$$z = |z|\, e^{i\vartheta} \quad ; \quad W^m = z$$

$$W_k = \sqrt[m]{|z|}\; e^{i\vartheta/m}\; \xi_k^{(m)} \qquad k = 0, 1, 2, \dots, m-1$$

$$\xi_k^{(m)} = e^{i\frac{2k\pi}{m}} \quad , \; k = 0, 1, 2, \dots, m-1$$

$$\left(\xi_k^{(m)}\right)^m = 1 \quad , \; k = 0, 1, 2, \dots, m-1$$

$$W_k = \sqrt[m]{|z|}\; e^{i\vartheta/m}\; e^{i\frac{2k\pi}{m}} \; , \; k = 0, 1, 2, \dots, m-1$$

$$m = 2 \quad , \quad W^2 = z \quad , \quad z = |z|\, e^{i\vartheta}$$

$$W_{\pm} = \pm\sqrt{|z|}\; e^{i\vartheta/2} \qquad \boxed{\left\{\xi_k^{(2)}\right\}_{k=0,1} = \{1, -1\}}$$

$$z = i \longrightarrow z = e^{i\pi/2} \longrightarrow W_\pm = \pm e^{i\frac{\pi}{4}}$$

$$W^2 = i$$

$$z = -3e^{i\pi/3} \longrightarrow z = 3e^{i\pi + i\pi/3} = 3e^{i\frac{4}{3}\pi}$$

$$W^2 = -3e^{i\pi/3}$$

$$W_\pm = \pm\sqrt{3}\, e^{i\frac{4}{6}\pi} = \pm\sqrt{3}\, e^{i\frac{2}{3}\pi}$$

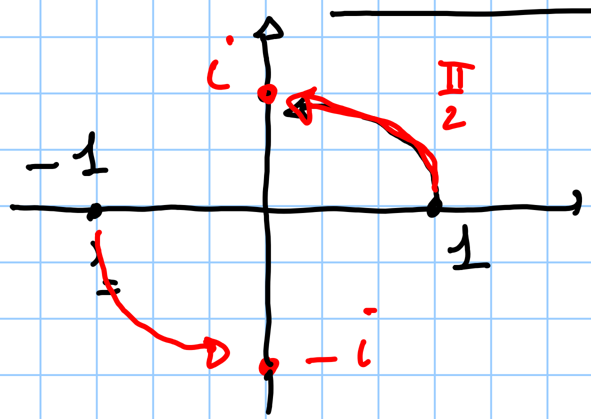
$$W^2 = -1$$

$$z = -1 = e^{i\pi}$$

$$W_k = e^{i\pi/2}\, \xi_k^{(2)}, \quad k = 0, 1$$

$$\xi_0^{(2)} = 1, \quad \xi_1^{(2)} = -1$$

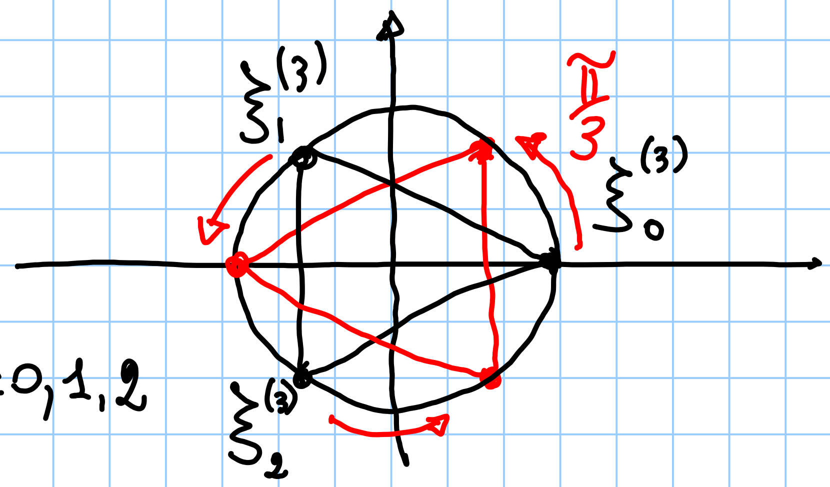
$$W_\pm = \pm e^{i\pi/2} = \pm i$$



$$W^3 = -1$$

$$z = -1 = e^{i\pi}$$

$$\left\{\xi_k^{(3)}\right\}_{k=0,1,2} = \left\{1, e^{i\frac{2}{3}\pi}, e^{i\frac{4}{3}\pi}\right\}$$



