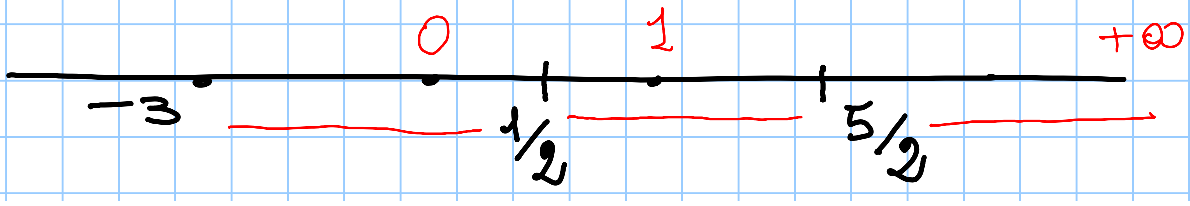
$$\int_{-3}^{+\infty} \frac{e^{-x^2} \log(|x|)}{|x-1|^{3/2}}\, dx$$

$$f(x) = \frac{e^{-x^2} \log(|x|)}{|x-1|^{3/2}}$$

$$\int_{-3}^{+\infty} f(x)\, dx = \int_{-3}^{1/2} f(x)\, dx + \int_{1/2}^{5/2} f(x)\, dx$$

$$+ \int_{5/2}^{+\infty} f(x)\, dx$$



$$f(x) \underset{x\to 0}{=} \frac{(1+o(1))\log(|x|)}{(1+o(1))} = \log(|x|)\,(1+o(1))$$

$$|f(x)| = \log\frac{1}{|x|}(1+o(1))$$

$$g(x) = \frac{1}{\sqrt{|x|}} \qquad \lim_{x\to 0} \frac{|f(x)|}{g(x)} =$$

$$= \lim_{x\to 0} \sqrt{|x|}\,\log\left(\frac{1}{|x|}\right)(1+o(1)) = 0^+$$

"$f \leq g$"

$$Y = \frac{1}{|x|} \qquad \lim_{x\to 0} \sqrt{|x|}\,\log\left(\frac{1}{|x|}\right)$$

$$= \lim_{Y\to +\infty} \frac{\log(Y)}{\sqrt{Y}} = 0^+$$

T. CONFRONTO ASINTOTICO

con $L = 0$, $\int_{-3}^{1/2} g(x)\,dx$ CONVERGE

$\Longrightarrow \int_{-3}^{1/2} |f(x)|\,dx$ CONVERGE

$\Longrightarrow \int_{-3}^{\frac{1}{2}} f(x)\,dx$ CONVERGE

$$f(x) \underset{x\to 1}{=} \frac{e^{-(1+o(1))^2} \log(|x|)}{|x-1|^{3/2}}$$

$$\log(|x|) \underset{x\to 1}{=} \log(x) =$$

$$h(x) = h(1) + h'(1)(x-1) + o(x-1)$$

$$= \log(1) + \frac{1}{x}\Big|_{x=1} (x-1) + o(x-1)$$

$$= (x-1) + o(x-1) =$$

$$= (x-1)(1+o(1)), \quad x \to 1$$

$$\log(1+y) \underset{y\to 0}{=} y + o(y),$$

$$x = 1 + y$$

$$\log(x) = \log(1 + (x-1)) =$$

$$= x - 1 + o(x-1), \quad x \to 1$$

$$|f(x)| \underset{x\to 1}{=} \frac{e^{-(1+o(1))^2} \left| (x-1) + o(x-1) \right|}{|x-1|^{3/2}} =$$

$$= \frac{e^{-1}(1+o(1)) \, |(x-1)| \, (1+o(1))}{|x-1|^{3/2}} =$$

$$= e^{-1}(1+o(1)) \frac{|x-1|}{|x-1|^{3/2}}, \quad x \to 1$$

$$g(x) = \frac{|x-1|}{|x-1|^{3/2}} = \frac{1}{\sqrt{|x-1|}} = \frac{1}{|x-1|^{\frac{1}{2}}}$$

$$\int_{1/2}^{3/2} \frac{1}{|x-1|^{\alpha}}\, dx \quad \text{CONVERGE} \iff \alpha < 1$$

$$\lim_{x \to 1} \frac{|f(x)|}{g(x)} = e^{-1}$$

T. del CONFRONTO ASINTOTICO con $L \in (0, +\infty)$

$$\int_{1/2}^{3/2} g(x)\, dx \quad \text{CONVERGE}$$

$$\Longrightarrow \int_{1/2}^{3/2} |f(x)|\, dx \quad \text{CONVERGE}$$

$$\Longrightarrow \int_{1/2}^{3/2} f(x)\, dx \quad \text{CONVERGE}$$

$$f(x) \underset{x\to+\infty}{=} \frac{e^{-x^2}\log(|x|)}{|x-1|^{3/2}} =$$

$$= \frac{e^{-x^2}\log(x)}{x^{3/2}(1+o(1))}$$

$$g(x) = \frac{1}{x^2}$$

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)} = \lim_{x\to+\infty} e^{-x^2}\sqrt{x}\log(x) = 0^+$$

