu_{n_j} - P'_{t'}\mu_{n_j}| \le Ct^{-1}$$

provided one choses j large enough. It follows that there exists a sequence t_j and a measure μ such that $\lim_{j\to\infty} |\mu - P_{t_j}\mu_{n_j}| = 0$. $\square$

Appendix B. Some technical facts

The first assertion follows directly from Lemma 1.9. For the second we need a well known result.

Theorem B.1 (Analytic Fredholm theorem–finite rank)**.** *Let D be an open connected subset of $\mathbb{C}$ and $\mathcal{B}$ a Banach space. Let $F : \mathbb{C} \to L(\mathcal{B}, \mathcal{B})$ be an analytic operator-valued function such that $F(z)$ is finite rank for each $z \in D$. Then, one of the following two alternatives holds true*

- *$(\mathbb{1} - F(z))^{-1}$ exists for no $z \in D$*
- *$(\mathbb{1} - F(z))^{-1}$ exists for all $z \in D \backslash S$ where S is a discrete subset of D (i.e. S has no limit points in D). In addition, if $z \in S$, then 1 is an eigenvalue for $F(z)$ and the associated eigenspace has finite multiplicity.*

The proof is the same as for the Analytic Fredholm alternative for compact operators in Hilbert spaces given in [63, Theorem VI.14] (since compact operators in Hilbert spaces can always be approximated by finite rank ones). In fact the theorem holds also for compact operators in Banach spaces but the proof is a bit more involved.

Let $T'_{n,t} := (T')^n P_t$, clearly such an operator is finite rank, in addition

$$\|(T')^n \mu - T'_{n,t}\mu\| \le \sigma^n \|(\mathbb{1} - P_t)\mu\| + B|(\mathbb{1} - P_t)\mu| \le (1+4)\sigma^n\lambda^{-n}\|\mu\| + Bt^{-1}\|\mu\|.$$

By choosing $t = \sigma^n$ we have that there exists $C_1 > 0$ such that

$$\|(T')^n - T'_{n,t}\| \le C_1\sigma^n.$$

For each $z \in \mathbb{C}$ we can now write

$$\mathbb{1} - z(T')^n = (\mathbb{1} - z((T')^n - T'_{n,t})) - zT'_{n,t}.$$

Since

$$\|z((T')^n - T'_{n,t})\| \le |z|C_1\sigma^n < \frac{1}{2},$$

provided that $|z| \le \frac{1}{2C_1}\sigma^{-n}$. Given any z in the disk $D_n := \{|z| < \frac{1}{2C_1}\sigma^{-n}\}$ the operator $B(z) := \mathbb{1} - z((T')^n - T'_{n,t})$ is invertible.[14] Hence

$$\mathbb{1} - z(T')^n = \left(\mathbb{1} - zT'_{n,t}B(z)^{-1}\right)B(z) =: (\mathbb{1} - F(z))B(z).$$

By applying Theorem B.1 to $F(z)$ we have that the operator is either never invertible or not invertible only in finitely many points in the disk D_n. Since for $|z| < 1$ we have $(\mathbb{1} - z(T')^n)^{-1} = \sum_{k=0}^{\infty} z^k (T')^{nk}$, the first alternative cannot hold hence the Theorem follows.

[14] Clearly $B(z)^{-1} = \sum_{k=0}^{\infty}\left[z((T')^n - T'_{n,t})\right]^k$.

Appendix C. On the peripheral spectrum of the transfer operator

It is then natural to start looking at the eigenvalues of modulus one. By Lemma 1.10 and the usual fact about the spectral decomposition of the operators [40], follows that there exists a finite set $\Theta \subset [0, 2\pi)$ such that we can write[15]

$$T' = \sum_{\theta \in \Theta} e^{i\theta}\Pi_\theta + R$$

where Π_θ are finite rank operators and the spectral radius of R is strictly smaller than one. Moreover, $\Pi_\theta \Pi_{\theta'} = \delta_{\theta\theta'}\Pi_\theta$, $\Pi_\theta R = R\Pi_\theta = 0$. It follows that, for each $\theta \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} e^{-ik\theta}(T')^k = \begin{cases} \Pi_\theta & \text{if } \theta \in \Theta \\ 0 & \text{otherwise.} \end{cases}$$

