$$y''(x) + 4y(x) = x^2 \quad , \quad x \in \mathbb{R}$$

$$\lambda^2 + 4 = 0 \; ; \; \lambda_\pm = \pm 2i \; ;$$

$$y_1(x) = \cos(2x)$$

$$y_2(x) = \sin(2x)$$

$$0 \neq \pm 2i$$

$$g(x) = x^2 \longrightarrow \lambda_0 = 0 \quad , \quad \lambda_0 \neq \lambda_\pm$$

$$y_p(x) = Ax^2 + Bx + C$$

$$y_p'(x) = 2Ax + B$$

$$y_p''(x) = 2A$$

$$2A + 4(Ax^2 + Bx + C) = x^2$$

$$2A + 4C = 0 \qquad C = -\frac{1}{8}$$

$$4B = 0 \longrightarrow B = 0$$

$$4A = 1 \qquad A = \frac{1}{4}$$

$$y_p(x) = \frac{x^2}{4} - \frac{1}{8}$$

$$y(x) = c_1 \cos(2x) + c_2 \sin(2x) + \frac{x^2}{4} - 1/8$$

$$\begin{cases} y'' + 4y = x^2 \\ y(0) = \frac{1}{2} \\ y'(0) = 0 \end{cases}$$

$$y'(x) = -2c_1 \sin(2x) + 2c_2 \cos(2x) + \frac{x}{2}$$

$$\begin{cases} c_1 + 0 - \frac{1}{8} = \frac{1}{2} \\ 0 + 2c_2 = 0 \end{cases}$$

$$c_1 = 5/8 \quad , \quad c_2 = 0$$

$$y(x) = 5/8 \cos(2x) + \frac{x^2}{4} - \frac{1}{8}$$

$$\begin{cases} y'' + 4y' + 4y = e^{-2x}, \quad x \in \mathbb{R} \\ y(0) = 1 \\ y'(0) = -2 \end{cases}$$

$$\lambda^2 + 4\lambda + 4 = 0 \quad ; \quad (\lambda + 2)^2 = 0$$

$$\lambda_+ = \lambda_- = -2$$

$$y_1(x) = e^{-2x} \quad ; \quad y_2(x) = x e^{-2x}$$

$$g(x) = e^{-2x} \longrightarrow \lambda_0 = -2$$

$$\lambda_0 = \lambda_+ = \lambda_-$$

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x} + y_p(x)$$

$$y_p(x) = x^2 \left(A e^{-2x} \right)$$

$$y_p(x) = A x^2 e^{-2x}$$

$$y_p'(x) = 2Axe^{-2x} - 2Ax^2e^{-2x}$$

$$y_p''(x) = 2Ae^{-2x} - 8Axe^{-2x}$$

$$+ 4Ax^2e^{-2x};$$

$$4Ax^2\cancel{e^{-2x}} - 8Ax\cancel{e^{-2x}} + 2A\cancel{e^{-2x}}$$

$$+ 4\left(2Ax\cancel{e^{-2x}} - 2Ax^2\cancel{e^{-2x}}\right)$$

$$+ 4Ax^2\cancel{e^{-2x}} = \cancel{e^{-2x}}$$

$$\cancel{4Ax^2} - \cancel{8Ax} + 2A + \cancel{8Ax} - \cancel{8Ax^2}$$

$$+ \cancel{4Ax^2} = 1$$

$$A = \frac{1}{2}$$

$$y_p(x) = \frac{1}{2}x^2e^{-2x}$$

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x} + \frac{1}{2} x^2 e^{-2x}$$

$$y'(x) = \dots$$

$$y'(0) = \left(-2c_1 e^{-2x} + c_2 e^{-2x}\right)\Big|_{x=0} =$$

$$= -2c_1 + c_2$$

$$\begin{cases} y(0) = 1 \\ y'(0) = -2 \end{cases} \qquad \begin{cases} c_1 = 1 \\ -2c_1 + c_2 = -2 \end{cases}$$