T. CONFRONTO ASINTOTICO

CON $L = 0$ $\int_{\frac{5}{2}}^{+\infty} g(x)\,dx$

"$f \leq g$" CONVERGE

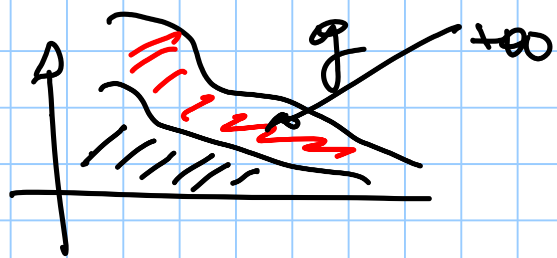
$\int_{\frac{5}{2}}^{+\infty} f(x)\,dx$ CONVERGE

$g(x) = \frac{1}{\sqrt{x}}$ ← SE AVESSIMO USATO

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = 0$$

T. CONFRONTO ASINTOTICO, $L=0$

"$f \leq g$"



IN QUESTO CASO

$$\int_{5/2}^{+\infty} g(x)\,dx = +\infty$$

NON DICE NIENTE ↪ ? $\int_{5/2}^{+\infty} f(x)\,dx \leq +\infty$?

$$I_\alpha = \int_0^{+\infty} e^{-\alpha x}\,dx$$

$\alpha \leq 0 \Rightarrow e^{-\alpha x} \geq 1 \quad \forall x \in (0, +\infty)$

$\int_0^{+\infty} 1 \cdot dx$ DIVERGE $\Rightarrow$

Se $\alpha \leq 0$ I_α DIVERGE

$\boxed{\alpha > 0}$

$$\int_0^{+\infty} e^{-\alpha x} dx = \lim_{t \to +\infty} \int_0^t e^{-\alpha x} dx =$$

$$= \lim_{t \to +\infty} \frac{1}{-\alpha} e^{-\alpha x} \Big|_0^t =$$

$$= \lim_{t \to +\infty} -\frac{1}{\alpha} \left(e^{-\alpha t} - 1 \right) = \frac{1}{\alpha}$$

AVREMMO POTUTO CONFRONTARE PER ESEMPIO CON $1/x^2$, $x \geq 1$

$$f_\alpha(x) = e^{-\alpha x}, \quad \alpha > 0$$

$$g(x) = \frac{1}{x^2}, \quad x \geq 1$$

$$\lim_{x \to +\infty} \frac{f_\alpha(x)}{g(x)} = \lim_{x \to +\infty} x^2 e^{-\alpha x} = 0$$

$\int_1^{+\infty} g(x)\,dx$ CONVERGE $\Longrightarrow$ $\int_1^{+\infty} f(x)\,dx$ CONVERGE

$$\int_0^{+\infty} e^{-\alpha x} x^k \, dx$$

$\forall k \in \mathbb{N}, \ \forall \alpha > 0$

E' CONVERGENTE

CONFRONTO ASINTOTICO CON $g(x) = \frac{1}{x^2}, \ x \geq 1$

Ex.

$$\int_0^1 x^\alpha \log\left(\sqrt{x} + \sqrt[3]{x}\right) dx$$

$$f_\alpha(x) = x^\alpha \log\left(\sqrt{x} + \sqrt[3]{x}\right)$$

Se $\alpha > 0$ $\lim_{x \to 0^+} f_\alpha(x) = 0$

f_α si estende per continuità

con $f_\alpha(0) = 0$ e $f \in \mathcal{R}(0,1)$

Se $\alpha \le 0$

$$f_\alpha(x) = \frac{\log(\sqrt{x} + \sqrt[3]{x})}{x^{|\alpha|}}$$

e Se $\alpha \le -1$ $(|\alpha| \ge 1)$

$$\lim_{x \to 0^+} \frac{|f_\alpha(x)|}{\frac{1}{x^{|\alpha|}}} = \lim_{x \to 0^+} |\log(\sqrt{x} + \sqrt[3]{x})| = +\infty$$

" $|f_\alpha| \ge \frac{1}{x^{|\alpha|}}$ "

$\int_0^1 \frac{1}{x^{|\alpha|}}\, dx$ DIVERGE Se $|\alpha| \ge 1$

$\Longrightarrow \int_0^1 |f_\alpha(x)|\, dx$ DIVERGE

$$\int_0^1 f_\alpha(x)\, dx = \int_0^{x_0} f_\alpha(x)\, dx + \underbrace{\int_{x_0}^1 f_\alpha(x)\, dx}_{\text{FINITO}}$$

$$\underbrace{\int_0^{x_0} f_\alpha(x)\, dx}_{\text{DIVERGE}} = - \underbrace{\int_0^{x_0} |f_\alpha(x)|\, dx}_{\text{DIVERGE}}$$

Se $\alpha \in (-1, 0)$, $|\alpha| \in (0,1)$

$g_\beta(x) = \frac{1}{x^\beta}$, $\beta \in (|\alpha|, 1)$

" $\frac{1}{x^{|\alpha|+\varepsilon}}$ " $\beta = |\alpha| + \varepsilon < 1$, $\varepsilon > 0$

$$\lim_{x \to 0^+} \frac{|f_\alpha(x)|}{\frac{1}{x^\beta}} = \lim_{x \to 0^+} x^{\beta - |\alpha|} \left| \log(\sqrt{x} + \sqrt[3]{x}) \right| = 0$$