$$W_k = e^{i\frac{\pi}{3}} \, \xi_3^{(k)}, \quad k=0,1,2$$

$$= \left\{e^{i\frac{\pi}{3}}, \; e^{i\pi}, \; e^{i\frac{5}{3}\pi}\right\}$$

TEOREMA FONDAMENTALE dell'ALGEBRA

$$\mathbb{N} \ni m \geq 1, \quad \{a_k\}_{k=0,1,2,\dots,m} \subset \mathbb{C}$$

$$P_m(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m$$

SE $a_n \neq 0$ ALLORA L'EQUAZIONE

$$P_n(z) = 0$$

ha ESATTAMENTE n SOLUZIONI in $\mathbb{C}$.

(Contate con la loro molteplicità)

IN PARTICOLARE $\exists \{z_1, z_2, \dots, z_n\}$:

(non NECESSARIAMENTE DISTINTI)

$$P_n(z) = a_n (z - z_1)(z - z_2) \dots (z - z_n).$$

$$P_3(z) = z^3 - 1$$

$$P_3(z) = 0 \longrightarrow z_1 = 1, \; z_2 = e^{i\frac{2}{3}\pi}, \; z_3 = e^{i\frac{4}{3}\pi}$$

$$z^3 - 1 = (z - 1)\left(z - e^{i\frac{2}{3}\pi}\right)\left(z - e^{i\frac{4}{3}\pi}\right)$$ Ex

$$P_5(z) = (z-2i)^3 (z+i)(z-1+3i)$$

$z_1 = z_2 = z_3 = 2i$ (2i HA MOLTEPLICITÀ 3)

$z_4 = -i$, $z_5 = 1-3i$

SOLUZIONE EQUAZIONI DI GRADO $n = 2$ in $\mathbb{C}$

$n = 2 \quad \{a, b, c\} \subset \mathbb{C}, \quad a \neq 0$

- $az^2 + bz + c = 0 \iff$

$a^2z^2 + abz + ac = 0 \iff$

$a^2z^2 + abz + \left(\frac{b^2}{4}\right) - \left(\frac{b^2}{4}\right) + ac = 0 \iff$

- $\left(az + \frac{b}{2}\right)^2 = \frac{b^2}{4} - ac = \frac{b^2 - 4ac}{4}$

$\left(az + \frac{b}{2}\right)^2 = \frac{\Delta}{4}, \quad \Delta = b^2 - 4ac$

$$W^2 = \frac{\Delta}{4} \longrightarrow W_\pm = \pm \frac{\sqrt{|\Delta|}}{2} e^{i\theta/2}$$

$$\theta = \arg(\Delta)$$

$$W_\pm = \left(a z_\pm + \frac{b}{2} \right) = \pm \frac{\sqrt{|\Delta|}}{2} e^{i\theta/2}$$

$$z_\pm = -\frac{b}{2a} \pm \frac{\sqrt{|\Delta|}}{2a} e^{i\frac{\theta}{2}}$$

$$z_\pm = \frac{-b \pm \sqrt{|\Delta|}\, e^{i\theta/2}}{2a}$$

$$\Delta = b^2 - 4ac \quad , \theta = \arg(\Delta)$$

- $\Delta > 0 \Longleftrightarrow \Delta \in \mathbb{R}^+ \Longrightarrow \theta = 0 \,, |\Delta| = \Delta$

$$z_\pm = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$\boxed{\text{RADICI REALI DISTINTE}}$

- $\Delta < 0 \Longleftrightarrow \Delta \in \mathbb{R}^- \Longrightarrow \theta = \pi$

$$z_\pm = \frac{-b \pm \sqrt{|\Delta|}\, e^{i\frac{\pi}{2}}}{2a}$$

$\boxed{\text{RADICI COMPLESSE CONIUGATE}}$

$$z_\pm = -\frac{b}{2a} \pm i\,\frac{\sqrt{|\Delta|}}{2a} = x_0 \pm i y_0$$

$$x_0 = -\frac{b}{2a} \qquad y_0 = \frac{\sqrt{|\Delta|}}{2a}$$

$$a z^2 + bz + c = a\,(z - z_+)(z - z_-)$$

$$\boxed{\Delta < 0}$$

$$a z^2 + bz + c = \frac{1}{a}\left[\left(az + \frac{b}{2}\right)^2 - \frac{\Delta}{4}\right] =$$

