COUPLED MAP LATTICES WITHOUT CLUSTER EXPANSION

GERHARD KELLER AND CARLANGELO LIVERANI

Abstract. We present an approach to the investigation of the statistical properties of weakly coupled map lattices that avoids completely cluster expansion techniques. Although here it is implemented on a simple case we expect similar strategies to be applicable in a much larger class of situations.

1. Introduction

Since their introduction coupled map lattice systems have raised considerable interest both in the theoretical physics and in the mathematical community (see [14, 5] for a review). The first rigorous results on their statistical properties, contained in the pioneering works [6, 3], where based on a cluster expansion approach. Such an approach is the one mainly employed also in much of the related work, either by using a Markov partition on the single map to view the system as a two dimensional spin systems (e.g. [6, 22, 26, 10, 11, 13, 12]) or by viewing the transfer operator itself, when applied to the constant function one, as the object to expand (e.g. [3, 21, 1, 8]). This point of view has proven very powerful and has yielded rather general results [4] and very complete ones in the analytic setting [1, 8, 23, 2]. Nevertheless, it imposes serious limitations on the smoothness requirement of the systems, in particular Lasota-Yorke like maps and Laplacian like coupling seem outside its realm of application.

To our knowledge, the only papers trying to develop alternative techniques, able to overcome the above mentioned limitations, are the ones by G.Keller and collaborators. In particular, [18] where a general setting is proposed but no uniqueness or mixing properties of the invariant measures could be proved, [19] where satisfactory results are obtained for the case of unidirectional coupling, and the Ph.D. thesis of M.Schmitt [24, 25] where, for the first time in this setting, uniqueness and mixing is proved, but only under extremely strong expansion and distortion assumptions on the local map. See also [15, 16, 17] for other relevant aspects of this approach.

In the present paper, we show how to overcome Schmitt' s small distortion and large expansion assumptions obtaining in such a way the first reasonably general alternative to the cluster expansion approaches. The results are comparable to the ones in [4], yet the argument is considerably simpler. In order to keep the exposition as simple as possible we do not attempt to cover Lasota-Yorke type maps and Laplacian like couplings but leave this for another publication.

Date: May 10, 2004.

1991 *Mathematics Subject Classification.* 37D20, 37D35, 37A25, 37A60, 82C20.

Key words and phrases. Coupled map lattice, transfer operator, spectral gap, exponential decay of correlations, spatio-temporal chaos.

It is a pleasure to thank V.Baladi, J.Bricmont, H.H.Rugh for many discussions through the years.

The paper is self-contained and it is organized as follows. Section 2 contains a precise description of the class of systems considered and states (a bit loosely) the main result of the paper. Section 3 describes the functional space that will be used to study the transfer operator and details some of its properties. Section 4 states exactly the technical results needed to prove the main theorem. Section 5 is devoted to obtaining a Lasota-Yorke type inequality. Section 6 estimates the distance, in an appropriate norm, between the unperturbed and the coupled systems. Section 7 contains the proof of the crucial technical results: the spectral gap of the transfer operator. Finally, section 8 contains the proof, and makes precise the statement, of the main theorem, based on the previous results.

2. The model

Let $\Omega := \mathbb{T}^{\mathbb{Z}}$ and define the unperturbed dynamics $F_0 : \Omega \to \Omega$ as

$$F_0(x)_i := T(x_i)$$

where $T \in \mathcal{C}^2(\mathbb{T}, \mathbb{T})$ is a smooth expanding map and $|DT| \geq \lambda > 1$. Next, consider a diffeomorphism $G_\varepsilon : \Omega \to \Omega$ defined by $G_\varepsilon(x)_i = x_i + \varepsilon \sum_{|j| \leq r} \alpha_j g(x_{i+j})$, where $g \in \mathcal{C}^2(\mathbb{T}, \mathbb{R})$.[1]

Finally, we define the relevant dynamics

$$F_\varepsilon(x) := G_\varepsilon(F_0(x)). \tag{2.1}$$

The main result of the paper can be summarized as follows.

Theorem 2.1. *For each ε small enough, there exists a probability measure μ_ε, with finite dimensional marginals absolutely continuous with respect to Lebesgue, invariant with respect to F_ε. Such a measure is the unique invariant measure in a class of* sufficiently regular *measures. In addition, μ_ε is exponentially mixing in space and time and it is $\varepsilon(\ln \varepsilon^{-1})^2$* close *to the unperturbed one. Its entropy density relative to Lebesgue measure is finite.*

Remark 2.2. *The meaning of* sufficiently regular *is made precise in section 3; in fact, μ_ε belongs to the space $\mathcal{B}_1$ defined there. The notion of* close *is defined in Lemma 8.3. The first statement of the Theorem has no pretense of being optimal. In fact, E. and M. Järvenpää have shown that, in general, it is not possible to hope for a unique invariant measure in the class of measures with finite dimensional marginals absolutely continuous with respect to Lebesgue* [9]*. In our opinion a physically natural identification of a class of measures in which unicity holds is still missing.*

3. A function space

Let $\Lambda \subset \mathbb{Z}$ be boxes,[2] $\mathcal{F}_\Lambda$ and $\overline{\mathcal{F}}_\Lambda$ be the subsets of $\mathcal{C}^1(\Omega, \mathbb{R})$ and $\mathcal{C}^1(\Omega, \mathbb{R}^\Lambda)$, respectively, of functions depending only on the variables in the box Λ. Let $\mathcal{M}(\Omega)$

[1] Clearly one can consider a more general case, provided the range remains bounded by r (for example one can drop the smoothness requirement to $\mathcal{C}^{1+\alpha}$), yet let us make our life easy by looking at a concrete example.

[2] By "boxes" we mean sets of the type $I \cap \mathbb{Z}$, where $I \subset \mathbb{R}$ is a closed connected set. By $|\Lambda|$ we will denote the cardinality of the box Λ. The empty set is also considered a box.

be the space of signed measures on Ω.[3] Define

$$
\begin{aligned}
|\mu|_\Lambda &:= \sup_{\substack{\varphi\in\mathcal{F}_\Lambda\\ |\varphi|_\infty\le 1}} \mu(\varphi)\\
\|\mu\|_\Lambda &:= \sup_{\substack{\varphi\in\overline{\mathcal{F}}_\Lambda\\ \sum_{k\in\Lambda}|\varphi_k|_\infty\le 1}} \sum_{k\in\Lambda}\mu(\partial_k\varphi_k)\\
|\mu|_\theta &:= \sup_\Lambda \theta^{|\Lambda|}|\mu|_\Lambda\\
\|\mu\|_\theta &:= \sup_\Lambda \theta^{|\Lambda|}\|\mu\|_\Lambda + |\mu(1)|,
\end{aligned}
\tag{3.1}
$$

for each $\theta\in(0,1]$. As usual we consider the Banach spaces $\mathcal{B}_{w,\theta}$ and $\mathcal{B}_\theta$ obtained by completing the space $\{\mu\in\mathcal{M}(\Omega) \ : \ |\mu|_\theta+\|\mu\|_\theta<\infty\} =: \mathcal{M}_\theta(\Omega)$ with respect to the norms $|\cdot|_\theta$ and $\|\cdot\|_\theta$, respectively. Observe that $|\mu|_\theta\le|\mu|_1$ and $\|\mu\|_\theta\le\|\mu\|_1$ for all $\theta\in(0,1]$.