" $|f_\alpha| \le \frac{1}{x^\beta}$ ", $\beta < 1$

$\int_0^1 \frac{1}{x^\beta} dx$ CONVERGE

$\Longrightarrow \int_0^1 |f_\alpha(x)| dx$ CONVERGE

$\Longrightarrow \int_0^1 f_\alpha(x) dx$ CONVERGE

$\int_0^1 f_\alpha(x) dx$ CONVERGE

$\Updownarrow$

$\alpha > -1$

$$\alpha = -5/6$$

$$\int_0^1 \frac{1}{x^{5/6}} \log\left(\sqrt{x} + \sqrt[3]{x}\right) dx =$$

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{1}{x^{5/6}} \log\left(\sqrt{x} + \sqrt[3]{x}\right) dx$$

$$\int \frac{1}{x^{5/6}} \log\left(\sqrt{x} + \sqrt[3]{x}\right) dx =$$

$$\frac{x^{1/6}}{1/6} \log\left(\sqrt{x} + \sqrt[3]{x}\right) - \underbrace{\int \frac{x^{1/6}}{1/6} \frac{\left(\sqrt{x} + \sqrt[3]{x}\right)'}{\left(\sqrt{x} + \sqrt[3]{x}\right)} dx}_{I}$$

$$I = 6 \int \sqrt[6]{x} \frac{1}{\sqrt{x} + \sqrt[3]{x}} \cdot \left(\frac{1}{2\sqrt{x}} + \frac{1}{3x^{2/3}}\right) dx$$

$$x = t^6 \qquad dx = 6t^5 dt$$

$$= 6\int \frac{t}{t^3+t^2}\cdot\left(\frac{1}{2t^3}+\frac{1}{3t^4}\right)6t^5 dt$$

$$6\int \frac{t}{t^3+t^2}\,3t^2 dt + 6\int \frac{t}{t^3+t^2}\cdot 2t\, dt$$

$$= 18\int \frac{t}{t+1}\,dt + 12\int \frac{dt}{t+1} =$$

$$= 18\int dt - 18\int \frac{dt}{t+1} + 12\int \frac{dt}{t+1} =$$

$$= 18t - 6\log(1+t) =$$

$$= 18\sqrt[6]{x} - 6\log\left(1+\sqrt[6]{x}\right)$$

$$\lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \frac{1}{x^{5/6}} \log\left(\sqrt{x} + \sqrt[3]{x}\right) dx$$

$$\int \frac{1}{x^{5/6}} \log\left(\sqrt{x} + \sqrt[3]{x}\right) dx =$$

$$6x^{\frac{1}{6}} \log\left(\sqrt{x} + \sqrt[3]{x}\right)$$

$$- \left(18 \sqrt[6]{x} - 6 \log\left(1 + \sqrt[6]{x}\right)\right)$$

$$\lim_{\varepsilon \to 0^+} \Big(6 \log(2) - 18 + 6 \log(2)$$

$$- 6 \varepsilon^{\frac{1}{6}} \log\left(\sqrt{\varepsilon} + \sqrt[3]{\varepsilon}\right)$$

$$+ 18 \varepsilon^{\frac{1}{6}} - 6 \log\left(1 + \varepsilon^{\frac{1}{6}}\right)\Big) =$$

$$= 12 \log 2 - 18$$

$$\lim_{\varepsilon \to 0^+} \varepsilon^{\frac{1}{6}} \log\left(\sqrt{\varepsilon} + \sqrt[3]{\varepsilon}\right) = 0$$

$$\int_0^{+\infty} \underbrace{\frac{e^{-(1+\alpha)x}\left(x+\sqrt{e^x-1}\right)}{\left(x\log\left(1+\frac{1}{x}\right)\right)^{\alpha}}}_{f_\alpha(x)}\,dx$$

$$f_\alpha(x) \underset{x\to 0^+}{=} \frac{(1+o(1))\left(x+\sqrt{x+o(x)}\right)}{\left(x\log\left(\frac{1}{x}(1+o(1))\right)\right)^{\alpha}} =$$

$\log\frac{1}{x} = |\log(x)|$
$0<x<1$

$$= \frac{\sqrt{x}\,(1+o(1))}{(x|\log(x)|)^{\alpha}} = \frac{1}{x^{\alpha-\frac{1}{2}}}\,\frac{1+o(1)}{|\log(x)|^{\alpha}}$$

Se $\alpha \leq 0$ $\lim_{x\to 0^+} f_\alpha(x) = 0$

e $f \in \mathcal{R}(0,t)$ $\forall\, t>0$

Se $\alpha > 0$, $\alpha - \frac{1}{2} < 1$ $\left(0 < \alpha < \frac{3}{2}\right)$

$$\lim_{x \to 0^+} \frac{f_\alpha(x)}{\frac{1}{x^{\alpha - \frac{1}{2}}}} = \lim_{x \to 0^+} \frac{1}{|\log(x)|^\alpha} = 0^+$$

T. CONFRONTO ASINTOTICO CON

$L = 0$, "$f_\alpha(x) < \frac{1}{x^{\alpha - \frac{1}{2}}}$"

$$\int_0^1 \frac{1}{x^{\alpha - \frac{1}{2}}}\, dx$$ CONVERGE se $\alpha - \frac{1}{2} < 1$

$$\int_0^1 f_\alpha(x)\, dx$$ CONVERGE

Se $\alpha = \frac{3}{2}$ $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{x|\log(x)|^{3/2}}$

$$\int_0^{1/2} \frac{dx}{x\,|\log(x)|^{\beta}}$$ CONVERGE $\beta > 1$

DIVERGE $\beta \leq 1$.