Also, by Lemma 1.9 follows $\|\Pi_\theta \mu\| \le C|\mu|$. Since Π_θ is a finite rank projector, there must exist $\mu_{\theta,l} \in \mathcal{B}$, $\ell_{\theta,l} \in \mathcal{B}'$ such that $\Pi_\theta = \sum_l \mu_{\theta,l} \otimes \ell_{\theta,l}$, moreover $T'\mu_{\theta,l} = e^{i\theta}\mu_{\theta,l}$ and $\ell_{\theta,l}(T'\mu) = e^{i\theta}\ell_{\theta,l}(\mu)$ for all $\mu \in \mathcal{B}$. Hence, it must be $|\ell_{\theta,l}(\mu)| \le C|\mu| = C\int |h_\mu| dm$. Since $L^\infty(X, m)$ is the dual of L^1, it follows that there exists $\bar{\ell}_{\theta,l} \in L^\infty(X, m)$ such that

$$\ell_{\theta,l}(\mu) = \int \bar{\ell}_{\theta,l} h_\mu = \mu(\bar{\ell}_{\theta,l}).$$

Hence, for each $\mu \in \mathcal{B}$,

$$\mu(\bar{\ell}_{\theta,l}) = \ell_{\theta,l}(\mu) = e^{-i\theta}\ell_{\theta,l}(T'\mu) = e^{-i\theta}T'\mu(\bar{\ell}_{\theta,l}) = e^{-i\theta}\mu(\bar{\ell}_{\theta,l} \circ T).$$

The above implies that $\bar{\ell}_{\theta,l} \circ T = e^{-i\theta}\bar{\ell}_{\theta,l}$ Lebesgue a.s.. Let us set $\mu_* := \Pi_0 m$.

Lemma C.1. *For each $\ell \in L^\infty(X, m)$ such that $\ell \circ T = \ell$, m-a.s., if we define the measure $\mu(\varphi) := \mu_*(\ell\varphi)$, then μ is invariant and $\mu \in \mathcal{B}$.*

Proof. First of all notice that $T'\mu(\varphi) = \mu_*(\ell \cdot \varphi \circ T) = \mu_*((\ell\varphi) \circ T) = \mu_*(\ell\varphi) = \mu(\varphi)$, that is μ is an invariant measure. Next, for each $\varepsilon > 0$ there exists $\ell_\varepsilon \in L^\infty$ such that $|\ell_\varepsilon|_\infty \le 2|\ell|_\infty$ and $\mu_*(|\ell - \ell_\varepsilon|) + m(|\ell - \ell_\varepsilon|) \le \varepsilon$. Then, setting $\mu_\varepsilon(\varphi) := \mu_*(\ell_\varepsilon\varphi)$

$$|(T')^n\mu(\varphi) - (T')^n\mu_\varepsilon(\varphi)| \le \varepsilon|\varphi|_\infty$$

implies

$$|\Pi_0\mu_\varepsilon - \mu| \le \limsup_{n\to\infty} \left| \frac{1}{n}\sum_{k=0}^{n-1} e^{-ik\theta}(T')^k(\mu_\varepsilon - \mu) \right| \le \varepsilon$$

Hence, for each $\varphi \in \mathcal{C}^1$, $|\varphi|_\infty \le 1$,

$$\mu(\partial_{x_k}\varphi) = \lim_{\varepsilon\to 0} \Pi_0\mu_\varepsilon(\partial_{x_k}\varphi) \le \lim_{\varepsilon\to 0} \|\Pi_0\mu_\varepsilon\| \le C\lim_{\varepsilon\to 0} |\mu_\varepsilon| \le C.$$

□

Thus, for each $p \in \mathbb{N}$ and $\theta \in \Theta$, the measure $\mu_{p,\theta}(\varphi) := \mu_*(\bar{\ell}^p_{\theta,i}\varphi)$ is in $\mathcal{B}$ and $T'\mu_{p,\theta} = e^{ip\theta}\mu_{p,\theta}$. But this implies that $\{p\theta\}_{p\in\mathbb{N}} \subset \sigma_{\mathcal{B}}(T') \cap \{|z| = 1\}$ and since the latter is finite it must be $\theta = 2\pi\frac{s}{t}$ for some $s, t \in \mathbb{N}$. We have just proven the following

Lemma C.2. *The peripheral spectrum of T', $\sigma_{\mathcal{B}}(T') \cap \{|z| = 1\}$, is the fine union of cyclic groups.*

---

[15] Remark that there cannot be Jordan blocks with eigenvector of modulus one, since this would imply that $\|(T')^n\|$ grows polynomially, contrary to Lemma 1.9.

