$$y'' + 4y = \sin(x)$$

$$\lambda_0 = i$$

$$\begin{cases} y'' + 4y = 0 \\ \lambda^2 + 4 = 0 \end{cases} \longrightarrow \lambda_\pm = \pm 2i$$

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

$$y_1(x) = \cos(2x)$$

$$y_2(x) = \sin(2x)$$

$$\lambda_0 \neq \lambda_\pm$$

$$\rightarrow y_p(x) = A\cos(x) + B\sin(x)$$

$$y_p'(x) = -A\sin(x) + B\cos(x)$$

$$y_p''(x) = -A\cos(x) - B\sin(x)$$

$$-A\cos(x) - B\sin(x) + 4\left(A\cos(x) + B\sin(x)\right) = \sin(x)$$

$$3A\cos(x) + 3B\sin(x) = \sin(x)$$

$$3A = 0$$

$$3B = 1 \qquad B = \frac{1}{3}$$

$$y_p(x) = \frac{1}{3}\sin(x)$$

$$y(x) = c_1\cos(2x) + c_2\sin(2x) + \frac{1}{3}\sin(x)$$

$$y'' + 4y = \cos(2x)$$

$$g(x) \rightarrow \lambda_0 = 2i$$

$$\lambda_\pm = \pm 2i \qquad \boxed{\lambda_0 = \lambda_+}$$

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

$$y_1(x) = \cos(2x)$$

$$y_2(x) = \sin(2x)$$

$$y_p(x) = x\left(A\cos(2x) + B\sin(2x)\right)$$

$$y_p'(x) = A\cos(2x) + B\sin(2x) + x\left(-2A\sin(2x) + 2B\cos(2x)\right)$$

$$y_p''(x) = 2\left(-2A\sin(2x) + 2B\cos(2x)\right) + x\left(-4A\cos(2x) - 4B\sin(2x)\right);$$

$$-4A\sin(2x) + 4B\cos(2x) - \cancel{4Ax\cos(2x)} - \cancel{4Bx\sin(2x)} + \cancel{4Ax\cos(2x)} + \cancel{4Bx\sin(2x)} = \cos(2x)$$