$$\int_0^{1/2} \frac{dx}{x\,|\log(x)|^{\frac{3}{2}}}$$ CONVERGE

$$\int_0^{1/2} f_\alpha(x)\,dx$$ CONVERGE

Se $\alpha \leq \frac{3}{2}$

Se $\alpha > \frac{3}{2}$, $\alpha - \frac{1}{2} > 1$,

$$g_\beta(x) = \frac{1}{x^\beta} , \quad \beta \in \left(1, \alpha - \frac{1}{2}\right)$$

$\beta = \alpha - \frac{1}{2} - \varepsilon$, $\alpha - \frac{1}{2} - \varepsilon > 1$

$$\lim_{x \to 0^+} \frac{f_\alpha(x)}{g_\beta(x)} = \left\{ \alpha - \frac{1}{2} - \beta > 0 \right\}$$

$$= \lim_{x \to 0^+} \frac{1}{x^{\alpha - \frac{1}{2} - \beta} |\log(x)|^\alpha} = \frac{1}{0^+} = +\infty$$

$\int_0^{\frac{1}{2}} g_\beta(x)\,dx$ DIVERGE

$\Rightarrow \int_0^{\frac{1}{2}} f_\alpha(x)\,dx$ DIVERGE

$$\int_0^{1/2} f_\alpha(x)\,dx \quad \text{CONVERGE} \iff \alpha \leq \frac{3}{2}$$

$$f_\alpha(x) \underset{x \to +\infty}{=} \frac{e^{-(1+\alpha)x}\, e^{\frac{x}{2}}\,(1+o(1))}{\left(x \cdot \left(\frac{1}{x}+o\left(\frac{1}{x}\right)\right)\right)^\alpha} =$$

$$= e^{-x\left(\alpha+\frac{1}{2}\right)}(1+o(1))$$

CONVERGE $\iff \alpha+\frac{1}{2}>0$

$$\int_1^{+\infty} f_\alpha(x)\,dx \quad \text{CONVERGE}$$

SSE $\alpha > -\frac{1}{2}$

$$\int_0^{+\infty} f_\alpha(x)\,dx \quad \text{CONVERGE}$$

SE E SOLO SE $-\frac{1}{2} < \alpha \leq \frac{3}{2}$

$$\int_0^{+\infty} \left(x + \sqrt{e^x - 1} \right) e^{-x} \, dx =$$

$$\boxed{\alpha = 0} \quad \int_0^{+\infty} x e^{-x} \, dx + \int_0^{+\infty} \sqrt{e^x - 1} \; e^{-x} \, dx =$$

$$\int_0^{+\infty} x e^{-x} = \lim_{M \to +\infty} \int_0^M x e^{-x} = \lim_{M \to +\infty} - \int_0^M x \left(e^{-x}\right)' dx$$

$$= \lim_{M \to +\infty} - x e^{-x} \Big|_0^M + \int_0^M e^{-x} \, dx$$

$$= \lim_{M \to +\infty} \left(-M e^{-M} - e^{-M} + 1 \right) = 1$$

$$\int_0^{+\infty} \sqrt{e^x - 1} \; e^{-x} \, dx =$$

$$e^x = t, \quad x = \log(t), \quad dx = \frac{dt}{t}$$

$$\int_1^{+\infty} \sqrt{t-1} \quad \frac{dt}{t^2} =$$

$s = \sqrt{t-1}$, $t-1 = s^2$, $dt = 2s\,ds$

$$\int_0^{+\infty} \frac{s \cdot 2s\,ds}{(1+s^2)^2} = 2\int_0^{+\infty} \frac{s^2\,ds}{(1+s^2)^2} =$$

$$2 \lim_{M \to +\infty} \int_0^M \frac{s^2}{(1+s^2)^2}\,ds =$$

$$= 2 \lim_{M \to +\infty} \int_0^M \frac{s\left(\frac{s^2}{2}\right)'\,ds}{(1+s^2)^2} =$$

$$= 2 \lim_{M \to +\infty} \frac{1}{2} \int_0^M s \left(-\frac{1}{1+s^2}\right)' ds =$$

$$= \lim_{M \to +\infty} \left(- \frac{s}{1+s^2} \Big|_0^M + \int_0^M \frac{ds}{1+s^2} \right) =$$

$$= \lim_{M \to +\infty} \left(- \frac{M}{1+M^2} + \operatorname{arctg}(M) \right) = \frac{\pi}{2}$$

$$\int_0^{+\infty} \left(x + \sqrt{e^x - 1} \right) e^{-x} \, dx = 1 + \frac{\pi}{2}$$