Let us list some properties of $\mathcal{B}_{w,\theta}$ and $\mathcal{B}_\theta$.

Lemma 3.1. *For each $\theta\in(0,1)$ and $\mu\in\mathcal{B}_\theta$ holds true*

$$|\mu|_\theta \le 2(1-\theta)^{-1}\|\mu\|_\theta.$$

Proof. For each $\Lambda=[q,p]\subset\mathbb{Z}$ and $\varphi\in\mathcal{F}_\Lambda$ we can write

$$
\begin{aligned}
\varphi =& \varphi - \int\varphi dx_q + \sum_{k=q+1}^{p}\left[\int\varphi dx_q\cdots dx_{k-1} - \int\varphi dx_q\cdots dx_k\right] + \int\varphi dx_q\cdots dx_p\\
=:& \sum_{k=q}^{p}\varphi_k + \int\varphi dx_q\cdots dx_p.
\end{aligned}
$$

Note that $\varphi_k\in\mathcal{F}_{\{k,\dots,p\}}$, that $|\varphi_k|_\infty\le 2|\varphi|_\infty$, and that $\int\varphi_k dx_k=0$. Accordingly, the function[4] $\Phi_k(x):=\int_0^{x_k}\varphi_k(x_{\neq k},y)dy$ is a well defined function on Ω belonging to $\mathcal{F}_{\{k,\dots,p\}}$. We can thus compute

$$
\begin{aligned}
\theta^{|\Lambda|}\mu(\varphi) &= \theta^{|\Lambda|}\sum_{k=q}^{p}\mu(\partial_{x_k}\Phi_k) + \theta^{|\Lambda|}\mu\left(\int\varphi dx_q\dots dx_p\right)\\
&\le |\varphi|_\infty\left\{2\sum_{k=0}^{|\Lambda|-1}\theta^k\|\mu\|_\theta + |\mu(1)|\right\}.
\end{aligned}
$$

□

Remark 3.2. *If $\mu\in\mathcal{B}_{w,\theta}$ is positive,[5] then μ is a positive measure on Ω and $|\mu|_\theta=|\mu|_{\theta=1}=\mu(1)$.[6] On the other hand, if μ is a measure, then $|\mu|_\theta\le|\mu|(1)$. In particular, $|\mu|_1$ coincides with the total variation norm of the measure μ.*

---

[3] Here we consider Borel measures where the topology is the product one.

[4] By $(x_{\neq k},y)$ we mean the point $\xi\in\Omega$ such that $\xi_i=x_i$ for all $i\neq k$ and $\xi_k=y$. Similar, self evident, notations will be used in the following without further warning.

[5] That is, for each box $\Lambda\subset\mathbb{Z}$ and $\varphi\in\mathcal{F}_\Lambda$, $\varphi\ge0$ implies $\mu(\varphi)\ge0$.

[6] Indeed, for each $\varphi\in\mathcal{F}_\Lambda$, $\mu(\varphi)\le\mu(|\varphi|_\infty)=|\varphi|_\infty\mu(1)$. But $\mu(1)\le|\mu|_\theta$, hence μ is a continuous functional on the continuous functions (since the local functions $\cup_{\Lambda\subset\mathbb{Z}}\mathcal{F}_\Lambda$ form an algebra that separate the points and thus are dense in the continuous one by Stone-Weierstrass).

Lemma 3.3. *If $\mu \in \mathcal{B}_\theta$ is positive its restriction μ_Λ to any box $\Lambda \subset \mathbb{Z}$ is a measure whose entropy relative to Lebesgue can grow at most as $|\Lambda|^2$. If $\theta = 1$ it grows at most like $|\Lambda|$.*

Proof. First of all note that if $\mu \in \mathcal{B}_\theta$, then, for each box $\Lambda \subset \mathbb{Z}$, it induces the linear functional $\mu_\Lambda : \mathcal{C}^0(\mathbb{T}^\Lambda, \mathbb{R}) \to \mathbb{R}$ defined by

$$\mu_\Lambda(\varphi) := \mu(\tilde{\varphi}) \quad \forall \varphi \in \mathcal{C}^0(\mathbb{T}^\Lambda, \mathbb{R}),$$

where $\tilde{\varphi} \in \mathcal{C}^0(\Omega, \mathbb{R})$ is defined by $\tilde{\varphi}(x) := \varphi(x_{\in\Lambda})$. Clearly $|\mu_\Lambda(\varphi)| \leq |\mu|_\Lambda |\varphi|_\infty$, that is μ_Λ is a measure on $\mathbb{T}^\Lambda$. In addition, it satisfies

$$\sup_{\substack{\varphi \in \mathcal{C}^1(\mathbb{T}^\Lambda, \mathbb{R}^\Lambda) \\ \sum_{i\in\Lambda} |\varphi_i|_\infty \leq 1}} \sum_i \mu_\Lambda(\partial_i \varphi_i) \leq \theta^{-|\Lambda|} \|\mu\|_\theta$$

which readily implies that μ_Λ is absolutely continuous with respect to Lebesgue and that the density is a function of bounded variation with variation bounded by $\theta^{-|\Lambda|}\|\mu\|_\theta$, e.g. see [20].