$$c_1 = 1 \qquad c_2 = 0$$

$$y(x) = e^{-2x} + \frac{x^2}{2} e^{-2x}$$

$$y'''(x) + y(x) = 0$$

$$\lambda^3 + 1 = 0 \quad ;$$

$$\lambda^3 = -1 \quad ; \quad \lambda_k = e^{i\frac{\pi}{3}} e^{i\frac{2k\pi}{3}} \quad k = 0, 1, 2$$

$$\lambda_0 = e^{i\pi/3} \qquad \lambda_1 = e^{i\pi} \qquad \lambda_2 = e^{i\frac{5}{3}\pi}$$

$$\lambda_1 = -1 \;, \quad \lambda_\pm = e^{\pm i\frac{\pi}{3}} =$$

$$= \cos\left(\frac{\pi}{3}\right) \pm i \sin\left(\frac{\pi}{3}\right)$$

$$= \frac{1}{2} \pm i \frac{\sqrt{3}}{2}$$

$$y_1(x) = e^{\lambda_1 x} = e^{-x}$$

$$y_2(x) = e^{\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2} x\right)$$

$$y_3(x) = e^{\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2} x\right)$$

$$y(x) = c_1 e^{-x} + c_2 e^{x/2} \cos\left(\frac{\sqrt{3}}{2} x\right) + c_3 e^{x/2} \sin\left(\frac{\sqrt{3}}{2} x\right)$$

$$z^2 = 2 + 3i \quad ; \quad z^2 = x^2 - y^2 + 2ixy$$

$$\begin{cases} x^2 - y^2 = 2 \\ 2xy = 3 \end{cases} \qquad \begin{cases} x^2 - \frac{9}{4}\frac{1}{x^2} = 2 \\ y = \frac{3}{2x} \end{cases}$$

$$4x^4 - 8x^2 - 9 = 0 \quad ; \quad t = x^2$$

$$4t^2 - 8t - 9 = 0 ,$$

$$t_\pm = \frac{4 \pm \sqrt{16+36}}{4} = \frac{4 \pm 2\sqrt{13}}{4}$$

$$= \frac{2 \pm \sqrt{13}}{2}$$

$$x^2 = t_+ \quad ; \quad x = \pm \frac{\sqrt{2+\sqrt{13}}}{\sqrt{2}}$$

$$y = \pm \frac{3}{2} \frac{\sqrt{2}}{\sqrt{2+\sqrt{13}}}$$

$$\begin{cases} y'(x) = y(x) + \dfrac{e^{x}(x+2)}{\sqrt{x^2+4x+5}} \\ y(0) = 0 \end{cases}$$

$$a_0(x) = 1, \quad a_1(x) = \frac{e^{x}(x+2)}{\sqrt{x^2+4x+5}}$$

$$\int a_0(x)\,dx = \int dx = x + C$$

$$A(x) = x$$

$$y(x) = c e^{A(x)} + e^{A(x)} \int_0^x e^{-A(t)} a_1(t)\,dt$$

$$= c e^{x} + e^{x} \int_0^x e^{-t} \frac{e^{t}(t+2)}{\sqrt{t^2+4t+5}}\,dt =$$

$$= ce^x + e^x \int_0^x \frac{(t+2)}{\sqrt{(t+2)^2+1}}\, dt$$

$$y(0) = 0$$

$$c + 1 \cdot 0 = 0 \longrightarrow c = 0$$

$$y(x) = e^x \int_0^x \frac{(t+2)}{\sqrt{(t+2)^2+1}}\, dt$$

$$\int_0^x \frac{t+2}{\sqrt{1+(t+2)^2}}\, dt = \int_2^{2+x} \frac{y\, dy}{\sqrt{1+y^2}} =$$

$$t+2 = y$$

$$dt = dy$$

$$= \sqrt{1+y^2}\,\Big|_2^{2+x} = \sqrt{1+(2+x)^2} - \sqrt{5}$$

$$= \sqrt{x^2+4x+5} - \sqrt{5}$$

$$y(x) = e^{x}\left(\sqrt{x^2+4x+5} - \sqrt{5}\right)$$

$$x \longrightarrow x$$

$$\begin{cases} y'(x) = \dfrac{3x^2-1}{x^3-x}\, y(x) + x^3 \\ y\left(\frac{1}{3}\right) = y_0 \end{cases}$$