$$-4A\sin(2x)+4B\cos(2x)=\cos(2x)$$

$$A=0 \qquad B=\frac{1}{4}$$

$$y_p(x)=\frac{x}{4}\sin(2x)$$

$$y(x)=c_1\cos(2x)+c_2\sin(2x)+\frac{x}{4}\sin(2x)$$

$$y''+4y=x\cos(2x)$$

$$\lambda_0=2i$$

$$\boxed{\lambda_0=\lambda_+}$$

$$\lambda_\pm=\pm 2i$$

$$y_1(x)=\cos(2x),\ y_2(x)=\sin(2x)$$

$$y(x)=c_1y_1(x)+c_2y_2(x)+y_p(x)$$

$$y_p(x) = x\Big((Ax+B)\cos(2x) + (Cx+D)\sin(2x)\Big)$$

$$y_p'(x) = \ldots$$

$$y_p''(x) = \ldots \longrightarrow \text{SOSTITUIRE in}$$

$$y_p'' + 4y_p = x\cos(2x)$$

DETERMINARE A, B, C, D

$$y_p(x) = \frac{x}{16}\cos(2x) + \frac{x^2}{8}\sin(2x)$$

$$y'' + 3y' = \underbrace{(x^3 - 1)e^x}$$

$g(x) \to \lambda_0 = 1$

$g(x) = P_3(x)\, e^{\alpha x}$

$$\boxed{\lambda_0 \neq 0, \ \lambda_0 \neq -3}$$

$$\lambda^2 + 3\lambda = 0$$

$$\lambda_1 = 0 \qquad \lambda_2 = -3$$

$$Y_1(x) = e^{\lambda_1 x} = 1$$

$$Y_2(x) = e^{\lambda_2 x} = e^{-3x}$$

$$Y'' + 3Y' = 0$$

$$Y(x) = c_1 \cdot Y_1(x) + c_2 Y_2(x) + Y_p(x) =$$

$$= c_1 + c_2 e^{-3x} + Y_p(x)$$

$$Y_p(x) = (A + Bx + Cx^2 + Dx^3)\, e^{x}$$

$$Y_p'(x) = (B + 2Cx + 3Dx^2) e^{x} +$$

$$(A + Bx + Cx^2 + Dx^3) e^{x}$$

$$Y_p''(x) = (2C + 6Dx) e^{x} +$$

$$+ 2(B + 2Cx + 3Dx^2) e^{x} +$$

$$+\left(A+Bx+Cx^2+Dx^3\right)e^x \quad ;$$

$$\left[2C+6Dx+2B+4Cx+6Dx^2+\right.$$
$$\left.+A+Bx+Cx^2+Dx^3\right]\cancel{e^x}$$

$$+3\left[B+2Cx+3Dx^2+A+Bx+Cx^2+Dx^3\right]\cancel{e^x}$$
$$=\left(x^3-1\right)\cancel{e^x}$$

$$2C+2B+A+3(B+A)=-1$$
$$6D+4C+B+3(2C+B)=0$$
$$6D+C+\quad 3(3D+C)=0$$
$$D\quad +3D\quad =1$$

$$\begin{cases} 2C+5B+4A=-1 \\ 10C+4B+6D=0 \\ 4C\quad +15D=0 \\ D=\frac{1}{4} \end{cases}$$

$$\begin{cases} 4A = -1 - 5B - 2C = \dots \\ 4B = -6D - 10C = -\frac{6}{4} + \frac{10 \cdot 15}{16} = \boxed{\frac{63}{8}} \\ C = -\frac{15}{4}D = -\frac{15}{16} \\ D = \frac{1}{4} \end{cases}$$

$$B = \frac{63}{32}$$

$$A = -\frac{287}{128}$$

$$Y(x) = c_1 + c_2 e^{-3x} + \left(-\frac{287}{128} + \frac{63}{32}x - \frac{15}{16}x^2 + \frac{x^3}{4}\right)e^x$$

$$Y'' + 3Y' = 0 \qquad Y_1 = 1 \qquad Y_2 = e^{-3x}$$

SI PUO'
DIMINUIRE IL GRADO
DELL'EQUAZIONE DIFFERENZIALE
PERCHE' NON C'E' IL TERMINE Y

$$z(x) = y'(x) \longrightarrow z' + 3z = 0 \longrightarrow$$

$$z(x) = \tilde{c}_2 e^{-3x}$$

$$y(x) = \tilde{c}_2 \int e^{-3x} dx = -\frac{\tilde{c}_2}{3} e^{-3x} + c_1$$

$$y(x) = c_1 + c_2 e^{-3x}$$

$$y'' + a y' = f(x)$$

$$z(x) = y'(x) \longrightarrow z'(x) + a\, z(x) = f(x)$$

1° ORDINE
NON OMOGENEA

$$z(x)$$

$$y(x) = \int z(x)\, dx$$

$$y'' - y = x + 2e^{-x}$$

$$x \to \lambda_{0,1} = 0 \qquad 2e^{-x} \to \lambda_{0,2} = -1$$

Equazione LINEARE (NON OMOGENEA)