C.1. **Dynamical properties.**

Lemma C.3. *If the map T is topologically transitive then 1 is a simple eigenvalue for T'. If all the powers of T are topologically transitive, then $\{1\}$ is the all peripheral spectrum.*

Proof. We do the proof only for $d = 1$, as in higher dimension it is more complex (see footnote below). If one it is not simple, then there exists an invariant set A, $\mu_*(A) \notin \{0, 1\}$. But then $\mathbb{1}_A \in BV$ which implies that A contains an open set, the same applies to A^c (this is true only for $d = 1$).[16] But then, by topological transitivity, there is an orbit that visits such opens sets, hence the sets are not invariant. The same argument applied to T^n concludes the Lemma. □

In conclusion, we have obtained conditions under which the system has a unique invariant measure μ_* absolutely continuos w.r.t. Lebesgue. In addition, there exists $\rho > 0$ such that for each $\mu \in \mathcal{B}$ we have

$$\|(T')^n\mu - \mu_*\| \leq C\|\mu\|e^{-\rho n}.$$

C.2. **Birkhoff averages.** From now on we assume that one is simple and is the only eigenvalue of modulus one. Let $f \in L^\infty(X, m)$, and let $\hat{f} = f - \mu_*(f)$, then

$$m(\hat{f}_n^2) = \frac{1}{n^2}\left[\sum_{k=0}^{n-1} m(\hat{f}^2 \circ T^k) + 2\sum_{j>k=0}^{n-1} m(\hat{f} \circ T^j \hat{f} \circ T^k)\right] \leq Cn^{-1}|f|_\infty.$$

By Chebyshev inequality, we have

$$m(\{x \;:\; |\hat{f}_n| \leq L^{-1}\}) \leq C\frac{L^2}{n}.$$

The above, by Borel-Cantelli, implies[17]

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k(x) = \mu_*(f) \quad m\text{-almost surely.}$$

That is μ_* is a physical measure (also SRB) and the unique one. In fact one can obtain much sharper results on the behavior of the $\hat{f}_n$ (large deviaitons).

---

[16] In higher dimensions one can have a Cantor like set with characteristic function in BV. Hence one must either use a different functional space (a convenient one in this respect has been introduced in [66]) or use explicitly the dynamics: for example note the one can easily bound the ε neighborhood of the boundary of the partition and that, by a commonly used argument, implies that there is a large measure of point with an open neighborhood whose preimages are all away from singularities. One can then proceed to prove that on such open sets the density must be continuos showing that any invariant set must contain an open set.

[17] Actually one must apply Borel-Cantelli with some care (but this is a quite standard an general strategy):

Consider the set $\mathcal{N} := \{4^k + j2^k \;:\; k \in \mathbb{N}\; j < 3 \cdot 2^k\}$, then

$$\sum_{l\in\mathcal{N}} m(\{x \;:\; |\hat{f}_l| \leq L^{-1}\}) \leq CL^2\sum_{k=0}^{\infty}\sum_{j=0}^{3\cdot 2^k} 4^{-k} \leq CL^2\sum_{k=0}^{\infty} 3 \cdot 2^{-k} < \infty.$$

Hence Borel-Cantelli imply that every infinite sequence in $\mathcal{N}$ converges. Next notice that

$$|\hat{f}_n - \hat{f}_{n+m}| \leq |f|_\infty \frac{m}{n}$$

which readily imply the wanted result.

References

[1] Baladi, Viviane, *Positive transfer operators and decay of correlations.* Advanced Series in Nonlinear Dynamics, **16**. World Scientific Publishing Co., Inc., River Edge, NJ, 2000.

[2] Viviane Baladi. Anisotropic Sobolev spaces and dynamical transfer operators: C^∞ foliations. In *Algebraic and topological dynamics*, volume 385 of *Contemp. Math.*, pages 123–135. Amer. Math. Soc., Providence, RI, 2005, MR2180233, Zbl pre02236663.

[3] V. Baladi, M. Degli Eposti, S. Isola, E. Järvenpää, A. Kupiainen, *The spectrum of weakly coupled map lattices,* J. Math. Pures Appl. **77** (1998), 539–584.

[4] V. Baladi, H.H Rugh, *Floquet spectrum of weakly coupled map lattices,* Commun. Math. Phys.**220** (2001), 561–582.

[5] V.Baladi, L.-S.Young, *On the spectra of randomly perturbed expanding maps*, Comm. Math. Phys., **156**:2 (1993), 355-385; **166**:1 (1994), 219–220.

