$$y'(x) = -\frac{2}{3} x^{-\frac{1}{3}} y(x) + 4x \,,$$

$$a_0(x) = -\frac{2}{3} \frac{1}{x^{\frac{1}{3}}} \,, \quad a_1(x) = 4x$$

$$\int a_0(x)\,dx = -\frac{2}{3} \int x^{-\frac{1}{3}}\,dx =$$

$$= -\frac{2}{3}\,\frac{3}{2}\, x^{2/3} + c = -x^{\frac{2}{3}} + c$$

$$A(x) = -x^{2/3}$$

$$y(x) = e^{-x^{2/3}} \int e^{x^{2/3}} 4x\,dx =$$

$$= 4 e^{-x^{2/3}} \int x e^{x^{2/3}}\,dx$$

$$\int x\, e^{x^{2/3}}\, dx = \int t^{3/2} e^{t} \tfrac{3}{2} t^{\frac{1}{2}}\, dt$$

$$t = x^{2/3} \; ; \quad x = \pm t^{3/2}$$

$\boxed{x > 0}$ $\quad dx = \frac{3}{2} t^{\frac{1}{2}} dt \; , \quad x = t^{3/2}$

$$= \tfrac{3}{2}\int t^2 e^t\, dt = \frac{3}{2} t^2 e^t - 3\int t\, e^t dt$$

$$= \frac{3}{2} t^2 e^t - 3t\, e^t + 3\int e^t\, dt =$$

$$= \frac{3}{2} t^2 e^t - 3t e^t + 3\, e^t + c =$$

$$= \tfrac{3}{2} x^{4/3} e^{x^{2/3}} - 3\, x^{2/3} e^{x^{2/3}} + 3\, e^{x^{2/3}} + c$$

$$y(x) = 4\left(\tfrac{3}{2} x^{4/3} - 3x^{2/3} + 3\right) + c\, e^{-x^{2/3}}.$$

Ex VERIFICARE CHE NEL CASO $x < 0$ E' LO STESSO

$$\begin{cases} y'(x) = -\dfrac{\cos(x)}{2+\sin(x)}\,y(x) + \cos(x) \\ y(0) = 3 \end{cases}$$

$$a_0(x) = -\frac{\cos(x)}{2+\sin(x)} \quad , \quad \operatorname{dom}(a_0) = \mathbb{R}$$

$2+\sin(x) \geq 1$

$$a_1(x) = \cos(x) \quad , \quad \operatorname{dom}(a_1) = \mathbb{R}$$

$$I = \mathbb{R}$$

$$\int a_0(x)\,dx = -\int \frac{\cos(x)}{2+\sin(x)}\,dx =$$

$$= -\log(2+\sin(x)) + c$$

$$A(x) = -\log(2+\sin(x))$$

$\operatorname{dom}(A) = \mathbb{R}$; $2+\sin(x) \geq 1$ $\forall x \in \mathbb{R}$

$$y(x) = c\,e^{A(x)} + e^{A(x)}\int_0^x e^{-A(t)}\cos(t)\,dt$$

$$= c\,\frac{1}{2+\sin(x)} + y_p(x)$$

$$y(0) = 3\,; \qquad \frac{c}{2} + 0 = 3 \rightarrow c = 6$$

$$y_p(x) = e^{A(x)}\int_0^x \big(2+\sin(t)\big)\cos(t)\,dt$$

$$\int_0^x \big(2+\sin(t)\big)\cos(t)\,dt =$$

$$= 2\sin(x) + \int_0^x \sin(t)\cos(t)\,dt =$$

$$= 2\sin(x) + \frac{1}{2}\sin^2(x) = \frac{\sin(x)}{2}\big(4+\sin(x)\big)$$

$$y(x) = \frac{6}{2+\sin(x)} + \frac{1}{2}\,\frac{\sin(x)\left(4+\sin(x)\right)}{2+\sin(x)}$$

EQUAZIONI LINEARI OMOGENEE del 2° ORDINE A COEFFICENTI COSTANTI

$$y'' - 2y' - 3y = 0$$

$$y(x) = e^{\lambda x}, \quad y'(x) = \lambda e^{\lambda x}$$

$$y''(x) = \lambda^2 e^{\lambda x}$$

$$\lambda^2 \cancel{e^{\lambda x}} - 2\lambda \cancel{e^{\lambda x}} - 3 \cancel{e^{\lambda x}} = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