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

$$\lambda_{\pm} = \pm 1 ; \quad y_1(x) = e^{x} , \quad y_2(x) = e^{-x}$$

$$\longrightarrow y_p(x) = y_{p,1}(x) + y_{p,2}(x)$$

$$y''_{p,1} - y_{p,1} = x$$

$$y''_{p,2} - y_{p,2} = 2e^{-x}$$

Ex

$$y''(x) = x^2$$

$$y'(x) = \int x^2 \, dx = c_2 + x^3/3$$

$$y(x) = \int \left(c_2 + x^3/3 \right) dx =$$

$$c_1 + c_2 x + x^4/12 =$$

$$c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

$$y'' = x^2 \qquad , \quad \lambda_0 = 0$$

$$\lambda^2 = 0 \qquad \lambda_1 = 0 = \lambda_2$$

$$y_1(x) = 1 \quad , \quad y_2(x) = x\, y_1(x) = x$$

$$y(x) = c_1 + c_2 x + y_p(x)$$

$$y_p(x) = x^2 \left(A x^2 + B x + C \right)$$

$$y_p(x) = Ax^4 + Bx^3 + Cx^2$$

$$y_p'(x) = 4Ax^3 + 3Bx^2 + 2Cx$$

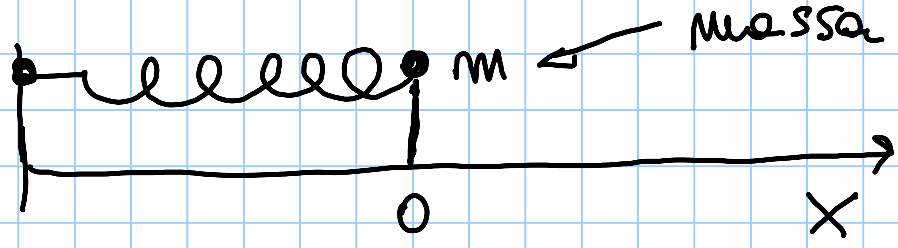
$$y_p''(x) = 12Ax^2 + 6Bx + 2C.$$

$$12Ax^2 + 6Bx + 2C = x^2 \Longrightarrow A = \frac{1}{12}$$

$$y'' - 2y' + y = xe^x$$

Ex

OSCILLATORE ARMONICO



$x(t)$ = posizione della TESTA della molla al tempo t

$x=0$: POSIZIONE di RIPOSO

Se $x \neq 0$ la molla ESERCITA una forza di richiamo PROPORZIONALE ad x,

$$F(x) = -kx \, , \quad k > 0$$

k COSTANTE (DIPENDE DALLA MOLLA)

LEGGE di NEWTON

$$m\ddot{x}(t) = F(x(t))$$

$\ddot{x}(t) \doteq$ ACCELERAZIONE al Tempo t

$$\ddot{x}(t) = \frac{d^2 x(t)}{dt^2}$$

$$m\frac{d^2 x(t)}{dt^2} = -kx(t)$$

$$\begin{cases} \ddot{x}(t) + \frac{k}{m}x(t) = 0 \\ x(0) = x_0 \\ \dot{x}(0) = y_0 \end{cases}$$

Equazione CARATTERISTICA $\lambda^2 = -\frac{k}{m}$, $\lambda = \pm i\sqrt{\frac{k}{m}}$

$\omega := \sqrt{\frac{k}{m}}$ PULSAZIONE

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$\dot{x}(t) = -\omega A \sin(\omega t) + \omega B \cos(\omega t)$$

$$\begin{cases} A = x_0 \\ \omega B = y_0 \end{cases} \longrightarrow \quad \begin{aligned} A &= x_0 \\ B &= y_0/\omega \end{aligned}$$

$$x(t) = x_0 \cos(\omega t) + \frac{y_0}{\omega} \sin(\omega t)$$

MOTO ARMONICO

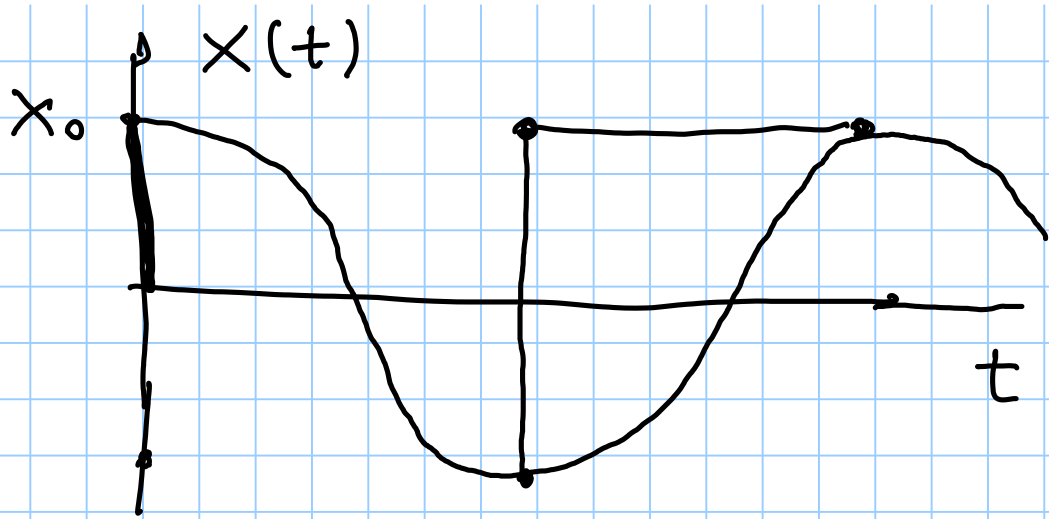
Se $y_0 = 0 \Rightarrow \dot{x}(0) = 0$ VELOCITÀ INIZIALE NULLA