$$= \frac{a^{\cancel{2}}}{\cancel{a}}\left[\left(z + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right] =$$

$$= a\left[\left(z + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right], \forall z \in \mathbb{C}$$

$$z = x \in \mathbb{R}$$

$$a x^2 + bx + c =$$

$$a\left[\left(x+\frac{b}{2a}\right)^2-\frac{\Delta}{4a^2}\right] = \left\{\begin{matrix}\Delta<0\\ \Delta=-|\Delta|\end{matrix}\right\}$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2+\frac{|\Delta|}{4a^2}\right] =$$

$$= a\left[(x-x_0)^2+y_0^2\right]$$

$$x_0=-\frac{b}{2a}, \quad y_0^2=\frac{|\Delta|}{4a^2}$$

$$iz^2-2iz+1=0$$

$$z_\pm = \frac{-(-2i)\pm\sqrt{|\Delta|}\,e^{i\theta/2}}{2i}$$

$$\Delta = b^2-4ac = (-2i)^2-4i = -4-4i$$

$$= -4(1+i)$$

$$|\Delta| = \sqrt{16+16} = 4\sqrt{2} = \boxed{2^{5/2}}$$

$$\begin{cases} \theta = -\arccos\left(\frac{-4}{4\sqrt{2}}\right) = -\arccos\left(-\frac{1}{\sqrt{2}}\right) \\ y = -4 < 0 \qquad\qquad = -\frac{3}{4}\pi \underset{\text{mod } 2\pi}{=} \boxed{\frac{5\pi}{4}} \end{cases}$$

$$\Delta = |\Delta| e^{i\frac{5}{4}\pi} = 2^{5/2} e^{i\frac{5}{4}\pi}$$

$$\sqrt{|\Delta|} e^{i\theta/2} = 2^{5/4} e^{i\frac{5}{8}\pi}$$

$$z_\pm = \frac{2i \pm 2^{5/4} e^{i\frac{5}{8}\pi}}{2i} = 1 \pm \frac{2^{1/4}}{i} e^{i\frac{5}{8}\pi} =$$

$$= 1 \pm 2^{1/4} e^{i\frac{5}{8}\pi - i\frac{\pi}{2}} = 1 \pm 2^{1/4} e^{i\frac{\pi}{8}} =$$

$$= 1 \pm 2^{1/4} \cos\left(\frac{\pi}{8}\right) \pm i\, 2^{1/4} \sin\left(\frac{\pi}{8}\right)$$

$$\cos(x) = 2\cos^2\left(\frac{x}{2}\right) - 1$$

$$\cos^2\left(\frac{x}{2}\right) = \frac{\cos(x)+1}{2}$$

$x = \frac{\pi}{4}$

$$\cos^2\left(\frac{\pi}{8}\right) = \frac{\cos\left(\frac{\pi}{4}\right) + 1}{2}$$

$$= \frac{\frac{\sqrt{2}}{2} + 1}{2} = \frac{\sqrt{2} + 2}{4}.$$

$$\cos\left(\frac{\pi}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{2}$$

$$\sin\left(\frac{\pi}{8}\right) = \frac{\sqrt{2-\sqrt{2}}}{2}$$

FACOLTATIVO

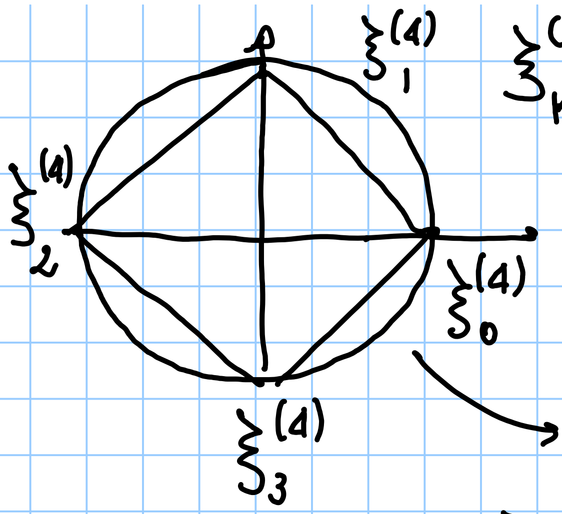
$$x^4+1 = (x^2+B_1x+C_1)(x^2+B_2x+C_2)$$