Thus it make sense to talk of relative entropy. Since $\ln x$ is convex, calling m_Λ the Lebesgue measure on $\mathbb{T}^\Lambda$, by Jensen inequality

$$\mu_\Lambda(\ln h_\Lambda) = p\mu_\Lambda(\ln h_\Lambda^{\frac{1}{p}}) \leq p \ln m_\Lambda(h_\Lambda^{1+\frac{1}{p}}).$$

By choosing $p = |\Lambda| - 1$, the embedding theorem of $BV(\mathbb{T}^\Lambda)$ into $L^{1+\frac{1}{|\Lambda|-1}}(\mathbb{T}^\Lambda)$ (see [7] page 140) yields

$$\mu_\Lambda(\ln h_\Lambda) \leq |\Lambda| \ln \|\mu\|_\Lambda \leq |\Lambda| \left[\ln \|\mu\|_\theta + |\Lambda| \ln \theta^{-1}\right].$$

□

Remark 3.4. *The space $\mathcal{B}_\theta$ contains objects that are not measures.*[7]

4. Technical results

Theorem 2.1 rests on the following result.

Theorem 4.1. *There exists $\theta_1 \in (0,1)$ and $\lambda_* > 1$ such that, for each $\theta \in (\theta_1, 1)$, there exists $\varepsilon_\theta > 0$ and $C_\theta > 0$ such that, for each $\mu \in \mathcal{B}_\theta$, $\mu(1) = 0$, and $0 \leq \varepsilon < \varepsilon_\theta$ holds true*

$$\|F_{\varepsilon *}^n \mu\|_\theta \leq C_\theta \lambda_*^{-n} \|\mu\|_\theta.$$

The proof of the above theorem can be found in section 7. It relies on the following two basic results proved in section 5 and section 6, respectively.

Lemma 4.2. *For each $\sigma \in (\lambda^{-1}, 1)$ there exist $\theta_0 \in (0,1)$ and $B, \varepsilon_0 \geq 0$ such that, for each $\theta \in [\theta_0, 1]$, $\varepsilon \in [0, \varepsilon_0]$ and $\mu \in \mathcal{B}_\theta$ holds true*

$$|F_{\varepsilon *}\mu|_\theta \leq \theta^{-2r} |\mu|_\theta$$
$$\|F_{\varepsilon *}\mu\|_\theta \leq \sigma\theta^{-2r} \|\mu\|_\theta + B|\mu|_\theta.$$

[7] Just consider $h \in BV(\mathbb{T}, \mathbb{R})$ of zero average, but such that $\int |h| = 1$, and the sequence of measures

$$\mu_q(\varphi) = \int_{\mathbb{T}^\mathbb{Z}} \prod_{j=-q}^{q} h(x_j)\varphi(x)dx,$$

then $\sum_{q=1}^\infty \theta^{-q}\mu_q$ clearly belongs to $\mathcal{B}_\theta$ but it cannot be a measure.

In addition, if μ is a measure, then, for each $n \in \mathbb{N}$,

$$|F^n_{\varepsilon *}\mu|_\theta \leq |\mu|_{\theta=1}.$$

We may choose θ_0 so close to 1 that

$$\sigma_0 := \sigma\theta_0^{-2r} < 1 \; . \tag{4.1}$$

Remark 4.3. *Note that Lemma 4.2, contrary to what happens for the uncoupled case, implies only that the spectral radius of F_ε on $\mathcal{B}_\theta$ is bounded by θ^{-2r}, instead than one. Yet, like in the uncoupled case, $1 \circ F_\varepsilon = 1$, hence one must belong to the spectrum of $F_{\varepsilon *}$. All the subsequent work is devoted to bound the rest of the spectrum and show that it is contained in a disk of radius strictly less than one.*

Let $\Lambda \subset \mathbb{Z}$ be a box and define

$$G_{\varepsilon,\Lambda}(x)_i := \begin{cases} x_i + \varepsilon \sum\limits_{\substack{|j|\leq r \\ i+j\in\Lambda}} \alpha_j g(x_{i+j}) & \forall i \in \Lambda \\ x_i & \forall i \notin \Lambda. \end{cases} \tag{4.2}$$

Define as well $F_{\varepsilon,\Lambda} := G_{\varepsilon,\Lambda} \circ F_0$.

Lemma 4.4. *There exists $D \geq 0$ such that, for each $\varepsilon \in [0, \varepsilon_0]$, $\mu \in \mathcal{B}_\theta$, $\theta \in (0,1]$, and all boxes $\Lambda' \subset \Lambda \subset \mathbb{Z}$ holds true*

$$|G_{\varepsilon *}\mu - G_{\varepsilon,\Lambda' *}\mu|_\Lambda \leq D(|\Lambda\backslash\Lambda'| + r)\varepsilon(\|\mu\|_{\Lambda+r} + |\mu|_{\Lambda+r}) \; .$$

5. Proofs: Lasota-Yorke type inequality

This section is devoted to the proof of Lemma 4.2. As usual in this type of theory the proof consists just of a, more or less lengthy, computation.

The first inequality is very simple. Indeed for each $\varphi \in \mathcal{F}_\Lambda$, $F_{\varepsilon *}\mu(\varphi) = \mu(\varphi \circ F_\varepsilon)$, but $\varphi \circ F_\varepsilon \in \mathcal{F}_{\Lambda+r}$ and $|\varphi \circ F_\varepsilon|_\infty = |\varphi|_\infty$, hence $|F_{\varepsilon *}\mu|_\Lambda \leq |\mu|_{\Lambda+r}$, which implies the result.[8] Analogously, the last inequality follows from Remark 3.2

$$|F^n_{\varepsilon *}\mu|_\theta \leq |F^n_{\varepsilon *}\mu|_{\theta=1} \leq |\mu|_{\theta=1}.$$

The second inequality requires more work. We start by writing, for $\varphi \in \overline{\mathcal{F}}_\Lambda$,[9]

$$\begin{aligned} \sum_i F_{\varepsilon *}\mu(\partial_i\varphi_i) &= \sum_i \mu((\partial_i\varphi_i) \circ F_\varepsilon) \\ &= \sum_i \mu(\partial_i((DF_\varepsilon)^{-1}\varphi \circ F_\varepsilon)_i) - \sum_{ij} \mu(\varphi_j \circ F_\varepsilon \, \partial_i((DF_\varepsilon)^{-1})_{ij}). \end{aligned} \tag{5.1}$$

Next we must estimate the norms of the functions

$$\begin{aligned} \psi_i &:= \sum_j ((DF_\varepsilon)^{-1})_{ij}\varphi_j \circ F_\varepsilon \\ \phi_i &:= \varphi_i \circ F_\varepsilon \sum_j \partial_j((DF_\varepsilon)^{-1})_{ji}. \end{aligned} \tag{5.2}$$

---

[8]Here, and in the following, given a box $\Lambda = \{q, \dots, p\}$ and $k \in \mathbb{Z}$ by $\Lambda + k$ we mean the box $\{q-k, \dots, p+k\}$, clearly $\Lambda - k = \emptyset$ if $2k > |\Lambda|$.