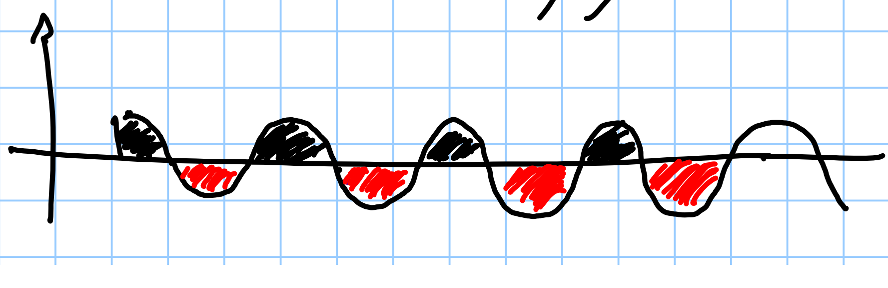
FACOLTATIVO

INTEGRALI IMPROPRI
CON TERMINI
CHE NON HANNO
SEGNO FISSATO
per "$x \to b^-$"

$$\int_1^{+\infty} \cos(x)\,dx = \lim_{t \to +\infty} \int_1^t \cos(x)\,dx$$

$$= \lim_{t \to +\infty} \sin(x)\Big|_1^t =$$

$$= \lim_{t \to +\infty} \big(\sin(t) - \sin(1)\big) \nexists$$



RICORDIAMO CHE SE
$f: [a,b) \to \mathbb{R}$, $b \in (a,+\infty]$,
$f \in \mathcal{R}(a,t)$, $\forall t \in (a,b)$
ALLORA L'INTEGRALE
IMPROPRIO $\int_a^b f(x)dx$ si dice
ASSOLUTAMENTE
CONVERGENTE
SE E' CONVERGENTE
$$\int_a^b |f(x)|\,dx.$$
IN PARTICOLARE
Se $\int_a^b f(x)\,dx$ è
ASSOLUTAMENTE

CONVERGENTE, ALLORA

$\int_a^b f(x)\,dx$ CONVERGE

$$\int_1^{+\infty} \frac{\sin(x)}{x^4}\,dx$$

$$\int_1^{+\infty} \frac{|\sin(x)|}{x^4}\,dx$$

$$0 \leq \frac{|\sin(x)|}{x^4} \leq \frac{1}{x^4}$$

$\int_1^{+\infty} \frac{|\sin(x)|}{x^4}\,dx$ CONVERGE

$\Longrightarrow$ $\int_1^{+\infty} \frac{\sin(x)}{x^4}\,dx$ CONVERGE

$$\int_a^b f(x)\,dx \quad , \quad \int_a^b |f(x)|\,dx$$

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$$

$$\left|\int_a^b f(x)\,dx\right| = \left| \int_a^b f^+(x)\,dx - \int_a^b f^-(x)\,dx \right|$$

$$\leq \left|\int_a^b f^+(x)\,dx\right| + \left|\int_a^b f^-(x)\,dx\right| = \int_a^b |f(x)|\,dx$$

$$\left| \int_1^{+\infty} \frac{\sin(x)}{x^4}\,dx \right| \leq \underline{\int_1^{+\infty} \frac{|\sin(x)|}{x^4}\,dx}$$

SE $\int_a^b |f(x)|\,dx$ CONVERGE

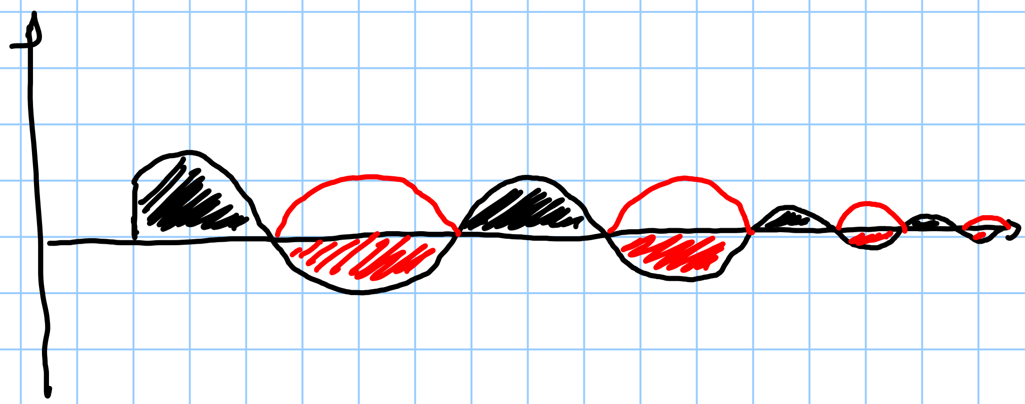
$$\Downarrow$$

$\int_a^b f(x)\,dx$ CONVERGE

$\exists$ ESEMPI di $f(x)$!

$$\int_a^b f(x)\,dx \text{ CONVERGE}$$

MA $$\int_a^b |f(x)|\,dx \text{ DIVERGE}$$



$$f(x) = \frac{\sin(x)}{\sqrt{x}} \quad , x \in [1, +\infty)$$

$$\int_1^{+\infty} \frac{\sin(x)}{\sqrt{x}}\,dx \quad \text{CONVERGE}$$

$$\lim_{t \to +\infty} \int_1^t \frac{\sin(x)}{\sqrt{x}} = \left\{ \begin{array}{c} \text{Integriamo} \\ \text{PER PARTI} \end{array} \right\}$$