$$z^4+1 = (z-z_0)(z-z_1)(z-z_2)(z-z_3)$$

$$z^4=-1 \quad , \quad -1 = e^{i\pi}$$

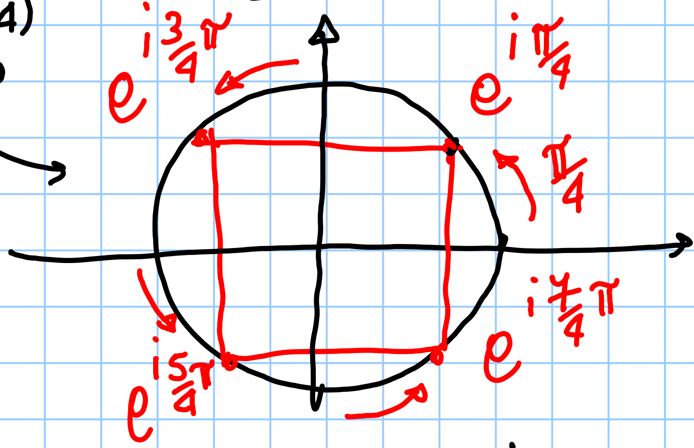
$$z_k = e^{i\frac{\pi}{4}} \, e^{i\frac{2k\pi}{4}} \quad k=0,1,2,3$$

$$\{z_k\}_{k=0..3} = \{e^{i\frac{\pi}{4}}, e^{i\frac{3}{4}\pi}, e^{i\frac{5}{4}\pi}, e^{i\frac{7}{4}\pi}\}$$



$$\xi_k^{(4)} = e^{i\frac{2k\pi}{4}}, \quad k = 0,1,2,3$$

$$= \{1, i, -1, -i\}$$



$$e^{i\frac{7}{4}\pi} = e^{-i\frac{\pi}{4}}, \quad e^{i\frac{5}{4}\pi} = e^{-i\frac{3}{4}\pi}$$

$$(z^4+1) = (z-z_0)(z-z_1)(z-z_2)(z-z_3)$$

$$= \left(z-e^{i\frac{\pi}{4}}\right)\left(z-e^{i\frac{3}{4}\pi}\right)\left(z-e^{-i\frac{3}{4}\pi}\right)\left(z-e^{-i\frac{\pi}{4}}\right)$$

$$= \left(z-e^{i\frac{\pi}{4}}\right)\left(z-e^{-i\frac{\pi}{4}}\right)\left(z-e^{i\frac{3}{4}\pi}\right)\left(z-e^{-i\frac{3}{4}\pi}\right)$$

$$= \left(z^2-\left(e^{i\frac{\pi}{4}}+e^{-i\frac{\pi}{4}}\right)z+1\right)\left(z^2-\left(e^{i\frac{3}{4}\pi}+e^{-i\frac{3}{4}\pi}\right)z+1\right)$$

$$e^{i\theta}e^{-i\theta} = e^{i\theta-i\theta} = e^0 = 1$$

$$\left\{ z + \bar{z} = 2\,\mathrm{Re}(z) \right\}$$

$$= \left(z^2 - 2\,\mathrm{Re}\left(e^{i\frac{\pi}{4}}\right) z + 1 \right) \left(z^2 - 2\,\mathrm{Re}\left(e^{i\frac{3}{4}\pi}\right) z + 1 \right)$$

$$= \left(z^2 - \sqrt{2}\, z + 1 \right) \left(z^2 + \sqrt{2}\, z + 1 \right)$$

$$z^4 + 1 = \left(z^2 - \sqrt{2}\, z + 1 \right) \left(z^2 + \sqrt{2}\, z + 1 \right), \ \forall z \in \mathbb{C}$$

$$\longrightarrow \quad z = x \in \mathbb{R}$$

$$x^4 + 1 = \underbrace{\left(x^2 - \sqrt{2}\, x + 1 \right)}_{\Delta < 0} \underbrace{\left(x^2 + \sqrt{2}\, x + 1 \right)}_{\Delta < 0}$$

$$\int \frac{dx}{1 + x^4} = \int \frac{Ax + B}{x^2 - \sqrt{2}x + 1}\, dx + \int \frac{Cx + D}{x^2 + \sqrt{2}x + 1}\, dx$$

$$z_0 : z_0^4 = -1 \ ; \quad \overline{z_0^4} = \overline{-1} = -1$$

$$z_0 = \rho e^{i\theta}, \quad z_0^4 = \rho^4 e^{i4\theta},$$

$$\rho^4 e^{-i4\theta} = \overline{z_0^4} = (\overline{z_0})^4 \Rightarrow (\overline{z_0})^4 = -1$$