$$a_0(x) = \frac{3x^2-1}{x^3-x}, \quad a_1(x) = x^3$$

$$\mathrm{dom}(a_0) = (-\infty,-1) \cup (-1,0) \cup (0,1) \cup (1,+\infty)$$

$$\frac{1}{3} \in (0,1) \longrightarrow I = (0,1)$$

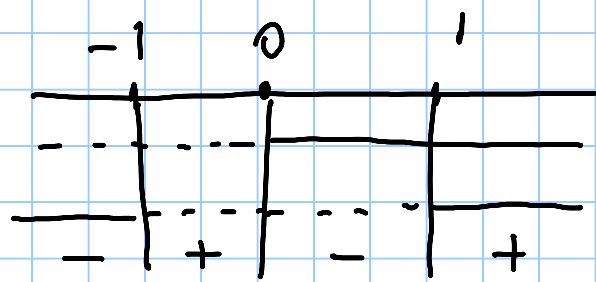
$$\int a_0(x)\,dx = \int \frac{3x^2-1}{x^3-x}\,dx =$$

$$= \log|x^3-x| + c$$

$$A(x) = \log|x^3-x| \quad ;$$

$$x \in (0,1)$$

$$x^3 - x = x(x^2-1)$$



$$|x^3-x| = x - x^3 \quad , x \in (0,1)$$

$$A(x) = \log(x-x^3) \quad , x \in (0,1)$$

$$y(x) = c\,e^{A(x)} + e^{A(x)} \int_{\frac{1}{3}}^{x} e^{-A(t)} a_1(t)\, dt =$$

$$= c\left(x - x^3\right) + \left(x - x^3\right) \int_{\frac{1}{3}}^{x} \frac{t^3}{t - t^3}\, dt$$

$$x \in (0,1)$$

$$y\left(\tfrac{1}{3}\right) = y_0$$

$$c\left(\tfrac{1}{3} - \tfrac{1}{27}\right) = y_0 \; ; \; \left(\tfrac{9}{27} - \tfrac{1}{27}\right) c = y_0$$

$$c = \tfrac{27}{8} y_0$$

$$\int_{\frac{1}{3}}^{x} \frac{t^3}{t - t^3}\, dt = \int_{\frac{1}{3}}^{x} \frac{t^3 - t + t}{t - t^3}\, dt =$$

$$= \int_{\frac{1}{3}}^{x} \left(-1 + \frac{t}{t-t^3} \right) dt = -\left(x - \frac{1}{3}\right) + \int_{\frac{1}{3}}^{x} \frac{dt}{1-t^2} \;;$$

$$\int_{\frac{1}{3}}^{x} \frac{dt}{1-t^2} = -\int_{\frac{1}{3}}^{x} \left(\frac{1}{2} \frac{1}{t-1} - \frac{1}{2} \frac{1}{t+1} \right) dt$$

$$= -\frac{1}{2} \log \left| \frac{t-1}{t+1} \right| \Bigg|_{\frac{1}{3}}^{x} = \left\{ \begin{array}{l} \frac{1}{3} \in (0,1), \\ x \in (0,1) \Rightarrow t \in (0,1) \end{array} \right.$$

$$\text{Se } t \in (0,1), \; \frac{t-1}{t+1} < 0 \Big\} =$$

$$= -\frac{1}{2}\log\left(\frac{1-t}{t+1}\right)\Bigg|_{\frac{1}{3}}^{x} = -\frac{1}{2}\log\left(\frac{1-x}{x+1}\right)$$

$$+\ \frac{1}{2}\log\frac{2/3}{4/3} = -\frac{1}{2}\log\left(\frac{1-x}{x+1}\right) - \frac{1}{2}\log(2)$$