[9]We slightly abuse notation and consider $\mathcal{C}^1(\Omega, \mathbb{R}^\Lambda)$ as a subspace of $\mathcal{C}^1(\Omega, \mathbb{R}^\mathbb{Z})$, i.e. if $\varphi \in \mathcal{C}^1(\Omega, \mathbb{R}^\Lambda)$, then φ can be considered an element of $\mathcal{C}^1(\Omega, \mathbb{R}^\mathbb{Z})$ with $\varphi_i = 0$ if $i \notin \Lambda$.

First notice that $DG_\varepsilon = \mathbf{Id} + \varepsilon A$, thus

$$(DF_\varepsilon)^{-1} = (DF_0)^{-1}(\mathbf{Id} + \varepsilon A \circ F_0)^{-1} = (DF_0)^{-1}\sum_{n=0}^{\infty} \varepsilon^n A^n \circ F_0 \tag{5.3}$$

provided ε is small enough, where DF_0 is a diagonal matrix.

Note that $(A^n)_{ij} = 0$ if $|i-j| > (n+1)r$. Thus if $\varphi \in \overline{\mathcal{F}}_\Lambda$, then $\psi^n := DF_0^{-1} A^n \circ F_0\, \varphi \circ F_\varepsilon \in \overline{\mathcal{F}}_{\Lambda+(n+1)r}$. Since[10]

$$|\psi^n|_{\ell^1} \leq \|A\|^n \lambda^{-1} |\varphi|_{\ell^1} \tag{5.4}$$

it follows

$$\begin{aligned}\theta^{|\Lambda|}\sum_i \mu(\partial_i\psi_i) &= \theta^{|\Lambda|}\sum_n \varepsilon^n \sum_i \mu(\partial_i \psi_i^n) \leq \theta^{|\Lambda|}\sum_n \varepsilon^n \|\mu\|_{\Lambda+(n+1)r}\lambda^{-1}\|A\|^n \\ &\leq \sum_n \varepsilon^n \theta^{-2(n+1)r}\lambda^{-1}\|A\|^n\|\mu\|_\theta \leq (1-\varepsilon\theta^{-2r}\|A\|)^{-1}\lambda^{-1}\theta^{-2r}\|\mu\|_\theta\ .\end{aligned} \tag{5.5}$$

Analogously, setting $\phi_i^n := \varphi_i \circ F_\varepsilon \sum_j \partial_j(((DF_0)^{-1}A^n \circ F_0)_{ji}$, there exist $C, D > 0$ such that

$$|\phi_i^n|_\infty \leq CnD^n|\varphi_i|_\infty\ .$$

Accordingly,

$$\theta^{|\Lambda|}\sum_i \mu(\varphi_i) = \theta^{|\Lambda|}\sum_{n=0}^{\infty}\varepsilon^n\sum_i \mu(\phi_i^n) \leq C\sum_{n=0}^{\infty}\theta^{-2(n+1)r}nD^n\varepsilon^n\sum_i |\varphi_i|_\infty |\mu|_\theta. \tag{5.6}$$

Using equations (5.5) and (5.6) in equation (5.1) the result follows, provided ε is small enough.

6. Proofs: a perturbation result

In this section we prove Lemma 4.4. Let $\Lambda' \subset \Lambda \subset \mathbb{Z}$ be two boxes. Let $\varphi \in \mathcal{F}_\Lambda$. Then

$$G_{\varepsilon *}\mu(\varphi) - G_{\varepsilon,\Lambda' *}\mu(\varphi) = \mu(\varphi \circ G_{\varepsilon,\Lambda+r} - \varphi \circ G_{\varepsilon,\Lambda'})\ .$$

Setting $G_\xi(x) := G_{\varepsilon,\Lambda'}(x) + \xi\{G_{\varepsilon,\Lambda+r}(x) - G_{\varepsilon,\Lambda'}(x)\}$ one can write

$$\begin{aligned}\varphi\circ G_{\varepsilon,\Lambda+r}(x) - \varphi \circ G_{\varepsilon,\Lambda'}(x) &= \int_0^1 \frac{d}{d\xi}\varphi(G_\xi(x))d\xi \\ &= \sum_{i\in\Lambda}\int_0^1 (\partial_{x_i}\varphi)(G_\xi(x))[G_{\varepsilon,\Lambda+r}(x)_i - G_{\varepsilon,\Lambda'}(x)_i]d\xi \\ &= \sum_{i\in\Lambda\setminus(\Lambda'-r)}\int_0^1 (\partial_{x_i}\varphi)(G_\xi(x))[G_{\varepsilon,\Lambda+r}(x)_i - G_{\varepsilon,\Lambda'}(x)_i]d\xi\end{aligned}$$

Before going further let us notice that $DG_{\varepsilon,\Lambda} = \mathbf{Id} + \varepsilon A_\Lambda$ where A_Λ has all zero entries apart from a diagonal $\Lambda \times \Lambda$ block. Accordingly,

$$\partial_{x_i}(\varphi \circ G_\xi) = \sum_{j\in\Lambda}(\partial_{x_j}\varphi)\circ G_\xi (DG_\xi)_{ji}$$

where

$$DG_\xi = \mathbf{Id} + \varepsilon A_{\Lambda'} + \xi\varepsilon(A_{\Lambda+r} - A_{\Lambda'}) =: \Xi\ .$$

[10] For $\varphi \in \mathcal{C}^0(\Omega, \mathbb{R}^{\mathbb{Z}})$, $|\varphi|_{\ell^1} := \sum_{i\in\mathbb{Z}}|\varphi_i|_\infty$. We let $\|A\| = \sup_x \|A(x)\|_{\ell^1}$, accordingly.