Se $P_N(z_0) = 0$ e P_N è un polinomio a Coefficenti REALI $\Rightarrow P_N(\overline{z_0}) = 0$

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

FORMULA di EULERO

$$\cos(\theta) = \sum_{k=1}^{m} (-1)^k \frac{\theta^{2k}}{(2k)!} + R_{2m}(\theta)$$

$$R_{2m}(\theta) = \left.\frac{D^{(2m+1)}(\cos(\theta))}{(2m+1)!}\right|_{\theta=c.}$$

$c \in$ "$(0,\theta)$"

$c \in$ "$(\theta,0)$"

$$\left| R_{2m}(\theta) \right| = \frac{\left| D^{(2m+1)} \cos(\theta) \Big|_{\theta=c} \right|}{(2m+1)!} \leq$$

$$\leq \frac{1}{(2m+1)!} \quad , \forall \theta \in \mathbb{R} \; !$$

$$\sum_{k=0}^{m} (-1)^k \frac{\theta^{2k}}{(2k)!} = \cos(\theta) + \underbrace{R_{2m}(\theta)}_{\xrightarrow[m \to +\infty]{} 0}$$

$$\boxed{\sum_{k=0}^{+\infty} (-1)^k \frac{\theta^{2k}}{(2k)!} := \lim_{m \to +\infty} \sum_{k=0}^{m} (-1)^k \frac{\theta^{2k}}{(2k)!}}$$

$$\sum_{k=0}^{+\infty} (-1)^k \frac{\theta^{2k}}{(2k)!} = \cos(\theta) \quad , \forall \theta \in \mathbb{R}$$

SERIE di TAYLOR del $\cos(\theta)$

$$\cos(\theta) = \sum_{K=0}^{+\infty} (-1)^k \frac{\theta^{2k}}{(2K)!}, \forall \theta \in \mathbb{R}$$

$$\sin(\theta) = \sum_{K=0}^{+\infty} (-1)^K \frac{\theta^{2K+1}}{(2K+1)!}, \forall \theta \in \mathbb{R}$$

$$e^{\theta} = \sum_{K=0}^{+\infty} \frac{\theta^k}{k!}, \forall \theta \in \mathbb{R}$$

$z \in \mathbb{C}$ DEFINIZIONE

$$e^{z} := \sum_{k=0}^{+\infty} \frac{(z)^K}{k!}, z \in \mathbb{C}$$

$z = i\theta$

$$e^{i\theta} := \sum_{K=0}^{+\infty} \frac{(i\theta)^k}{K!} = \sum_{K=0}^{+\infty} \frac{(i)^k \theta^K}{k!} =$$

$$= \sum_{m=0}^{+\infty} (i)^{2m} \frac{\theta^{2m}}{(2m)!} + \sum_{m=0}^{+\infty} (i)^{2m+1} \frac{\theta^{2m+1}}{(2m+1)!} =$$

$$= \left\{ (i)^{2m} = (i^2)^m = (-1)^m , \quad (i)^{2m+1} = i\,(i)^{2m} = i\,(-1)^m \right\} =$$

$$= \sum_{m=0}^{+\infty} (-1)^m \frac{\theta^{2m}}{(2m)!} + i \sum_{m=0}^{+\infty} (-1)^m \frac{\theta^{2m+1}}{(2m+1)!} =$$

$$\cos(\theta) + i\sin(\theta), \quad \forall \theta \in \mathbb{R}$$

$$e^z = e^{x+iy} \overset{\text{SERIE}}{=} e^x e^{iy} =$$

$$= e^x \cos(y) + i e^x \sin(y)$$

- $$\sum_{k=0}^{+\infty} \frac{(x+iy)^k}{k!} = \left(\sum_{k=0}^{+\infty} \frac{x^k}{k!}\right)\left(\sum_{k=0}^{+\infty} \frac{(iy)^k}{k!}\right)$$