[6] V. Baladi, M. Tsujii, *Anisotropic Hölder and Sobolev spaces for hyperbolic diffeomorphisms*, Ann. Inst. Fourier, **57** no. 1 (2007), p. 127-154.

[7] Baladi, Viviane; Tsujii, Masato, *Dynamical determinants and spectrum for hyperbolic diffeomorphisms.* Geometric and probabilistic structures in dynamics, 29–68, Contemp. Math., **469**, Amer. Math. Soc., Providence, RI, 2008.

[8] Baladi, Viviane; Gouezel, Sebastien, *Banach spaces for piecewise cone hyperbolic maps*, preprint arXiv:0907.1402 .

[9] Jean-Baptiste Bardet, G. Keller, *Phase transitions in a piecewise expanding coupled map lattice with linear nearest neighbour coupling*, Nonlinearity **19** (2006), 2193-2210.

[10] Michael Blank, Gerhard Keller, and Carlangelo Liverani. Ruelle-Perron-Frobenius spectrum for Anosov maps. *Nonlinearity*, 15:1905–1973, 2002, MR1938476, Zbl 1021.37015.

[11] J. Bricmont, A. Kupiainen, *Coupled analytic maps*, Nonlinearity **8** (1995), no. 3, 379–396.

[12] J. Bricmont, A. Kupiainen, *High temperature expansions and dynamical systems*, Comm. Math. Phys. **178** (1996), no. 3, 703–732.

[13] L.A. Bunimovich, Ya.G. Sinai, *Space-time chaos in coupled map lattices*, Nonlinearity **1** (1988), 491–516.

[14] J. Buzzi, *Absolutely continuous invariant probability measures for arbitrary expanding piecewise R- analytic mappings of the plane*, Ergodic Theory and Dynamical Systems **20** (2000), 697–708.

[15] Buzzi, Jérôme; Maume-Deschamps, Véronique, *Decay of correlations for piecewise invertible maps in higher dimensions*, Israel J. Math. **131** (2002), 203–220.

[16] Chernov, Nikolai; Markarian, Roberto *Chaotic billiards.* Mathematical Surveys and Monographs, **127**. American Mathematical Society, Providence, RI, 2006.

[17] Cowieson, William J. *Stochastic stability for piecewise expanding maps in* R^d, Nonlinearity **13** (2000), no. 5, 1745–1760.

[18] Cowieson, William J., *Absolutely continuous invariant measures for most piecewise smooth expanding maps.* Ergodic Theory Dynam. Systems **22** (2002), no. 4, 1061–1078.

[19] P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner and G. Vattay, *Chaos: Classical and Quantum*, `ChaosBook.org` (Niels Bohr Institute, Copenhagen 2003).

[20] Demers, Mark F.; Liverani, Carlangelo, *Stability of statistical properties in two-dimensional piecewise hyperbolic maps.* Trans. Amer. Math. Soc. **360** (2008), no. 9, 4777–4814.

[21] Dolgopyat, Dmitry, *On dynamics of mostly contracting diffeomorphisms.* Comm. Math. Phys. **213** (2000), no. 1, 181–201.

[22] D.Dolgopyat, C.Liverani, *Random Walk in Deterministically Changing Environment*, ALEA **4**, 89-116 (2008).

[23] P. Ferrero, B. Schmitt, *Ruelle's Perron-Frobenius Theorem and Projective Metrics*, Colloquia Mathematica Societas János Bolyai **27** Random Fields, Esztergom (Hungary) (1979)

[24] T. Fischer, H.H. Rugh, *Transfer operators for coupled analytic maps,* Ergod. Th.& Dynam. Sys. **20** (2000), 109–143.

[25] G. Gielis, R.S. MacKay, *Coupled map lattices with phase transition,* Nonlinearity **13** (2000), 867–888.

[26] S.Gouezel, C.Liverani *Banach Spaces adapted to Anosov Systems*, Ergodic Theory and Dynamical Systems, **26**, 1, 189–217, (2006).

[27] Gouezel, Sebastien; Liverani, Carlangelo, *Compact locally maximal hyperbolic sets for smooth maps: fine statistical properties.* J. Differential Geom. **79** (2008), no. 3, 433–477.