Hence,

$$\sum_k \partial_{x_k}(\Xi^{-1}_{ki}\,\varphi\circ G_\xi) = \sum_k (\partial_{x_k}\Xi^{-1}_{ki})\,\varphi\circ G_\xi + (\partial_{x_i}\varphi)\circ G_\xi\ .$$

Using this relation we can continue the previous chain of equalities

$$\begin{aligned}
&= \sum_{i\in\Lambda\setminus(\Lambda'-r);k}\int_0^1 \left\{\partial_{x_k}(\Xi^{-1}_{ki}\,\varphi\circ G_\xi) - (\partial_{x_k}\Xi^{-1}_{ki})\varphi\circ G_\xi\right\}[G_{\varepsilon,\Lambda+r}(x)_i - G_{\varepsilon,\Lambda'}(x)_i]d\xi\\
&= \sum_{i\in\Lambda\setminus(\Lambda'-r)}\int_0^1 \sum_k \partial_{x_k}\left\{(\Xi^{-1}_{ki}\,\varphi\circ G_\xi)[G_{\varepsilon,\Lambda+r}(x)_i - G_{\varepsilon,\Lambda'}(x)_i]\right\}d\xi\\
&\quad - \sum_{i\in\Lambda\setminus(\Lambda'-r)}\int_0^1 \sum_k (\partial_{x_k}\Xi^{-1}_{ki})\varphi\circ G_\xi[G_{\varepsilon,\Lambda+r}(x)_i - G_{\varepsilon,\Lambda'}(x)_i]d\xi\\
&\quad - \sum_{i\in\Lambda\setminus(\Lambda'-r)}\int_0^1 \sum_k (\Xi^{-1}_{ki}\,\varphi\circ G_\xi)\partial_{x_k}[G_{\varepsilon,\Lambda+r}(x)_i - G_{\varepsilon,\Lambda'}(x)_i]d\xi\ .
\end{aligned}$$

Since all the functions inside the integrals are in $\mathcal{F}_{\Lambda+r}$ and bounded by a constant times ε, it follows

$$\begin{aligned}
G_{\varepsilon*}\mu(\varphi) - G_{\varepsilon,\Lambda'*}\mu(\varphi) &= \mu(\varphi\circ G_{\varepsilon,\Lambda+r} - \varphi\circ G_{\varepsilon,\Lambda'})\\
&\le D\varepsilon(|\Lambda\setminus\Lambda'| + r)(\|\mu\|_{\Lambda+r} + |\mu|_{\Lambda+r})\ .
\end{aligned}$$

As $\varphi\in\mathcal{F}_\Lambda$ is arbitrary, this finishes the proof of Lemma 4.4.

7. Proofs: spectral gap

The results developed in the previous sections are sufficient to prove the theorem stated in section 4.

Proof of Theorem 4.1. First iterate Lemma 4.2 to write, for each $n, m\in\mathbb{N}$,

$$\|F^{n+m}_{\varepsilon*}\mu\|_\theta \le \sigma_0^n\|F^m_{\varepsilon*}\mu\|_\theta + \theta^{-2rn}B_1|F^m_{\varepsilon*}\mu|_\theta \tag{7.1}$$

where $B_1 = (1-\sigma_0)^{-1}B$. Notice that (7.1), setting $m=0$, implies

$$\|F^n_{\varepsilon*}\mu\|_\theta \le (\sigma_0^n + (1-\theta)^{-1}B_1\theta^{-2rn})\|\mu\|_\theta \le C\frac{\theta^{-2rn}}{1-\theta}\|\mu\|_\theta \tag{7.2}$$

for some $C>0$ and all $n\in\mathbb{N}$ and $\varepsilon\in[0,\varepsilon_0]$. To conclude we need a careful estimate of the last term in (7.1).

Let $\Lambda = [p,q]$, $\Lambda'\in\{[p,q-1],[p,q]\}$. We start by noticing that

$$|G_{\varepsilon*}\mu - G_{\varepsilon,\Lambda'*}\mu|_\Lambda \le D(1+r)\varepsilon(\|\mu\|_{\Lambda+r} + |\mu|_{\Lambda+r}) \le D'\varepsilon\theta^{-(|\Lambda|+r)}(\|\mu\|_\theta + |\mu|_\theta)$$

by Lemma 4.4. Accordingly

$$\begin{aligned}
|F^m_{\varepsilon*}\mu - F^m_{\varepsilon,\Lambda'*}\mu|_\Lambda &\le \sum_{j=0}^{m-1}|F^{m-j-1}_{\varepsilon,\Lambda'*}F^{j+1}_{\varepsilon*}\mu - F^{m-j}_{\varepsilon,\Lambda'*}F^j_{\varepsilon*}\mu|_\Lambda\\
&\le \sum_{j=0}^{m-1}|(F_{\varepsilon*} - F_{\varepsilon,\Lambda'*})F^j_{\varepsilon*}\mu|_\Lambda \le D'\varepsilon\theta^{-(|\Lambda|+r)}\sum_{j=0}^{m-1}\|F_{0*}F^j_{\varepsilon*}\mu\|_\theta + |F^j_{\varepsilon*}\mu|_\theta\\
&\le D''\varepsilon\frac{\theta^{-(|\Lambda|+r(2m+1))}}{1-\theta}\|\mu\|_\theta
\end{aligned} \tag{7.3}$$

where we used (7.2) and Lemma 3.1 for the last inequality and D'' does not depend on θ.

The problem is then reduced to estimating $|F^m_{\varepsilon,\Lambda' *}\mu|_\Lambda$. Let $\varphi \in \mathcal{F}_\Lambda$ and define $\varphi_{m,q}(x) := \varphi([F^m_{\varepsilon,\Lambda'}(x)]_{\neq q}, x_q)$. Thus $\varphi \circ F^m_{\varepsilon,\Lambda'}(x) = \varphi_{m,q}(x_{\neq q}, T^m(x_q))$. On the other hand, calling h the invariant normalized density of T,

$$\begin{aligned}\varphi_{m,q}(x_{\neq q}, T^m(x_q)) =& \frac{d}{dx_q}\left\{\int_0^{x_q}\varphi_{m,q}(x_{\neq q}, T^m\xi)d\xi - x_q\int_{\mathbb{T}}\varphi_{m,q}(x_{\neq q}, T^m\xi)h(\xi)d\xi\right\}\\ &+\int_{\mathbb{T}}\varphi_{m,q}(x_{\neq q},\xi)h(\xi)d\xi\\ =&\frac{d}{dx_q}\int_{\mathbb{T}}\phi_{x_q}(\xi)\varphi_{m,q}(x_{\neq q}, T^m\xi)d\xi + \int_{\mathbb{T}}\varphi_{m,q}(x_{\neq q},\xi)h(\xi)d\xi,\end{aligned}$$

where $\phi_{x_q}(\xi) := \mathbf{1}_{[0,x_q]}(\xi) - x_q h(\xi)$, $\mathbf{1}_{[0,x_q]}$ being the characteristic function of the interval $[0, x_q]$. Accordingly, calling $\mathcal{L}$ the one dimensional transfer operator associated to the map T and setting $\bar\varphi_{\{q\}}(x) := \int_{\mathbb{T}}\varphi(x_{\neq q},\xi)h(\xi)d\xi \in \mathcal{F}_{\Lambda'}$,

$$\varphi_{m,q}(x_{\neq q}, T^m(x_q)) = \frac{d}{dx_q}\int_{\mathbb{T}}(\mathcal{L}^m\phi_{x_q})(\xi)\varphi_{m,q}(x_{\neq q},\xi)d\xi + \bar\varphi_{\{q\}}\circ F^m_{\varepsilon,\Lambda'}(x) \tag{7.4}$$

