LA DERIVATA di UNA funzione PARI è UNA funzione DISPARI

$f(x)$ PARI e DERIVABILE

$$f'(-x) = \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h}$$

$h = -t$

$$= \lim_{t \to 0} \frac{f(-x-t) - f(-x)}{-t} =$$

$$= -\lim_{t \to 0} \frac{f(x+t) - f(x)}{t} = -f'(x)$$

$$f(x) = \cos(x)\, e^{\frac{|\sin(x)|}{2}}$$

$dom(f) = \mathbb{R}$, PERIODICA

PERIODO 2π $\Rightarrow x \in [-\pi, \pi]$

PARI $\Rightarrow x \in [0, \pi]$

$$f(0) = 1 \; ; \; f(\pi) = -1$$

$$f(x) = \cos(x)\, e^{\frac{\sin(x)}{2}} \qquad x \in [0, \pi]$$

$$f'(x) = -\sin(x)\, e^{\frac{\sin(x)}{2}} + \frac{\cos^2(x)}{2}\, e^{\frac{\sin(x)}{2}} =$$

$$= \frac{e^{\frac{\sin(x)}{2}}}{2}\left[\cos^2(x) - 2\sin(x)\right] \geq 0$$

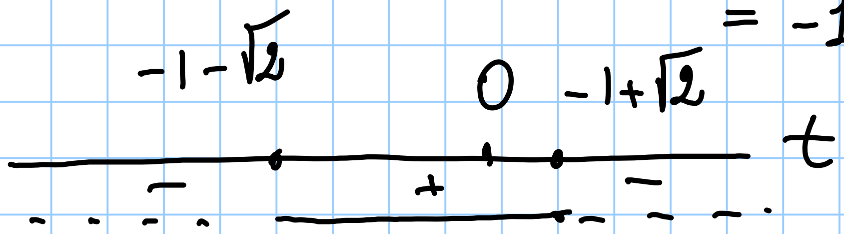
$$\cos^2(x) = 1 - \sin^2(x)$$

$$1 - 2\sin(x) - \sin^2(x) \geq 0$$

$$\sin(x) = t$$

$$-t^2 - 2t + 1 \geq 0 \qquad t_\pm = \frac{1 \pm \sqrt{1+1}}{-1}$$

$$= -1 \pm \sqrt{2}$$



$$-1-\sqrt{2} \leq t \leq \sqrt{2} - 1$$

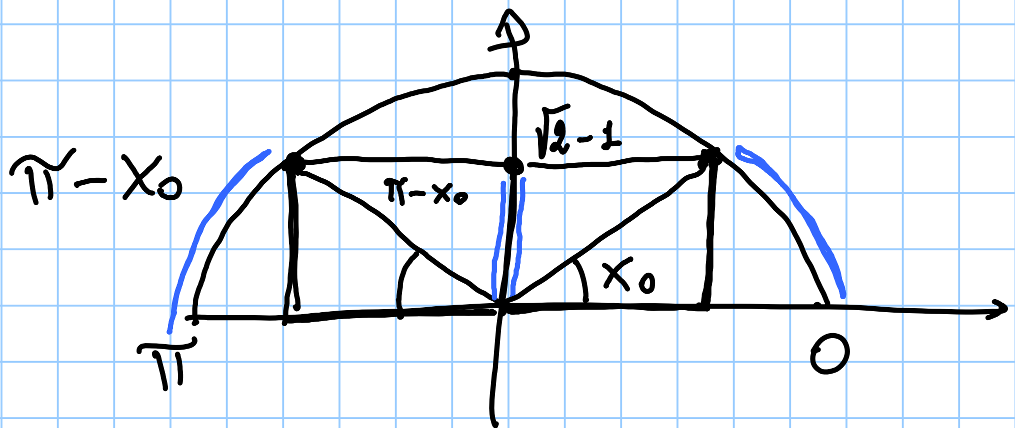
$$\underbrace{-1-\sqrt{2}}_{<-1} \leq \sin(x) \leq \sqrt{2} - 1 \qquad x \in [0, \pi]$$