EQUAZIONE CARATTERISTICA

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1+3}}{1} = 1 \pm 2 \begin{cases} 3 \\ -1 \end{cases}$$

$$\lambda_1 = 3$$

$$\lambda_2 = -1$$

$$y_1 = y_+(x) = e^{3x} = e^{\lambda_1 x}$$

$$y_2 = y_-(x) = e^{-x} = e^{\lambda_2 x}$$

$$y(x) = c_1 e^{3x} + c_2 e^{-x}$$

$$= c_1 y_1(x) + c_2 y_2(x)$$

$$\begin{cases} y'' - 2y' - 3y = 0 \\ y(1) = 2 \\ y'(1) = 0 \end{cases}$$

CONDIZIONI INIZIALI

$$y(x) = c_1 e^{3x} + c_2 e^{-x}$$

$$y(1) = c_1 e^{3} + c_2 e^{-1} = 2$$

$$y'(x) = 3c_1 e^{3x} - c_2 e^{-x}$$

$$y'(1) = 3c_1 e^{3} - c_2 e^{-1} = 0$$

$$\begin{cases} c_1 e^3 + c_2 e^{-1} = 2 \\ 3c_1 e^3 - c_2 e^{-1} = 0 \end{cases}$$

$$\begin{cases} 4 c_1 e^3 = 2 \\ 3 c_1 e^3 = c_2 e^{-1} \end{cases}$$

$$\begin{cases} c_1 = e^{-3}/2 \\ 3/2 = c_2 e^{-1} \end{cases} \qquad \begin{cases} c_1 = e^{-3}/2 \\ c_2 = 3/2 \, e \end{cases}$$

$$y(x) = \frac{1}{2} e^{3(x-1)} + \frac{3}{2} e^{-(x-1)}$$

$$y'' - 2y' + 2y = 0$$

$$\lambda^2 - 2\lambda + 2 = 0 \qquad \frac{\Delta}{4} = \frac{4-8}{4} = -1 < 0$$

$$\lambda_\pm = 1 \pm \sqrt{|-1|}\, e^{i\frac{\arg(-1)}{2}} =$$

$$= 1 \pm e^{i\frac{\pi}{2}} = 1 \pm i$$

$$y_+(x) = e^{\lambda_+ x} = e^{(1+i)x}$$

$$y_-(x) = e^{\lambda_- x} = e^{(1-i)x}$$

$$y_\pm(x) = e^{x \pm ix} = e^{x} e^{\pm ix} =$$

$$= e^{x}\left(\cos(x) \pm i \sin(x)\right)$$

FORMULA di EULERO

$$y_1(x) = e^x \cos(x) = \frac{(y_+ + y_-)}{2}$$

$$y_2(x) = e^x \sin(x) = \frac{y_+ - y_-}{2i}$$

$$y(x) = c_1 e^x \cos(x) + c_2 e^x \sin(x)$$

Se le radici sono complesse e coniugate, $\lambda_\pm = \alpha \pm i\beta$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}/\{0\}$

$$\Longrightarrow \begin{aligned} y_1(x) &= e^{\alpha x} \cos(\beta x) \\ y_2(x) &= e^{\alpha x} \sin(\beta x) \end{aligned}$$

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$\begin{cases} y'' - 2y' + 2y = 0 \\ y\left(\frac{\pi}{2}\right) = 0 \\ y'\left(\frac{\pi}{2}\right) = 1 \end{cases}$$

$$y(x) = c_1 e^x \cos(x) + c_2 e^x \sin(x)$$

$$y\left(\frac{\pi}{2}\right) = c_1 e^{\pi/2} \cdot 0 + c_2 e^{\pi/2} = 0$$

$$c_2 e^{\pi/2} = 0$$

$$\Longrightarrow \quad c_2 = 0$$

$$y(x) = c_1 e^x \cos(x)$$

$$y'(x) = c_1 e^x \cos(x) - c_1 e^x \sin(x)$$

$$y'\left(\frac{\pi}{2}\right) = c_1 e^{\pi/2} \cdot 0 - c_1 e^{\pi/2} = 1$$

$$c_1 = -e^{-\pi/2}$$

$$y(x) = -e^{(x-\pi/2)}\cos(x)$$

$$y'' + 2y' + y = 0$$

$$\lambda^2 + 2\lambda + 1 = 0 \qquad (\lambda+1)^2 = 0$$

$$\boxed{\lambda_\pm = -1}$$

RADICE DOPPIA o di MOLTEPLICITÀ 2.