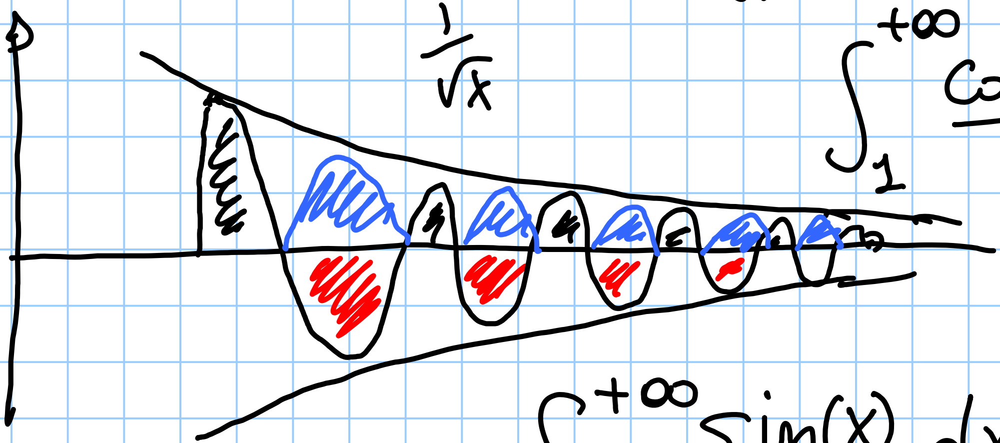
$$\lim_{t\to+\infty}\left[-\frac{\cos(x)}{\sqrt{x}}\Big|_1^t-\frac{1}{2}\int_1^t\frac{\cos(x)}{x^{3/2}}\,dx\right]$$

$$=\lim_{t\to+\infty}\left[-\frac{\cos(t)}{\sqrt{t}}+\cos(1)\right]-\frac{1}{2}\int_1^{+\infty}\frac{\cos(x)}{x^{3/2}}\,dx$$

$$=\cos(1)-\frac{1}{2}\int_1^{+\infty}\frac{\cos(x)}{x^{3/2}}\,dx$$

$$\left|\frac{\cos(x)}{x^{3/2}}\right|\le\frac{1}{x^{3/2}},\quad\int_1^{+\infty}\frac{1}{x^{3/2}}\,dx \text{ CONVERGE}$$

$$\Downarrow$$



$$\int_1^{+\infty}\frac{\cos(x)}{x^{3/2}}\,dx \text{ CONVERGE}$$

$$\int_1^{+\infty}\frac{\sin(x)}{\sqrt{x}}\,dx \quad \text{CONVERGE}$$

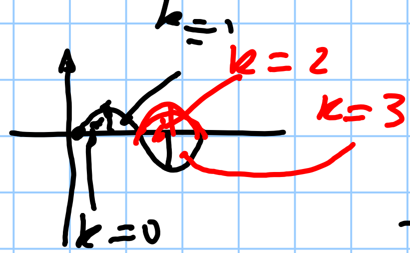
$$\int_1^{+\infty} \frac{|\sin(x)|}{\sqrt{x}}\,dx = \left\{ \text{DIVERGE} \right\} =$$

$$= \lim_{t \to +\infty} \int_1^{t} \frac{|\sin(x)|}{\sqrt{x}}\,dx \qquad \left[\begin{array}{l} \text{N.B.} \\ \text{IL} \\ \text{LIMITE} \\ \exists \end{array} \right.$$

$m \in \mathbb{N} \qquad t_m \geq m\frac{\pi}{2}$

$$\int_1^{t_m} \frac{|\sin(x)|}{\sqrt{x}}\,dx \geq \int_{\frac{\pi}{2}}^{m\frac{\pi}{2}} \frac{|\sin(x)|}{\sqrt{x}}\,dx$$

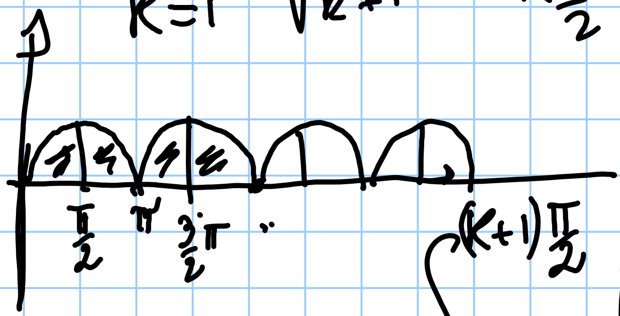
$$= \sum_{k=1}^{m-1} \int_{k\frac{\pi}{2}}^{(k+1)\frac{\pi}{2}} \frac{|\sin(x)|}{\sqrt{x}}\,dx \geq$$



$$\geq \left\{ \begin{array}{l} k\frac{\pi}{2} \leq x \leq (k+1)\frac{\pi}{2} \Rightarrow \dfrac{1}{x} \geq \dfrac{1}{(k+1)\frac{\pi}{2}} \\ \dfrac{1}{\sqrt{x}} \geq \sqrt{\dfrac{2}{\pi}}\,\dfrac{1}{\sqrt{k+1}} \end{array} \right\}$$

$$\geq \sum_{k=1}^{m-1} \sqrt{\frac{2}{\pi}} \int_{k\frac{\pi}{2}}^{(k+1)\frac{\pi}{2}} \frac{|\sin(x)|}{\sqrt{k+1}}\, dx =$$