$$y(x) = \frac{27}{8}y_0\left(x - x^3\right) - \left(x - x^3\right)\left(x - \frac{1}{3}\right)$$

$$- \frac{1}{2}\log\left(\frac{1-x}{x+1}\right) - \frac{1}{2}\log(2),$$

$$x \in (0,1)$$

$$\begin{cases} y'(x) = \dfrac{1}{x\log(x)}\, y(x) + \left(\log(x)\right)^4 \\ y\left(\frac{1}{e}\right) = -1 \end{cases}$$

$$a_0(x) = \frac{1}{x\log(x)} \; ; \; a_1(x) = \left(\log(x)\right)^4$$

$$dom(a_0) = (0,1) \cup (1,+\infty)$$

$$I = (0,1) \ni \frac{1}{e}$$

$$\int a_0(x)\,dx = \int \frac{dx}{x\log(x)} = \log|\log(x)| + c$$

$$A(x) = \log|\log(x)| = \log\left(\log\frac{1}{x}\right), \quad x \in (0,1)$$

$$y(x) = c\,e^{A(x)} + e^{A(x)} \int_{\frac{1}{e}}^{x} e^{-A(t)} a_1(t)\,dt =$$

$$= c\,|\log(x)| + |\log(x)| \int_{e^{-1}}^{x} \frac{(\log(t))^4}{|\log(t)|}\,dt$$

$$= -c\log(x) - \log(x) \int_{e^{-1}}^{x} \frac{(\log(t))^4}{-\log(t)}\,dt =$$

$$= -c\log(x) + \log(x) \int_{e^{-1}}^{x} \log^3(t)\,dt$$

$$y\left(\tfrac{1}{e}\right) = -1 \qquad -c\log\left(\tfrac{1}{e}\right) = -1$$

$$c = \frac{1}{\log\left(\frac{1}{e}\right)} = -\frac{1}{\log(e)} = -1$$

$$\int_{e^{-1}}^{x} \log^3(t)\,dt = t\log^3(t)\Big|_{e^{-1}}^{x} - \int_{e^{-1}}^{x} 3t\log^2(t)\frac{dt}{t}$$

$$= t\log^3(t)\Big|_{e^{-1}}^{x} - 3\int_{e^{-1}}^{x} \log^2(t)\,dt =$$

$$= t\log^3(t)\Big|_{e^{-1}}^{x} - 3t\log^2(t)\Big|_{e^{-1}}^{x} + 6\int_{e^{-1}}^{x}\log(t)\,dt$$

$$= t\log^3(t) - 3t\log^2(t) + 6t\log(t) - 6t\Big|_{e^{-1}}^{x}$$

$$= t\left(\log^3(t) - 3\log^2(t) + 6\log(t) - 6\right)\Big|_{e^{-1}}^{x} =$$

$$= x\left(\log^3(x) - 3\log^2(x) + 6\log(x) - 6\right)$$

$$- e^{-1}\left((-1)^3 - 3(-1)^2 + 6(-1) - 6\right) =$$

$$= x\left(\log^3(x) - 3\log^2(x) + 6\log(x) - 6\right)$$

$$+ \frac{16}{e}$$

$$y(x) = \log(x)\left(1 + 16e^{-1}\right) +$$

$$+ x\log(x)\left(\log^3(x) - 3\log^2(x) + 6\log(x) - 6\right)$$

$$x \in (0,1)$$

$$\begin{cases} y'(x) = \dfrac{\sqrt{x+2}}{\sqrt{2}\,x}\, y(x) + e^{\sqrt{2x+4}} \\ y(1) = 0 \end{cases}$$

$$a_0(x) = \frac{\sqrt{x+2}}{\sqrt{2}\,x}, \quad a_1(x) = e^{\sqrt{2(x+2)}}$$

$$\underline{I} = (0, +\infty) \ni 1$$

$$\int a_0(x)\,dx = \int \frac{\sqrt{x+2}}{\sqrt{2}\,x}\,dx =$$

$$\left\{ \sqrt{x+2} = y \qquad \begin{matrix} x+2 = y^2 \\ dx = 2y\,dy \end{matrix} \right\}$$