SEMPRE VERIFICATA

$$\Updownarrow$$

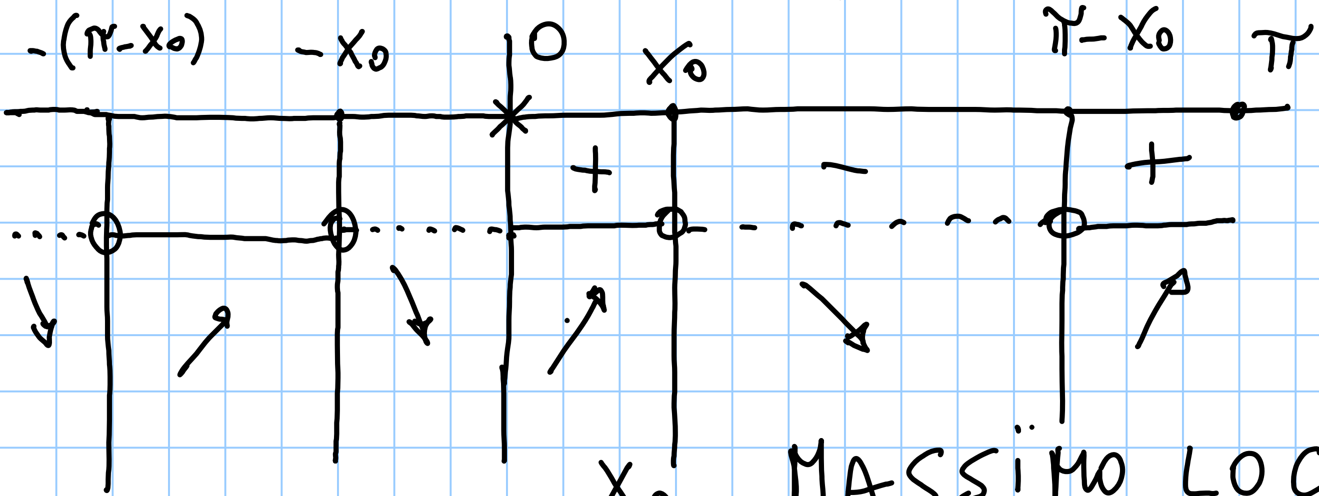
$$\sin(x) \leq \sqrt{2} - 1$$

$$x_0 = \arcsin\left(\sqrt{2} - 1\right)$$



$$0 \le x \le x_0$$

$$\pi - x_0 \le x \le \pi$$



x_0 MASSIMO LOCALE

$\pi - x_0$ MINIMO LOCALE

$x = 0$ MINIMO LOCALE

$$f(x_0) = \cos(x_0)\, e^{\frac{\sin(x_0)}{2}} =$$

$$\sin(x_0) = \sqrt{2} - 1;\quad \cos(x_0) = \sqrt{1 - \sin^2(x_0)} =$$

$$= \sqrt{1-(2+1-2\sqrt{2})} = \sqrt{2\sqrt{2}-2} = \sqrt{2}\,(\sqrt{2}-1)$$

$$f(x_0) = \sqrt{2}\,(\sqrt{2}-1)\, e^{(\sqrt{2}-1)/2}$$

$$f(\pi - x_0) = -\sqrt{2}\,(\sqrt{2}-1)\, e^{(\sqrt{2}-1)/2}$$

$$f(0) = 1 \quad , \quad f(\pi) = -1$$

x_0 MASSIMO , $\pi - x_0$ MINIMO

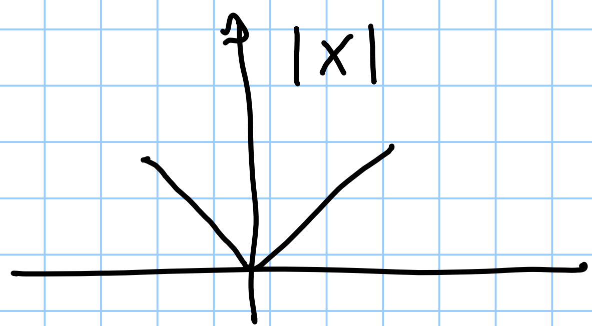
$$D^{(+)} f(0) = \lim_{x \to 0^+} f'(x) = \frac{1}{2}$$

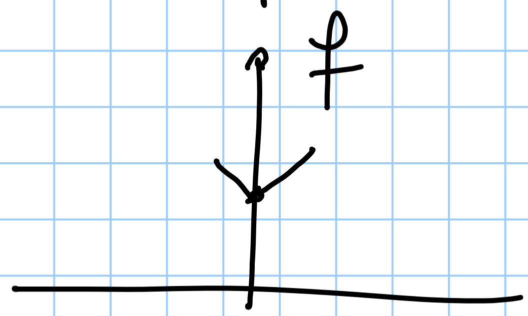
$$D^{(-)} f(\pi) = \lim_{x \to \pi^-} f'(x) = \frac{1}{2}$$