It is well known that $\mathcal{L}$, restricted to the zero average functions, has spectral radius $\sigma_1 < 1$. Thus, since ϕ_{x_q} is of zero average, there exists $A > 0$ such that

$$\|\mathcal{L}^m\phi_{x_q}\|_\infty \le \|\mathcal{L}^m\phi_{x_q}\|_{\mathrm{BV}} \le A\sigma_1^m\|\phi_{x_q}\|_{\mathrm{BV}} \le A'\sigma_1^m. \tag{7.5}$$

Thus,

$$\mu(\varphi\circ F^m_{\varepsilon,\Lambda'}) \le A'\sigma_1^m\|\mu\|_\Lambda + \mu(\bar\varphi_{\{q\}}\circ F^m_{\varepsilon,\Lambda'}). \tag{7.6}$$

In particular, the last term vanishes if $\Lambda = \{q\}$ and $\mu(1) = 0$. The key fact of the present approach is that the averaged test function no longer depends on the variable x_q: using the above inequality together with (7.3) we obtain

$$\begin{aligned}\theta^{|\Lambda|}|F^m_{\varepsilon *}\mu|_\Lambda &\le D''\varepsilon\frac{\theta^{-r(2m+1)}}{1-\theta}\|\mu\|_\theta + \theta^{|\Lambda|}|F^m_{\varepsilon,\Lambda' *}\mu|_\Lambda\\ &\le D''\varepsilon\frac{\theta^{-r(2m+1)}}{1-\theta}\|\mu\|_\theta + A'\sigma_1^m\|\mu\|_\theta + \theta\,\theta^{|\Lambda'|}|F^m_{\varepsilon,\Lambda' *}\mu|_{\Lambda'}\\ &\le 2D''\varepsilon\frac{\theta^{-r(2m+1)}}{1-\theta}\|\mu\|_\theta + A'\sigma_1^m\|\mu\|_\theta + \theta\,\theta^{|\Lambda'|}|F^m_{\varepsilon *}\mu|_{\Lambda'}\ .\end{aligned}$$

Hence, taking the supremum over all boxes Λ, we obtain for such μ

$$|F^m_{\varepsilon *}\mu|_\theta \le \left(2D''\varepsilon\frac{\theta^{-r(2m+1)}}{1-\theta} + A'\sigma_1^m\right)\|\mu\|_\theta + \theta\,|F^m_{\varepsilon,*}\mu|_\theta\ . \tag{7.7}$$

We are finally in the position to conclude the argument. Let $\mu \in \mathcal{B}_\theta$ such that $\mu(1) = 0$. Then equations (7.1), (7.2), and (7.7) yield

$$\|F^{n+m}_{\varepsilon *}\mu\|_\theta \le \left\{\sigma_0^n C\theta^{-2rm} + \left(2D''\varepsilon\frac{\theta^{-r(2m+1)}}{(1-\theta)^2} + \frac{A'\sigma_1^m}{1-\theta}\right)B_1\theta^{-2rm}\right\}\|\mu\|_\theta.$$

The last step is to choose properly n and m in the above inequality. For simplicity, rather than aiming at an optimal choice, we will make a very simple one: let

$\theta_1 \in [\theta_0, 1)$ be such that $\theta_1^{2r} > \lambda_*^{-2} > \max\{\sigma_0, \sigma_1\}$. Then, for $\theta \in [\theta_1, 1)$, we can choose $n = m = m_\theta =: \frac{1}{c} \ln(1-\theta)^{-1}$. Accordingly, by choosing c small enough,

$$\ln \frac{A'\sigma_1^{m_\theta}}{1-\theta} B_1 \theta^{-2rm_\theta} \leq -m_\theta \left\{ \ln \sigma_1^{-1} - 2r \ln \theta_1^{-1} - c \frac{\ln A' B_1}{\ln(1-\theta_1)^{-1}} - c \right\} < -m_\theta \ln \lambda_*^2.$$

Finally, we can choose $\varepsilon_\theta > 0$ so small that

$$\|F_{\varepsilon *}^{2m_\theta} \mu\|_\theta \leq \lambda_*^{-2m_\theta} \|\mu\|_\theta \quad (\varepsilon \in [0, \varepsilon_\theta], \mu \in \mathcal{B}_\theta, \mu(1) = 0) .$$

Hence we have the announced spectral gap. □

8. Proof of the main theorem

This concluding section is dedicated to the proof of Theorem 2.1. This is achieved in three Lemmata that also make the statement of the theorem precise. The proof is obtained by an approximation argument based on Theorem 4.1.

The class of *sufficiently regular* measures mentioned in Theorem 2.1 is the set $\mathcal{B}_{U,\theta}$ defined in the next Lemma.

Lemma 8.1. *Let* $\mathcal{B}_{U,\theta} := \overline{\mathcal{B}_\theta \cap \mathcal{C}^0(\Omega)^*}^{|\cdot|_1}$, *then there exists a unique invariant normalized element* $\mu_\varepsilon \in \mathcal{B}_{U,\theta}$. *In fact,* $\mu_\varepsilon \in \mathcal{B}_{\theta=1}$, *and it is a probability measure.*

Proof. It follows from Theorem 4.1 that there exists a unique $\mu_\varepsilon \in \mathcal{B}_\theta$ such that $F_{\varepsilon *}\mu_\varepsilon = \mu_\varepsilon$ and $\mu_\varepsilon(1) = 1$, moreover μ_ε is a probability measure. Indeed, let m be the infinite product Lebesgue measure on Ω. The sequence $\frac{1}{n} \sum_{k=0}^{n-1} F_{\varepsilon *}^k m$ is a weakly compact sequence of probability measures in $\mathcal{C}_0(\Omega)^*$. Let μ_* be an accumulation point. Clearly μ_* is an F_ε-invariant probability measure, and because $|m|_\theta = \|m\|_\theta = 1$, Lemma 4.2 yields, for $\theta \in (\theta_0, 1]$,

$$\limsup_k \|F_{\varepsilon *}^k m\|_\theta \leq B(1-\sigma_0)^{-1}$$

which implies $\mu_* \in \mathcal{B}_\theta$, $\|\mu_*\|_\theta \leq B(1-\sigma_0)^{-1}$, hence $\mu_* = \mu_\varepsilon$.