$$y_1 = e^{-x}$$

$$y_2 = x\,e^{-x} = x\,y_1$$

$$y_2' = y_1 + x\,y_1'$$

$$y_2'' = 2y_1' + x\,y_1''$$

$$y_2'' + 2y_2' + y_2 =$$

$$2y_1' + xy_1'' + 2(y_1 + xy_1') + xy_1 =$$

$$= 2y_1' + 2y_1 + x\underbrace{(y_1'' + 2y_1' + y_1)}_{0} =$$

$$= 2(y_1' + y_1) = 0$$

$$y' + y = 0$$

$$\lambda + 1 = 0 \leftarrow (\lambda + 1)^2 = 0$$

$$y(x) = c_1 e^{-x} + c_2 x e^{-x}$$
$$= c_1 y_1(x) + c_2 y_2(x)$$

$$\begin{cases} y'' + 2y' + y = 0 \\ y(2) = -1 \\ y'(2) = 3 \end{cases}$$

$$y(x) = c_1 e^{-x} + c_2 x e^{-x}$$

$$y'(x) = -c_1 e^{-x} - c_2 x e^{-x} + c_2 e^{-x}$$

$$\begin{cases} c_1 e^{-2} + c_2 \cdot 2 e^{-2} = -1 \\ -c_1 e^{-2} - c_2 \cdot 2 e^{-2} + c_2 e^{-2} = 3 \end{cases}$$

$$\begin{cases} c_1 + 2c_2 = -e^{2} \\ -c_1 - c_2 = 3e^{2} \end{cases}$$

$$c_2 = 2e^{2}$$

$$c_1 = -5e^{2}$$

$$y(x) = -5e^{2-x} + 2xe^{2-x}$$

Equazioni lineari del 2° ordine, non omogenee a coefficienti costanti

$$y'' + a y' + b y = g(x) \qquad (E)$$

$a \in \mathbb{R}$, $b \in \mathbb{R}$

$$g(x) = P_{k_1}(x)\, e^{\alpha x} \cos(\beta x) + Q_{k_2}(x)\, e^{\alpha x} \sin(\beta x)$$

P_{k_1} e Q_{k_2} sono POLINOMI di GRADO k_1 e k_2

$$\boxed{\lambda_0 = \alpha + i\beta}$$

$$g(x) = x^3 e^x \, ; \qquad \alpha = 1,\ \beta = 0 \quad P_3(x) = x^3$$

$$\lambda_0 = 1$$

$$g(x) = x^3 \cos(2x) \, ; \qquad \alpha = 0,\ \beta = 2 \quad P_3(x) = x^3$$

$$\lambda_0 = 2i$$

$$g(x) = x\, e^{-2x} \sin(x) \, ; \qquad \alpha = -2,\ \beta = 1 \quad Q_1(x) = x$$

$$\lambda_0 = -2 + i$$

$$g(x) = x^4 \, ; \ \lambda_0 = 0 \qquad \alpha = 0 = \beta$$

Siano y_1 e y_2 le due soluzioni della Equazione omogenea associata

$$y'' + ay' + by = 0$$

TUTTE LE SOLUZIONI di (E) SONO DELLA FORMA:

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

dove $y_p(x)$ è una SOLUZIONE qualunque di (E).

PROBLEMA: TROVARE $y_p(x)$

Siano λ_1 e λ_2 le sol. di $\lambda^2 + a\lambda + b = 0$

1) $\lambda_0 \neq \lambda_1$ e $\lambda_0 \neq \lambda_2$

$$y_p(x) = M(x)e^{\alpha x}\cos(\beta x) + N(x)e^{\alpha x}\sin(\beta x).$$

M e N sono Polinomi di GRADO Minore o uguale a $\max\{k_1, k_2\}$

2) $\lambda_1 \neq \lambda_2$ e $\lambda_0 = \lambda_1$ o $\lambda_0 = \lambda_2$

$$y_p(x) = x\left(M(x)e^{\alpha x}\cos(\beta x) + N(x)e^{\alpha x}\sin(\beta x)\right)$$

M e N COME sopra

3) $\lambda_1 = \lambda_2$ e $\lambda_0 = \lambda_1 = \lambda_2$

$$y_p(x) = x^2\left(M(x)e^{\alpha x}\cos(\beta x) +\right.$$

$$N(x)\, e^{\alpha x} \sin(\beta x) \Big)$$

M e N come sopra.