$D^{(+)}f(0) = \frac{1}{2}$ $\quad$ f è PARI $\Rightarrow$

f' è DISPARI

$\Rightarrow D^{(-)}f(0) = -\frac{1}{2}$





f PARI, $f : (-\varepsilon, \varepsilon) \to \mathbb{R}$,

$\exists D^{(+)}f(0) \Rightarrow D^{(-)}f(0) = -D^{(+)}f(0)$

$$D^{(+)}f(0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x}$$

$$D^{(-)}f(0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x} = \{y=-x\}$$

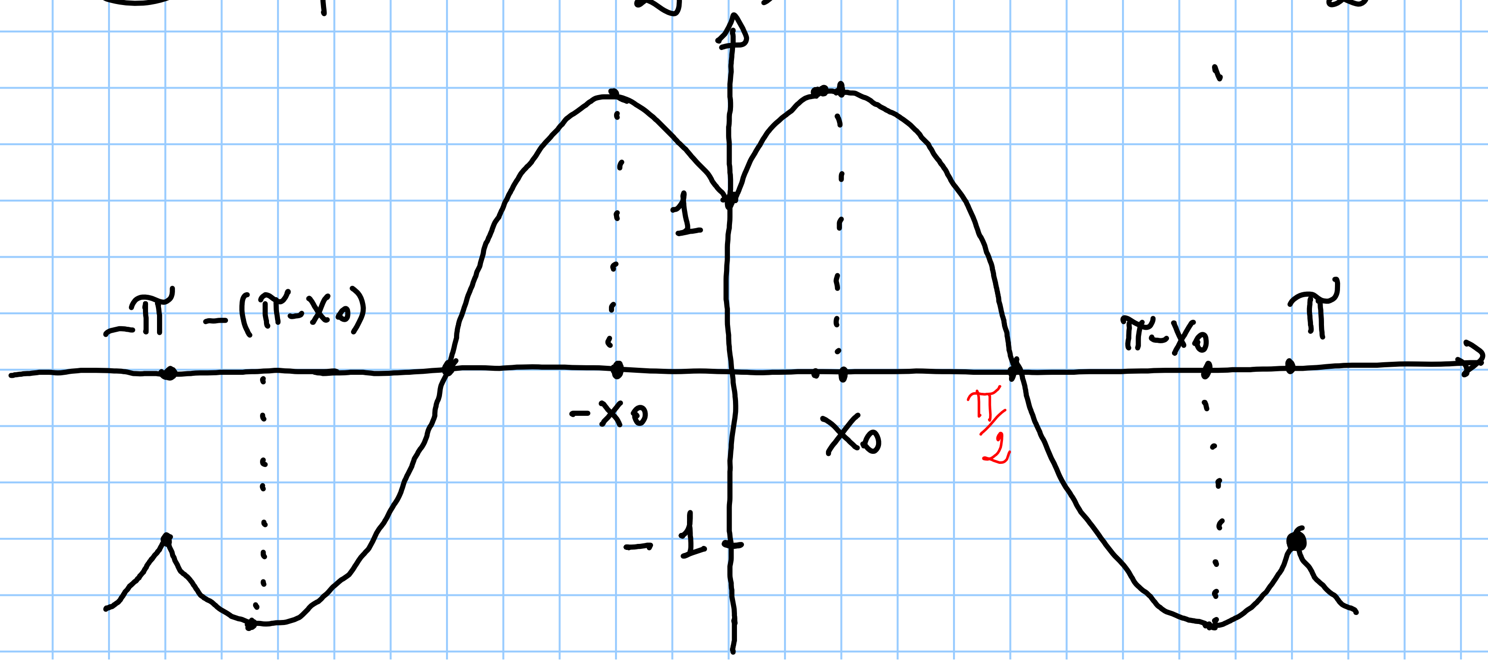
$$= \lim_{y\to 0^+} \frac{f(-y)-f(0)}{-y} = -\lim_{y\to 0^+} \frac{f(y)-f(0)}{y}$$

$$= -D^{(+)}f(0)$$

$$D^{(-)}f(0) = -\frac{1}{2}$$

$x=0$ e $x=\pi$
PUNTI ANGOLOSI

$$D^{(-)}f(\pi) = \frac{1}{2}, \quad D^{(+)}f(\pi) = -\frac{1}{2}$$



Ex

$$f''(x) = -\cos(x)\, e^{\frac{\sin(x)}{2}} - \frac{1}{2}\sin(x)\cos(x)\, e^{\frac{\sin(x)}{2}} .$$

$$- \cos(x)\sin(x)\, e^{\frac{\sin(x)}{2}} + \frac{\cos^3(x)}{4} e^{\frac{\sin(x)}{2}} =$$

$$= e^{\frac{\sin(x)}{2}} \left[-\cos(x) - \frac{3}{2}\sin(x)\cos(x) + \frac{\cos^3(x)}{4} \right]$$