[28] V.M. Gundlach, D.A. Rand, *Spatio-temporal chaos: 1. Hyperbolicity, structural stability, spatio-temporal shadowing and symbolic dynamics,* Nonlinearity **6** (1991), 165–200.
[29] V.M. Gundlach, D.A. Rand, *Spatio-temporal chaos: 2. Unique Gibbs states for higher-dimensional symbolic systems,* Nonlinearity **6** (1993), 201–214.
[30] V.M. Gundlach, D.A. Rand, *Spatio-temporal chaos: 3. Natural spatio-temporal measures for coupled circle map lattices,* Nonlinearity **6** (1993), 215–230.
[31] V.M. Gundlach, D.A. Rand, *Spatio-temporal chaos (Corrigendum),* Nonlinearity **9** (1996), 605–606.
[32] H.Hennion, *Sur un théorème spectral et son application aux noyaux Lipchitziens*, Proceedings of the American Mathematical Society, **118** (1993), 627–634.
[33] E. Järvenpää, *A note on weakly coupled expanding maps on compact manifolds,* Annales Academiæ Scientiarum Fennicæ Mathematica **24** (1999), 511-517.
[34] E. Järvenpää, M. Järvenpää, *On the definition of SRB-measures for coupled map lattices*, Comm. Math. Phys. **220** (2001), no. 1, 1–12.
[35] M. Jiang, *Equilibrium states for lattice models of hyperbolic type*, Nonlinearity **8** (1995), no. 5, 631–659.
[36] M. Jiang, *Equilibrium measures for coupled map lattices: existence, uniqueness and finite-dimensional approximations,* commun. Math. Phys. **193** (1998), 675-712.
[37] M. Jiang, *Sinai-Ruelle-Bowen measures for lattice dynamical systems*, J. Statist. Phys. **111** (2003), no. 3-4, 863–902.
[38] M. Jiang, A.E. Mazel, *Uniqueness and exponential decay of correlations for some two-dimensional spin lattice systems*, J. Statist. Phys. **82** (1996), no. 3-4, 797–821.
[39] M. Jiang, Ya. Pesin, *Equilibrium measures for coupled map lattices: existence, uniqueness and finite-dimensional approximations*, Comm. Math. Phys. **193** (1998), no. 3, 675–711.
[40] T.Kato, *Perturbation theory fro linear operators*, Springer, New York (1966).
[41] G.Keller, *Stochastic stability in some chaotic Dynamical Systems,* Monatshefte Math. **94** (1982), 313-333.
[42] G.Keller, C.Liverani, *Stability of the spectrum for transfer operators*, Annali della Scuola Normale Superiore di Pisa, Scienze Fisiche e Matematiche, (4) **XXVIII** (1999), 141-152.
[43] G. Keller, *Coupled map lattice via transfer operators on functions of bounded variation*, Stochastic and spatial structures of dynamical systems (Amsterdam, 1995), 71–80, Konink. Nederl. Akad. Wetensch. Verh. Afd. Natuurk. Eerste Reeks **45** North-Holland, Amsterdam, 1996.
[44] G. Keller, *Mixing for finite systems of coupled tent maps*, Tr. Mat. Inst. Steklova **216** (1997), Din. Sist. i Smezhnye Vopr., 320–326; translation in Proc. Steklov Inst. Math. 1997, no. 1 (216), 315–32
[45] G. Keller, *An ergodic theoretic approach to mean field coupled maps*, Fractal geometry and stochastics, II (Greifswald/Koserow, 1998), 183–208, Progr. Probab., **46**, Birkhäuser, Basel, 2000.
[46] G. Keller, M.Künzle, *Transfer operators for coupled map lattices*, Ergodic Theory Dynam. Systems **12** (1992), no. 2, 297–318.
[47] G. Keller, C. Liverani, *Coupled map lattices without cluster expansion*, Discrete and Continuous Dynamical Systems, 11, n.2,3, 325–335 (2004).
[48] G.Keller, C.Liverani, *A spectral gap for a one-dimensional lattice of coupled piecewise expanding interval maps*, to appear in Lecture Notes in Physics, Springer.
[49] G.Keller, C.Liverani, *Uniqueness of the SRB measure for piecewise expanding weakly coupled map lattices in any dimension*, Communications in Mathematical Physics, **262**, 1, 33–50, (2006).
[50] G. Keller, R. Zweimüller, *Unidirectionally coupled interval maps: between dynamics and statistical mechanics*, Nonlinearity **15** (2002), no. 1, 1–24.
[51] Yu.Kifer, *Random Perturbations of Dynamical Systems*, Progress in Probability and Statistics 16, Birkhäuser, Boston (1988).
[52] M. Künzle: Invariante Maße für gekoppelte Abbildungsgitter, Dissertation, Universität Erlangen (1993)
[53] C.Liverani, *Decay of Correlations*, Annals of Mathematics, **142**, pp. 239-301, (1995).
[54] C.Liverani, *Decay of Correlations in Piecewise Expanding maps*, Journal of Statistical Physics, **78**, 3/4, pp. 1111-1129, (1995).