If $\mu \in \mathcal{B}_{U,\theta}$ is invariant, then there exists a sequence of measures $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_\theta$ such that $\lim_{n\to\infty} |\mu_n - \mu|_{\theta=1} = 0$. Then

$$|F_{\varepsilon *}^k \mu_n - \mu|_\theta = |F_{\varepsilon *}^k (\mu_n - \mu)|_\theta \leq |F_{\varepsilon *}^k (\mu_n - \mu)|_{\theta=1} \leq |\mu_n - \mu|_{\theta=1}$$

and, taking the limit for $k \to \infty$,

$$|\mu_\varepsilon - \mu|_\theta \leq |\mu_n - \mu|_{\theta=1} \quad \forall\, n \in \mathbb{N} .$$

Thus $\mu = \mu_\varepsilon$. □

The above establishes the existence and uniqueness part of Theorem 2.1. The entropy density assertion follows from Lemma 3.3.

Lemma 8.2. *There exists* $\theta_* > 0$ *such that for each* $\theta \in (\theta_*, 1)$ *there exist* $\nu_\theta > 1$ *and* $C_\theta > 0$ *such that for each two boxes* Λ_1, Λ_2, *at a distance* d *and each* $\varphi_i \in \mathcal{F}_{\Lambda_i}$ *holds true*

$$|\mu_\varepsilon(\varphi_1\varphi_2) - \mu_\varepsilon(\varphi_1)\mu_\varepsilon(\varphi_2)| \leq \nu_\theta^{-d} C_\theta \theta^{-|\Lambda_1|-|\Lambda_2|} |\varphi_1|_{\mathcal{C}^0} |\varphi_2|_{\mathcal{C}^0}.$$

In addition, for each box Λ *and* $\varphi_1, \varphi_2 \in \mathcal{F}_\Lambda$ *holds true*

$$|\mu_\varepsilon(\varphi_1 \circ F_\varepsilon^n \varphi_2) - \mu_\varepsilon(\varphi_1)\mu_\varepsilon(\varphi_2)| \leq \lambda_*^{-n} C_\theta \theta^{-|\Lambda|} |\varphi_1|_{\mathcal{C}^0} |\varphi_2|_{\mathcal{C}^1}.$$

Proof. Since the φ_i depend on variables at a distance d it follows that $\varphi_i \circ F_\varepsilon^k$ depend on a disjoint set of variables for all $k \leq [\frac{d}{2r}] := k_d$. Accordingly,

$$\begin{aligned}\mu_\varepsilon(\varphi_1\varphi_2) &= F_{\varepsilon *}^{k_d} m(\varphi_1\varphi_2) + \mathcal{O}_\theta(\theta^{-|\Lambda_1|-|\Lambda_2|-d}\lambda_*^{-k_d}|\varphi_1|_\infty\,|\varphi_2|_\infty)\\ &= m(\varphi_1\circ F_\varepsilon^{k_d})m(\varphi_2\circ F_\varepsilon^{k_d}) + \mathcal{O}_\theta(\theta^{-|\Lambda_1|-|\Lambda_2|-d}\lambda_*^{-k_d}|\varphi_1|_\infty\,|\varphi_2|_\infty)\\ &= \mu_\varepsilon(\varphi_1)\mu_\varepsilon(\varphi_2) + \mathcal{O}_\theta(\theta^{-|\Lambda_1|-|\Lambda_2|-d}\lambda_*^{-k_d}|\varphi_1|_\infty\,|\varphi_2|_\infty).\end{aligned}$$

The result follows by choosing θ such that $\theta\lambda_*^{\frac{1}{2r}} =: \nu_\theta > 1$.

To prove the second inequality consider $\varphi_i \in \mathcal{F}_\Lambda$ and define

$$\mu_{\varphi_2}(\phi) := \mu_\varepsilon(\varphi_2\phi) - \mu_\varepsilon(\varphi_2)\mu_\varepsilon(\phi)\ .$$

Then $\mu_{\varphi_2}(1) = 0$ and for each $\Lambda' \subset \mathbb{Z}$ and $\phi \in \bar{\mathcal{F}}_{\Lambda'}$, $\sum_{i\in\Lambda'}|\phi_i|_\infty \leq 1$,

$$\sum_{i\in\Lambda'}\mu_{\varphi_2}(\partial_i\phi) = \sum_{i\in\Lambda'}\mu_\varepsilon(\varphi_2\partial_i\phi_i) - \sum_{i\in\Lambda'}\mu_\varepsilon(\varphi_2)\mu_\varepsilon(\partial_i\phi) = \mathcal{O}(|\varphi_2|_{\mathcal{C}^1}(|\mu_\varepsilon|_{\theta=1}+\|\mu_\varepsilon\|_{\theta=1}))$$

Accordingly,

$$\|\mu_{\varphi_2}\|_\theta \leq \|\mu_{\varphi_2}\|_{\theta=1} \leq C(|\mu_\varepsilon|_{\theta=1} + \|\mu_\varepsilon\|_{\theta=1})\,|\varphi_2|_{\mathcal{C}^1},$$

and, by Theorem 4.1,

$$\begin{aligned}\mu_\varepsilon(\varphi_1\circ F_\varepsilon^n\varphi_2) &= F_{\varepsilon *}^n\mu_{\varphi_2}(\varphi_1 - \mu_\varepsilon(\varphi_1)) + \mu_\varepsilon(\varphi_2)\mu_\varepsilon(\varphi_1)\\ &= \mu_\varepsilon(\varphi_2)\mu_\varepsilon(\varphi_1) + \mathcal{O}_\theta(\theta^{-|\Lambda|}\lambda_*^{-n}|\varphi_2|_{\mathcal{C}^1}|\varphi_1|_{\mathcal{C}^0}).\end{aligned}$$

□

The last assertion of Theorem 2.1 concerns the distance between the perturbed and unperturbed measure measured in the $|\cdot|_\theta$ norm.