$$= \sqrt{\frac{2}{\pi}} \sum_{k=1}^{m-1} \frac{1}{\sqrt{k+1}} \int_{k\frac{\pi}{2}}^{(k+1)\frac{\pi}{2}} |\sin(x)|\, dx =$$



$$\int_{k\frac{\pi}{2}}^{(k+1)\frac{\pi}{2}} |\sin(x)|\, dx = \int_{0}^{\frac{\pi}{2}} |\sin(x)|\, dx$$

$$= 1$$

$$= \sqrt{\frac{2}{\pi}} \sum_{k=1}^{m-1} \frac{1}{\sqrt{k+1}}$$

$$\sum_{k=1}^{m-1} \frac{1}{\sqrt{k+1}} = \overbrace{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \dots + \frac{1}{\sqrt{m-1}} + \frac{1}{\sqrt{m}}}^{m-1 \text{ TERMINI}}$$

$$> \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{m}} + \dots + \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{m}}$$

$$= \frac{n-1}{\sqrt{n}} = \sqrt{n} - \frac{1}{\sqrt{n}} \xrightarrow[n \to +\infty]{} +\infty$$

$$\lim_{n \to +\infty} \int_1^{t_n} \frac{|\sin(x)|}{\sqrt{x}} dx \geq \lim_n \sqrt{\frac{2}{\pi}} \frac{n-1}{\sqrt{n}} = +\infty$$

SE $\int_1^{+\infty} f(x)\,dx$ CONVERGE

E' VERO CHE

$f(x) \to 0$, $x \to +\infty$?

NO

$$\int_1^{+\infty} \sin(x^2)\,dx =$$

$x^2 = y \qquad x = \sqrt{y}$

$dx = \frac{1}{2\sqrt{y}} dy$

$$= \int_1^{+\infty} \frac{\sin(y)}{2\sqrt{y}} dy \quad \text{CONVERGE}$$

SE $\int_1^{+\infty} f(x)\,dx$ CONVERGE

e f è MONOTONA

$\Longrightarrow f \to 0,\ x \to +\infty.$

$$\Gamma(k+1) = \int_0^{+\infty} e^{-x} x^k\,dx$$

$k \in \mathbb{N} \cup \{0\}$

FUNZIONE GAMMA di EULERO

$$\Gamma(1) = \int_0^{+\infty} e^{-x}\,dx = 1$$

$k \geq 1$

$$\int_0^{+\infty} e^{-x} x^k\,dx = \int_0^{+\infty} x^k (-e^{-x})'\,dx$$

$$= \underbrace{\left(-x^k e^{-x}\right)\Big|_0^{+\infty}}_{=0} + \int_0^{+\infty} e^{-x} k x^{k-1}\,dx =$$

$$\lim_{t \to +\infty} \left(-x^k e^{-x}\right)_0^t = 0$$

$$= k \int_0^{+\infty} e^{-x} x^{k-1} dx = k\,\Gamma(k)$$

$$\Gamma(1) = 1$$

$$\Gamma(k+1) = k\,\Gamma(k) = k(k-1)\,\Gamma(k-1) =$$

$$= k(k-1)(k-2)\,\Gamma(k-2) \ldots\ldots$$

$$= k!$$

FUNZIONE GAMMA di EULERO

RAPPRESENTAZIONE INTEGRALE DEI NUMERI FATTORIALI

$$\boxed{0! := \Gamma(1) = 1}$$

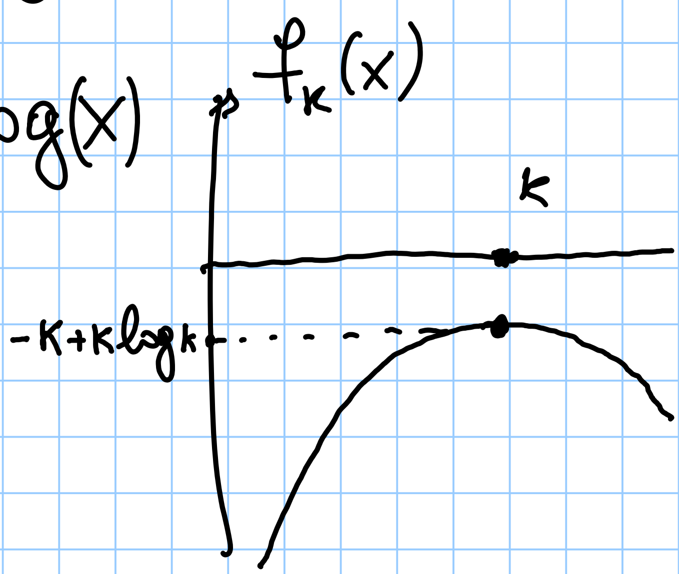
$$k! = \int_0^{+\infty} e^{-x} x^k \, dx = \Gamma(k+1)$$

$$k \in \mathbb{N} \cup \{0\}$$

FORMULA DI STIRLING (CENNI)

$$\int_0^{+\infty} e^{-x} x^k \, dx = \int_0^{+\infty} e^{f_k(x)} \, dx$$

$$f_k(x) = -x + k \log(x)$$



$$f_k'(x) = -1 + \frac{k}{x}$$

$$f_k'(k) = 0$$

$$> 0 \iff x < k$$

$$f_k(x) = f_k(k) + f_k'(k)(x-k) + \frac{1}{2} f''(k)(x-k)^2 + R_2(x,k)$$

$$= -k + k\log(k) - \frac{k}{2k^2}(x-k)^2 + R_2(x,k)$$

$$e^{f_k(x)} = e^{-k + k\log(k) - \frac{(x-k)^2}{2k} + R_2(x,k)}$$

$$= k^k e^{-k} e^{-\frac{(x-k)^2}{2k}} e^{R_2(x,k)}$$

$$\int_0^{+\infty} e^{-x} x^k \, dx = k^k e^{-k} \int_0^{+\infty} e^{-\frac{(x-k)^2}{2k} + \cancel{R_2(x,k)}} \, dx$$