[55] C.Liverani, *Rigorous numerical investigation of the statistical properties of piecewise expanding maps–A feasibility study*, Nonlinearity, 14, n. 3, pp. 463–490, (2001).
[56] C.Liverani, V. Maume-Deschamps, *Lasota-Yorke maps with holes: conditionally invariant probability measures and invariant probability measures on the survivor set*, **39** (3), 385-412 (2003).
[57] Carlangelo Liverani. Fredholm determinants, Anosov maps and Ruelle resonances. *Discrete Contin. Dyn. Syst.*, 13(5):1203–1215, 2005, MR2166265, Zbl pre02234438.
[58] Liverani, Carlangelo, *On contact Anosov flows.* Ann. of Math. (2) **159** (2004), no. 3, 1275–1312.
[59] Carlangelo Liverani and Masato Tsujii. Zeta functions and dynamical systems. *Nonlinearity*, 19:2467–2473, 2006, MR2260273.
[60] Ch. Maes, A. van Moffaert, *Stochastic stability of weakly coupled map lattices,* Nonlinearity **10** (1997), 715–730.
[61] J. Miller, D.A. Huse, *Macroscopic equilibrium from microscopic irreversibility in a chaotic coupled-map lattice*, Pys. Rev. E, **48**, 2528–2535 (1993).
[62] Ya.B. Pesin, Ya.G. Sinai, *Space-time chaos in chains of weakly interacting hyperbolic mappings*, adv.Sov.Math. **3** (1991), 165-198.
[63] M.Reed, B.Simon, *Methods of Modern Mathematical Physics. I-Functional Analysis*, Academic Press, New York (1980).
[64] H.H. Rugh, *Coupled maps and analytic function spaces*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 4, 489–535.
[65] Pollicott, Mark; Jenkinson, Oliver, *Computing invariant densities and metric entropy*, Comm. Math. Phys. **211** (2000), no. 3, 687–703.
[66] Saussol, Benoit, *Absolutely continuous invariant measures for multidimensional expanding maps*, Israel J. Math. **116** (2000), 223–248.
[67] M. Schmitt: BV-spectral theory for coupled map lattices, Dissertation, Universität Erlangen (2003). See also: Nonlinearity **17**, 671–690 (2004).
[68] M. Tsujii, *Absolutely continuous invariant measures for piecewise real-analytic expanding maps on the plane*, Communications in Mathematical Physics **208** (2000), 605–622.
[69] M. Tsujii, *Absolutely continuous invariant measures for expanding piecewise linear maps, Inventiones Mathematicae*, **143** (2001), 349–373.
[70] M. Tsujii, *Piecewise Expanding maps on the plane with singular ergodic properties*, Ergodic Theory and Dynamical Systems **20** (2000), 495–501.
[71] M. Tsujii, *Quasi-compactness of transfer operators for contact Anosov flows*, preprint.
[72] Varadhan, S. R. S. *Nonlinear diffusion limit for a system with nearest neighbor interactions. II.* Asymptotic problems in probability theory: stochastic models and diffusions on fractals (Sanda/Kyoto, 1990), 75–128, Pitman Res. Notes Math. Ser., 283, Longman Sci. Tech., Harlow, 1993.
[73] M.Viana, *Stochastic dynamics of deterministic systems*, Lecture Notes, XXI Braz. math. Colloq., IMPA, Rio de Janeiro (1997).
[74] D.L. Volevich, *Kinetics of coupled map lattices,* Nonlinearity **4** (1991), 37-45.
[75] D.L. Volevich, *Construction of an analogue of Bowen-Sinai measure for a multidimensional lattice of interacting hyperbolic mappings*, Russ. Acad. Math. Sbornink **79**, 347–363 (1994).
[76] L.-S.Young, *Statistical Properties of Dynamical Systems with Some Hyperbolicity,* Annals of Mathematics (2), **147**, (1998), no. 3, 585–650.
[77] L.-S.Young, *Recurrence times and rates of mixing,* Israel J. of Math., **110**, (1999), 153–188.

Carlangelo Liverani, Dipartimento di Matematica, II Universityà di Roma (Tor Vergata), Via della Ricerca Scientifica, 00133 Roma, Italy.
E-mail address: liverani@mat.uniroma2.it