Lemma 8.3. *There exist $\theta_* \in (0,1)$ such that, for each $\theta \in (\theta_*, 1)$ and $\varepsilon \in (0, \varepsilon_\theta)$, holds true*

$$|\mu_0 - \mu_\varepsilon|_\theta \leq C_\theta\varepsilon\ln\varepsilon^{-1}.$$

Proof. By Lemma 4.4, for each box Λ and each probability measure $\mu \in \mathcal{B}_{\theta=1}$,

$$|F_{0*}\mu - F_{\varepsilon *}\mu|_\Lambda \leq D\varepsilon(|\Lambda| + r)(\|\mu\|_{\Lambda+r} + 1) \leq D\varepsilon(|\Lambda| + r)(\|\mu\|_{\theta=1} + 1)\ .$$

Thus, observing that $|F_{0*}|_\theta = 1$,

$$|F_{0*}^n\mu_0 - F_{\varepsilon *}^n\mu_0|_\theta \leq \sum_{k=0}^{n-1}|(F_{0*} - F_{\varepsilon *})F_{\varepsilon *}^{n-k-1}\mu_0|_\theta \leq nC_\theta\varepsilon\ .$$

Accordingly,

$$|\mu_0 - \mu_\varepsilon|_\theta \leq |\mu_\varepsilon - F_{\varepsilon *}^n\mu_0|_\theta + |F_{\varepsilon *}^n\mu_0 - F_{0*}^n\mu_0|_\theta \leq C_\theta\{\lambda_*^{-n} + n\varepsilon\}$$

and the result follows by choosing n proportional to $\ln\varepsilon^{-1}$. □

References

[1] V.Baladi, M.Degli Esposti, S.Isola, E.Järvenpää, A.Kupiainen, *The spectrum of weakly coupled map lattices*, J. Math. Pures Appl. **77** (1998) 539–584.

[2] V.Baladi, H.H.Rugh, *Floquet spectrum of weakly coupled map lattices* Comm. Math. Phys. **220** (2001), no. 3, 561–582.

[3] J.Bricmont, A.Kupiainen, *Coupled analytic maps*, Nonlinearity **8** (1995), no. 3, 379–396.

[4] J.Bricmont, A.Kupiainen, *High temperature expansions and dynamical systems*, Comm. Math. Phys. **178** (1996), no. 3, 703–732.

[5] L.A.Bunimovich, *Coupled map lattices: one step forward and two steps back*, Chaos, order and patterns: aspects of nonlinearity—the "gran finale" (Como, 1993). Phys. D **86** (1995), no. 1-2, 248–255.

[6] L.A.Bunimovich, Ya.G.Sinai, *Space-time chaos in coupled map lattices*, Nonlinearity **1** (1988), 491-516.

[7] L.C.Evans, R.F.Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, New York (1992).

[8] T.Fischer, H.H.Rugh, *Transfer operators for coupled analytic maps*, Ergodic Theory Dynam. Systems **20** (2000), no. 1, 109–143.

[9] E.Järvenpää, M.Järvenpää, *On the definition of SRB-measures for coupled map lattices*, Comm. Math. Phys. **220** (2001), no. 1, 1–12.

[10] M.Jiang, *Equilibrium states for lattice models of hyperbolic type*, Nonlinearity **8** (1995), no. 5, 631–659.

[11] M.Jiang, *Sinai-Ruelle-Bowen measures for lattice dynamical systems*, J. Statist. Phys. **111** (2003), no. 3-4, 863–902.

[12] M.Jiang, A.E.Mazel, *Uniqueness and exponential decay of correlations for some two-dimensional spin lattice systems*, J. Statist. Phys. **82** (1996), no. 3-4, 797–821.

[13] M.Jiang, Ya.Pesin, *Equilibrium measures for coupled map lattices: existence, uniqueness and finite-dimensional approximations*, Comm. Math. Phys. **193** (1998), no. 3, 675–711.

[14] Kaneko K. (Ed.), *Theory and Applications of Coupled Map Lattices*, Wiley, 1993.

[15] G.Keller, *An ergodic theoretic approach to mean field coupled maps*, Fractal geometry and stochastics, II (Greifswald/Koserow, 1998), 183–208, Progr. Probab., **46**, Birkhuser, Basel, 2000.

[16] G.Keller, *Mixing for finite systems of coupled tent maps*, Tr. Mat. Inst. Steklova **216** (1997), Din. Sist. i Smezhnye Vopr., 320–326; translation in Proc. Steklov Inst. Math. 1997, no. 1 (216), 315–32

[17] G.Keller, *Coupled map lattice via transfer operators on functions of bounded variation*, Stochastic and spatial structures of dynamical systems (Amsterdam, 1995), 71–80, Konink. Nederl. Akad. Wetensch. Verh. Afd. Natuurk. Eerste Reeks **45** North-Holland, Amsterdam, 1996.

[18] G.Keller, M.Künzle, *Transfer operators for coupled map lattices*, Ergodic Theory Dynam. Systems **12** (1992), no. 2, 297–318.

[19] G.Keller, R.Zweimüller, *Unidirectionally coupled interval maps: between dynamics and statistical mechanics*, Nonlinearity **15** (2002), no. 1, 1–24.

[20] C.Liverani, *Invariant measures and their properties. A functional analytic point of view*, Dynamical Systems. Part II: Topological Geometrical and Ergodic Properties of Dynamics. Pubblicazioni della Classe di Scienze, Scuola Normale Superiore, Pisa. Centro di Ricerca Matematica "Ennio De Giorgi" : Proceedings. Published by the Scuola Normale Superiore in Pisa (2004).

[21] C.Maes, A.van Moffaert, *Stochastic stability of weakly coupled lattice maps*, Nonlinearity **10** (1997) 715–730.

[22] Ya.B.Pesin, Ya.G.Sinai, *Space-time chaos in chains of weakly interacting hyperbolic mappings*, adv.Sov.Math. **3** (1991), 165-198.

[23] H.H.Rugh, *Coupled maps and analytic function spaces*, Ann. Sci. cole Norm. Sup. (4) **35** (2002), no. 4, 489–535.

[24] M. Schmitt, *BV-Spectral Theory for Coupled Map Lattices*, Ph.D. Thesis, Erlangen (2003).

[25] M. Schmitt, *Spectral theory for nonanalytic coupled map lattices*, Nonlinearity 17 (2004), no. 2, 671–689.

[26] D.L.Volevich, *Construction of an analogue of Bowen-Sinai measure for a multidimensional lattice of interacting hyperbolic mappings*, Russ. Acad. Math. Sbornink **79**, 347–363 (1994).

Gerhard Keller, Mathematisches Institut, Universität Erlangen, Bismarckstr. $1\frac{1}{2}$, 91054 Erlangen, Germany
E-mail address: keller@mi.uni-erlangen.de

Carlangelo Liverani, Dipartimento di Matematica, II Università di Roma (Tor Vergata), Via della Ricerca Scientifica, 00133 Roma, Italy.
E-mail address: liverani@mat.uniroma2.it