TRASCURANDO IL TERMINE R_2

$$\int_0^{+\infty} e^{-\frac{(x-k)^2}{2k}} \, dx = \left\{ \frac{x-k}{\sqrt{k}} = y \right\}$$

$$\sqrt{k}\int_{-\sqrt{k}}^{+\infty} e^{-y^2/2}\,dy = \sqrt{k}\left(\int_{-\infty}^{+\infty} e^{-y^2/2}dy - \int_{-\infty}^{-\sqrt{k}} e^{-y^2/2}dy\right)$$

$$\int_{-\infty}^{+\infty} e^{-y^2/2}\,dy = \sqrt{2\pi}$$

$$\lim_{x\to-\infty}\frac{\int_{-\infty}^{x} e^{-y^2/2}\,dy}{e^{-x^2/2}} \overset{\text{"H"}}{=}$$

$$\lim_{x\to-\infty}\frac{e^{-x^2/2}}{-x\,e^{-x^2/2}} = 0$$

$$\int_{-\infty}^{-\sqrt{k}} e^{-y^2/2}\,dy = o\left(e^{-k/2}\right),\ k\to+\infty$$

ANCHE IL CONTRIBUTO DOVUTO AL TERMINE R_2 è $o(k^k e^{-k}\sqrt{k})$

$$\int_0^{+\infty} e^{-x} x^k \, dx = k^k e^{-k} \sqrt{2\pi k}\,\big(1 + o(1)\big)$$

FORMULA di STIRLING

$$k! = \sqrt{2\pi k}\; k^k e^{-k} \big(1 + o(1)\big), \quad k \to +\infty$$

NOTA: $o(1) = \dfrac{c + o(1)}{k^2}$, quindi

$$k! = k^k e^{-k} \sqrt{2\pi k} + c\, k^k e^{-k} \frac{1 + o(1)}{k^{3/2}}$$

IL TERMINE CHE TRASCURIAMO È MOLTO GRANDE

$$f_\alpha(x) = \frac{\log(\sqrt{x} + \sqrt[3]{x})}{x^{|\alpha|}}, \quad |\alpha| \in [0,1)$$

$$\int_0^1 f_\alpha(x)\,dx \quad ; \quad g_\alpha(x) = \frac{1}{x^{|\alpha|}}$$

$$\lim_{x \to 0} \frac{|f_\alpha(x)|}{g_\alpha(x)} = \lim_{x \to 0} \left| \log\left(\sqrt[3]{x} + o(\sqrt[3]{x})\right) \right|$$

$$= +\infty \qquad \text{" } f_\alpha \geq g_\alpha \text{ "}$$

$$\int_0^1 g_\alpha(x)\,dx \quad \text{converge} \qquad ?$$

IL CONFRONTO ASINTOTICO NON DICE NIENTE

$$\circledast \quad \int_0^{1/2} \frac{dx}{x\,|\log(x)|^\beta}$$

CONVERGE $\beta > 1$

DIVERGE $\beta \leq 1$

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^{1/2} \frac{dx}{x\,|\log x|^\beta} =$$

$\log x = t \quad , x = e^t$

$$\lim_{\varepsilon \to 0^+} \int_{\log \varepsilon}^{-\log 2} \frac{\cancel{e^t}\,dt}{\cancel{e^t}\,|t|^\beta} =$$

$t = -y$

$$\lim_{\varepsilon \to 0^+} \int_{\log 2}^{\log \frac{1}{\varepsilon}} \frac{dy}{y^\beta} =$$

$$\boxed{\beta \neq 1}$$

$$\lim_{\varepsilon \to 0^+} \frac{y^{1-\beta}}{1-\beta} \Bigg|_{\log 2}^{\log \frac{1}{\varepsilon}} =$$

$$\lim_{\varepsilon \to 0^+} \frac{\left(\log\left(\frac{1}{\varepsilon}\right)\right)^{1-\beta}}{1-\beta} - \frac{(\log 2)^{1-\beta}}{1-\beta} =$$

$$= \begin{cases} +\infty & \text{se } 1-\beta > 0, \ \beta < 1 \\ \dfrac{(\log 2)^{1-\beta}}{\beta - 1} & \text{se } 1-\beta < 0, \ \beta > 1 \end{cases}$$

Se $\beta = 1$

$$\lim_{\varepsilon \to 0^+} \int_{\log 2}^{\log \frac{1}{\varepsilon}} \frac{dy}{y} =$$

$$\lim_{\varepsilon \to 0^+} \log(y) \Big|_{\log 2}^{\log \frac{1}{\varepsilon}} = +\infty .$$