$$x(t) = x_0 \cos(\omega t) =$$

$$x_0 \cdot \cos\left(\sqrt{\tfrac{k}{m}}\, t\right)$$

Ampiezza dell'oscillazione =

$$2 \max_{\mathbb{R}} |x(t)| = 2\,|x_0|$$



OSCILLATORE ARMONICO SMORZATO

$$F(x) = -kx - \beta \dot{x}, \quad \beta > 0$$

FORZA di ATTRITO

$$m\ddot{x} = -kx - \beta\dot{x}$$

$$\begin{cases} \ddot{x} + \frac{\beta}{m}\dot{x} + \frac{k}{m}x = 0 \\ x(0) = x_0 \\ \dot{x}(0) = v_0 \end{cases}$$

$$\lambda^2 + \frac{\beta}{m}\lambda + \omega^2 = 0 \quad , \quad \omega = \sqrt{\frac{K}{m}}$$

$$2\lambda_\pm = -\frac{\beta}{m} \pm \sqrt{\left(\frac{\beta}{m}\right)^2 - 4\omega^2}$$

- $\left(\frac{\beta}{m}\right)^2 > 4\omega^2$;
- $\left(\frac{\beta}{m}\right)^2 < 4\omega^2$

$$\left(\frac{\beta}{m}\right)^2 > 4\omega^2 \quad , \quad \frac{\beta^2}{m^2} > 4\frac{K}{m}$$

$\boxed{\Delta > 0}$

$$\beta^2 > 4mk$$

$$\boxed{\beta > \sqrt{4mk}}$$

$$\lambda_1 = \frac{1}{2}\left(-\frac{\beta}{m} + \sqrt{\Delta}\right)$$

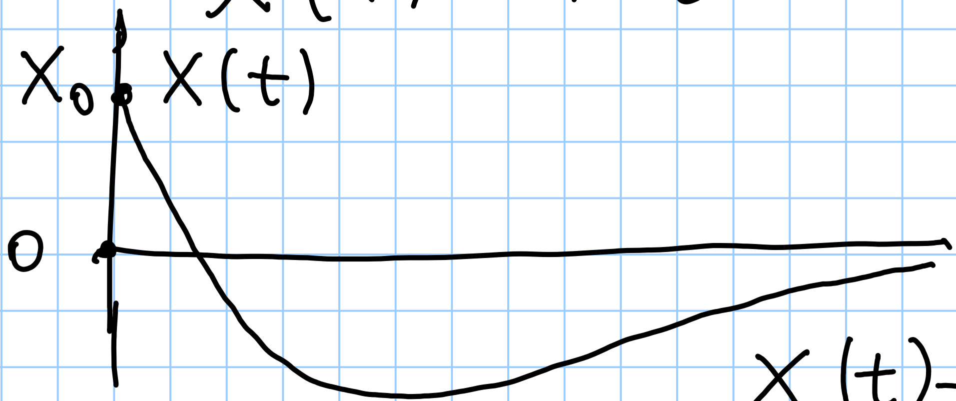
$$\lambda_2 = \frac{1}{2}\left(-\frac{\beta}{m} - \sqrt{\Delta}\right)$$

$$0 < \Delta = \left(\frac{\beta}{m}\right)^2 - 4\omega^2 < \left(\frac{\beta}{m}\right)^2$$