$$= \frac{1}{\sqrt{2}} \int \frac{y}{y^2-2} \cdot 2y\,dy = \sqrt{2} \int \frac{y^2}{y^2-2}\,dy =$$

$$= \sqrt{2}\,y + \sqrt{2} \int \frac{2\,dy}{y^2-2} =$$

$$= \sqrt{2}\,y + 2\sqrt{2} \int \frac{dy}{y^2-2} =$$

$$= \sqrt{2}\,y + \frac{2\sqrt{2}}{2\sqrt{2}} \int \left(\frac{1}{y-\sqrt{2}} - \frac{1}{y+\sqrt{2}} \right) dy$$

$$= \sqrt{2}\,y + \log \left| \frac{y-\sqrt{2}}{y+\sqrt{2}} \right| + c$$

$$= \sqrt{2}\,\sqrt{x+2} + \log \left| \frac{\sqrt{x+2} - \sqrt{2}}{\sqrt{x+2} + \sqrt{2}} \right| + c$$

$$= \sqrt{2x+4} + \log\left(\frac{\sqrt{x+2}-\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}\right) + c$$

$$x \in (0, +\infty)$$

$$A(x) = \sqrt{2x+4} + \log\left(\frac{\sqrt{x+2}-\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}\right)$$

$$y(x) = c\,e^{A(x)} + e^{A(x)} \int_1^x e^{-A(t)} a_1(t)\,dt$$

$$y(1) = 0 \quad ;$$

$$c\,e^{A(1)} + 0 = 0 \quad ; \quad c = 0$$

$$y(x) = e^{\sqrt{2x+4}} \frac{\sqrt{x+2}-\sqrt{2}}{\sqrt{x+2}+\sqrt{2}} \int_1^x \frac{\sqrt{t+2}+\sqrt{2}}{\sqrt{t+2}-\sqrt{2}}\,dt$$

$$\int_1^x \frac{\sqrt{t+2}+\sqrt{2}}{\sqrt{t+2}-\sqrt{2}}\,dt = \left\{\begin{array}{l}\sqrt{t+2}=y\\ t+2=y^2\end{array}\right.$$

$$\int_{\sqrt{3}}^{\sqrt{x+2}} \frac{y+\sqrt{2}}{y-\sqrt{2}}\cdot 2y\,dy = 2\int_{\sqrt{3}}^{\sqrt{x+2}} \frac{y\,(y+\sqrt{2})}{y-\sqrt{2}}\,dy$$

$$\left\{ y-\sqrt{2}=s \right\} = 2\int_{\sqrt{3}-\sqrt{2}}^{\sqrt{x+2}-\sqrt{2}} \frac{(s+\sqrt{2})(s+2\sqrt{2})}{s}\,ds$$

$$= 2\int_{\sqrt{3}-\sqrt{2}}^{\sqrt{x+2}-\sqrt{2}} \left(s+3\sqrt{2}+4/s\right)ds =$$

$$= 2\left[\left(\frac{\sqrt{x+2}-\sqrt{2}}{2}\right)^2 - \left(\frac{\sqrt{3}-\sqrt{2}}{2}\right)^2\right]$$

$$+ 6\sqrt{2}\left[\sqrt{x+2}-\sqrt{3}\right]$$

$$+ 8\log\left(\frac{\sqrt{x+2}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right) =$$

$$= x+2+2-2\sqrt{2}\sqrt{x+2}-3-2+2\sqrt{3}\sqrt{2}$$

$$+6\sqrt{2}\left[\sqrt{x+2}-\sqrt{3}\right]+8\log(\dots) =$$

$$= x-1 + \cancel{2\sqrt{2}}\left(\sqrt{3}-\sqrt{x+2}\right)$$

$$+ \underset{4}{\cancel{6}}\sqrt{2}\left(\sqrt{x+2}-\sqrt{3}\right)$$

$$+ 8\log\left(\frac{\sqrt{x+2}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right)$$

$y(x) = \dots\dots$ $\blacksquare$