$$= e^{\frac{\sin(x)}{2}} \cos(x) \left[-1 - \frac{3}{2}\sin(x) + \frac{1}{4}\cos^2(x) \right]$$

$$= e^{\frac{\sin(x)}{2}} \frac{\cos(x)}{4} \left[-4 - 6\sin(x) + 1 - \sin^2(x) \right]$$

$$= e^{\frac{\sin(x)}{2}} \frac{\cos(x)}{4} \left[-\sin^2(x) - 6\sin(x) - 3 \right] \geq 0$$

$$\sin(x) = t, \quad -t^2 - 6t - 3 \geq 0$$

$$t_{\pm} = \frac{3 \pm \sqrt{9-3}}{-1} = -3 \pm \sqrt{6}$$

$$-3 - \sqrt{6} \leq t \leq -3 + \sqrt{6}$$

$$-3-\sqrt{6} \leq \sin(x) \leq -3+\sqrt{6}$$

$$\Updownarrow$$

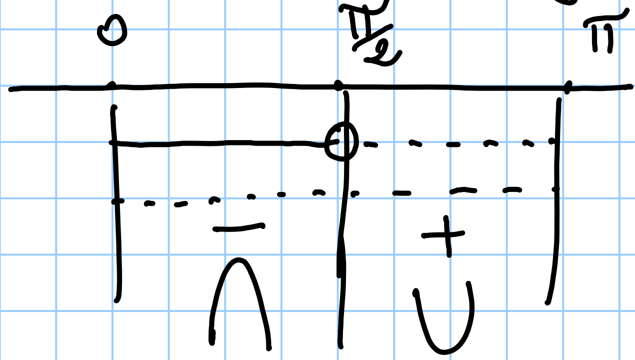
$x \in [0,\pi]$

$$\sin(x) \leq -3+\sqrt{6}$$

$2<\sqrt{6}<3$, $-1<\sqrt{6}-3<0$

$x \in [0,\pi]$ $\sin(x) \geq 0$.

$$\cos(x) > 0 \iff x \in \left[0,\tfrac{\pi}{2}\right)$$



$x_0 = \frac{\pi}{2}$ PUNTO di FLESSO

(DISCENDENTE

$f'\left(\frac{\pi}{2}\right) < 0$)

$$f\left(\frac{\pi}{2}\right) = 0$$

LA DERIVATA DEL POLINOMIO DI TAYLOR E' IL POLINOMIO DI TAYLOR DELLA DERIVATA?

$$\sin(x) = x - \frac{x^3}{3!} + o(x^3), \quad x \to 0$$

$$\cos(x) = \frac{d(\sin(x))}{dx} \overset{?}{=} \frac{d}{dx}\left(x - \frac{x^3}{3!}\right) + o(x^2)$$

$$= 1 - \frac{x^2}{2} + o(x^2)$$

$$f(x) = T_n(x) + R_n(x), \quad x \to 0$$

$$R_n(x) = o(x^n), \quad x \to 0$$

$$f'(x) \overset{?}{=} T_n'(x) + o(x^{n-1})$$

TEOREMA $f:(a,b)\longrightarrow \mathbb{R}$, $x_0 \in (a,b)$, f derivabile $n-1$ volte in (a,b) e n volte in x_0.

Allora $f'(x) = T_n'(x) + o((x-x_0)^{n-1})$, $x \to x_0$.

DIMOSTRAZIONE (S.P.D.G. $x_0 = 0$)

$$\frac{f'(x) - T_n'(x)}{x^{n-1}} \longrightarrow 0, \quad x \longrightarrow 0$$

$$\lim_{x\to 0} \frac{f'(x) - T_n'(x)}{x^{n-1}} =$$

$$= \lim_{x\to 0} \frac{f'(x) - \sum_{k=1}^{n} \frac{f^{(k)}(0)}{k!} k x^{k-1}}{x^{n-1}} \overset{H}{=}$$

$$= \lim_{x\to 0} \frac{f^{(2)}(x) - \sum_{k=2}^{m} \frac{f^{(k)}(0)}{k!} k(k-1)\, x^{k-2}}{(m-1)\, x^{m-2}} \overset{H}{=}$$

$$= \lim_{x\to 0} \cdots \overset{H}{=} \cdots \{ (m-2) \text{ VOLTE} \} \overset{H}{=}$$

$$= \lim_{x\to 0} \frac{f^{(m-1)}(x) - \sum_{k=m-1}^{m} \frac{f^{(k)}(0)}{k!} k(k-1)\cdots(k-m+1)\, x^{k-m+1}}{(m-1)!\, x} =$$

$$= \lim_{x\to 0} \frac{f^{(m-1)}(x) - f^{(m-1)}(0)}{(m-1)!\, x} - \frac{f^{(m)}(0)}{(m-1)!} = 0$$

$\blacksquare$

L'INTEGRALE DEL POLINOMIO DI TAYLOR E' IL POLINOMIO DI TAYLOR DELL'INTEGRALE?