$$\sqrt{\Delta} < \frac{\beta}{m}$$

$$\boxed{\lambda_2 < \lambda_1 < 0}$$

$$x(t) = A e^{\lambda_1 t} + B e^{\lambda_2 t}$$



$$x(t) \longrightarrow 0, \quad t \longrightarrow +\infty$$

NON CI SONO OSCILLAZIONI

$$\left(\frac{\beta}{m}\right)^2 < 4\omega^2$$

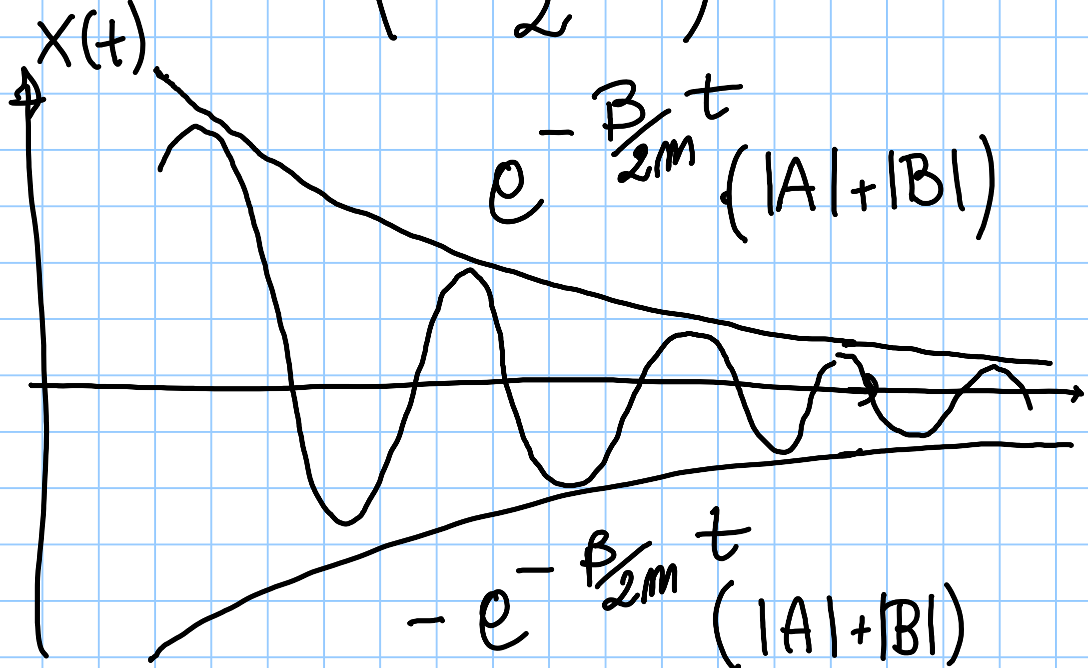
$$\boxed{\Delta < 0}$$

$$\beta < \sqrt{4mk}$$

$$\lambda_\pm = \frac{1}{2}\left(-\frac{\beta}{m} \pm i\sqrt{|\Delta|}\right)$$

$$x(t) = A e^{-\frac{\beta}{2m}t} \cos\left(\frac{\sqrt{|\Delta|}}{2} t\right) +$$

$$B e^{-\frac{\beta}{2m}t} \sin\left(\frac{\sqrt{|\Delta|}}{2} t\right)$$



OSCILLATORE ARMONICO FORZATO (RISONANZA)

$$\ddot{x} + \omega^2 x = \cos(\mu t), \quad \mu > 0$$

$$F(x) = -kx + m\cos(\mu t)$$

$\mu \neq \omega$

$\mu = \omega$

$$\mu \neq \omega; \quad x_p(t) = A_1 \cos(\mu t) + B_1 \sin(\mu t)$$

$$x(t) = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$$

$$|x(t)| \leq |c_1| + |c_2| + |A_1| + |B_1|$$

$\mu = \omega$

$$X_p(t) = t A_1 \cos(\omega t) + t B_1 \sin(\omega t)$$