Pluripotential theory in Kähler geometry

In the last 50 years pluripotential theory has played a central role in order to solve geometric problems, such as the existence of special metrics (e.g. Kähler-Einstein, csck) on a compact Kähler manifold. This course aims to present some recent developments in pluripotential theory. These new tools are so flexible that allow to study singular settings: we will be then able to work with a singular variety and/or to search for singular metrics.

The course will consist of 8 lectures, each of 1h30 for a total of 12 hours. I propose the course to run in December/January. In particular:

- 2 lectures in the week 20-24 December (e.g. the 20th and the 21th)
- 3 lectures in the week 10-14 January (e.g. the 10th, the 11th and the 14th)
- 3 lectures in the week 17-21 January (e.g. the 17th, the 18th and the 21th).

Here below a detailed description of the program for each lecture:

Lecture 1 We start with some preliminaries: compact Kähler manifolds, big cohomology classes, volume of a big class, quasi-plurisubharmonic functions and θ-plurisubharmonic (θ-psh for short) functions where $\{\theta\}$ is a big cohomology class, Monge-Ampère measure and Monge-Ampère energy classes. We then define the *singularity class* of a θ-plurisubharmonic function. We basically follow [BEGZ10], and [GZ17].

Lecture 2 We give some preliminaries on Monge-Ampère capacities in big cohomology classes together with a sketch of the proof of "Kołodziej Theorem". We follow the presentation in [EGZ09] and [GZ17].

Lecture 3 We consider and study a particular and important set of θ-psh functions: *envelopes*. More precisely, given h a (suitably regular) function we consider:

$$P_\theta(h) := (\sup\{u \quad \theta - \text{psh} \quad u \leq h\})^* .$$

We study the regularity properties of such functions and the properties of their Monge-Ampère measure. We follow [DDNL18a], [DDNL19] and [DNT20].

Lecture 4 We give a nice characterization of the Monge-Ampère class $\mathcal{E}$ (in the big setting) in terms of envelopes. We follow [DDNL18b].

Lecture 5 We study the space of the singularity classes of θ-psh functions. We follow [DDNL21].

Lectures 6 and 7 We study and solve complex Monge-Ampére equations with *prescribed singularities*. As corollary we show the existence of singular Kähler-Einstein metrics. We follow [DDNL19].

Lecture 8 We prove the log-concavity conjecture for the volume of $(1,1)$-currents. We follow [DDNL19].

References

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[DDNL18a] Tamás Darvas, Eleonora Di Nezza, and Chinh H. Lu. Monotonicity of nonpluripolar products and complex Monge-Ampère equations with prescribed singularity. *Anal. PDE*, 11(8):2049–2087, 2018.

[DDNL18b] Tamás Darvas, Eleonora Di Nezza, and Chinh H. Lu. On the singularity type of full mass currents in big cohomology classes. *Compos. Math.*, 154(2):380–409, 2018.

[DDNL19] Tamás Darvas, Eleonora Di Nezza, and Chinh H. Lu. Log-concavity of volume and complex Monge-Ampère equations with prescribed singularity. *Mathematische Annalen*, pages 1–38, 2019.

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[EGZ09] Philippe Eyssidieux, Vincent Guedj, and Ahmed Zeriahi. Singular Kähler-Einstein metrics. *J. Amer. Math. Soc.*, 22(3):607–639, 2009.

[GZ17] Vincent Guedj and Ahmed Zeriahi. *Degenerate complex Monge-Ampère equations*, volume 26 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2017.