O ANCHE COME RICAVARE IL POLINOMIO DI TAYLOR di

$\arctan(x)$, $x_0 = 0$

$$\cos(x) = 1 - \frac{x^2}{2} + o(x^2)$$

$$\sin(x) = \int_0^x \cos(y)\,dy = \int_0^x \left(1 - \frac{y^2}{2}\right) dy +$$

$$+ \int_0^x \underline{o(y^2)}\,dy = x - \frac{x^3}{3!} + o(x^3)$$

$$\lim_{x\to 0} \frac{\int_0^x o(y^2)\,dy}{x^3} \overset{H}{=} \lim_{x\to 0} \frac{o(x^2)}{3x^2} = 0$$

$$o(y^2) = g(y) = \cos(y) - \left(1 - \frac{y^2}{2}\right).$$

g è CONTINUA, quindi

$\int_0^x g(y)\,dy$ è DERIVABILE

$$\frac{d}{dx}\left(\int_0^x g(y)\,dy\right) = g(x).$$

$$f(x) = T_n(x) + R_n(x),$$

$$R_n(x) = o(x^n) \quad x \to 0$$

$$F(x) = \int_0^x f(y)\,dy \qquad ,$$

$$F(x) = \int_0^x T_n(y)\,dy + o(x^{n+1})$$

DIMOSTRAZIONE

$$\frac{\int_0^x f(y)\,dy - \int_0^x T_n(y)\,dy}{o(x^{n+1})} = o(1) \quad x \to 0$$

$$\lim_{x \to 0} \frac{\int_0^x f(y)\,dy - \int_0^x T_n(y)\,dy}{x^{n+1}} \overset{H}{=}$$

$$\lim_{x \to 0} \frac{f(x) - T_n(x)}{(n+1)\,x^n} = 0 \qquad \blacksquare$$

$$f(x) = \operatorname{arcTg}(x);$$

$$f'(x) = \frac{1}{1+x^2},$$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots + z^m + o(z^m)$$

$$z = -x^2;$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots + (-1)^m x^{2m} + o(x^{2m})$$

$$\operatorname{arcTg}(x) = \int_0^x \left(1 - y^2 + y^4 - y^6 + \dots + (-1)^m y^{2m}\right) dy + o(x^{2m+1})$$

$$T_m(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + (-1)^m \frac{x^{2m+1}}{2m+1} + o(x^{2m+2}), \quad m = 0, 1, 2 \dots$$

$$f(x) = e^{x^2 \log(1 + x(x+1))} + x \operatorname{arcTg}(x)$$

Calcolare T_5 $x_0 = 0$.

$$\log(1 + x + x^2) =$$

$$(x + x^2) - \frac{1}{2}(x + x^2)^2 + \frac{1}{3}(x + x^2)^3 +$$

$$+ o((x + x^2)^3) =$$

$$x + x^2 - \frac{1}{2}x^2 - x^3 + \frac{1}{3}x^3 + o(x^3)$$

$$e^{x^2 \log(1 + x + x^2)} = e^{x^3 + \frac{1}{2}x^4 - \frac{2}{3}x^5 + o(x^5)}$$

$$= 1 + \left(x^3 + \frac{x^4}{2} - \frac{2}{3}x^5 + o(x^5)\right) +$$

$$\frac{1}{2}\left(x^3 + o(x^3)\right)^2 + o\left(\left(x^3 + o(x^3)\right)^2\right) =$$

$$= 1 + x^3 + \frac{x^4}{2} - \frac{2}{3}x^5 + o(x^5)$$

$$x \, \operatorname{arcTg}(x) = x\left(x - \frac{x^3}{3} + \frac{x^5}{5} + o(x^6)\right)$$

$$= x\left(x - \frac{x^3}{3} + o(x^4)\right)$$

$$= x^2 - \frac{x^4}{3} + o(x^5)$$

$$T_5(x) = 1 + x^3 + \frac{x^4}{2} - \frac{2}{3}x^5 +$$

$$+ x^2 - \frac{x^4}{3} =$$

$$= 1 + x^2 + x^3 + \frac{1}{6}x^4 - \frac{2}{3}x^5$$