$$\int_1^{+\infty} \frac{dx}{x^\alpha} \quad \text{CONVERGE} \iff \alpha > 1$$

$$\int_1^{+\infty} \frac{dx}{x^\alpha} = \lim_{M\to+\infty} \int_1^M \frac{dx}{x^\alpha} = \{\alpha \neq 1\}$$

$$= \lim_{M\to+\infty} \frac{1}{1-\alpha} x^{1-\alpha} \Big|_1^M = \lim_{M\to+\infty} \frac{M^{1-\alpha}}{1-\alpha} - \frac{1}{1-\alpha}$$

$$= \begin{cases} \frac{1}{\alpha-1}, & \alpha > 1 \\ +\infty, & \alpha < 1 \end{cases}$$

Se $\alpha = 1$, $\lim_{M\to+\infty} \int_1^M \frac{dx}{x} = \lim_{M\to+\infty} \log(M) = +\infty$

$$\int_1^{+\infty} \frac{dx}{x^\alpha} \quad \text{CONVERGE} \iff \alpha > 1$$

$$\int_0^{1} \frac{dx}{x^\alpha} \quad \text{CONVERGE} \iff \alpha < 1$$

$$\int_1^{+\infty} \frac{2+\sin(x)}{\sqrt{x}}\, dx$$

$$f(x) = \frac{2+\sin(x)}{\sqrt{x}} \geq \frac{2-1}{\sqrt{x}} = g(x)$$

$$\int_1^{+\infty} g(x)\, dx = \int_1^{+\infty} \frac{dx}{\sqrt{x}}$$

DIVERGE

T. di CONFRONTO

$$\Longrightarrow \int_1^{+\infty} f(x)\, dx \quad \text{DIVERGE}$$

$$\int_{-1}^{0} \frac{|\sin(x+1)|}{\sqrt{|x|}} \; ;$$

$$0 \le f(x) = \frac{|\sin(x+1)|}{\sqrt{|x|}} \le \frac{1}{\sqrt{|x|}}, \forall\, x \in (-1,0)$$

$$= g(x)$$

$$\int_{-1}^{0} g(x)\, dx = \int_{-1}^{0} \frac{1}{\sqrt{|x|}}\, dx \text{, CONVERGE}$$

$\Longrightarrow$ T. di CONFRONTO

$$\int_{-1}^{0} f(x)\, dx \quad \text{CONVERGE}$$

$$\int_{-1}^{1} \frac{1}{|x|^{\alpha}}\, dx \quad \begin{cases} \nearrow \text{CONVERGE SE } \alpha < 1 \\ \searrow \text{DIVERGE SE } \alpha \ge 1 \end{cases}$$

$$\int_{x_0-1}^{x_0+1} \frac{1}{|x-x_0|^\alpha}\,dx \;\begin{cases} \nearrow \text{ CONVERGE SE } \alpha < 1 \\ \searrow \text{ DIVERGE SE } \alpha \geq 1 \end{cases}$$

IMPORTANTE

Se $f \in \mathcal{R}(a,t)$, $\forall t \in (a,b)$

$$0 \geq f(x) \geq g(x) \quad \forall x \in [x_0, b)$$

Si ha

$\int_a^b f(x)\,dx$ DIVERGE $\Rightarrow \int_a^b g(x)\,dx$ DIVERGE

$\int_a^b g(x)\,dx$ CONVERGE $\Rightarrow \int_a^b f(x)\,dx$ CONVERGE

$$\int_0^1 \frac{(-\cos(x))}{\sqrt[4]{x}}\,dx \quad ; \quad f(x) = \frac{-\cos(x)}{\sqrt[4]{x}}$$

$$x \in (0,1]$$

$$0 \geq f(x) = \frac{-\cos(x)}{\sqrt[4]{x}} \geq -\frac{1}{\sqrt[4]{x}} = g(x)$$

$$\int_0^1 -\frac{1}{\sqrt[4]{x}}\,dx = -\int_0^1 \frac{1}{\sqrt[4]{x}}\,dx$$

CONVERGE

quindi $\displaystyle\int_0^1 \frac{-\cos(x)}{\sqrt[4]{x}}\,dx$ CONVERGE

CONVIENE
USARE LA NOZIONE di
CONVERGENZA
ASSOLUTA

DEFINIZIONE

$f : [a, b) \to \mathbb{R}$, $b \in (a, +\infty]$,

$f \in \mathcal{R}(a, t)$, $\forall t \in (a, b)$

Si dice che L'INTEGRALE IMPROPRIO $\int_a^b f(x)\,dx$ è

ASSOLUTAMENTE CONVERGENTE

SE È CONVERGENTE

$$\int_a^b |f(x)|\,dx$$

TEOREMA

SE $\int_a^b f(x)\,dx$ è ASSOLUTAMENTE

CONVERGENTE, ALLORA

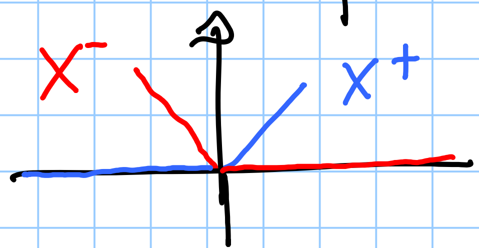
$\int_a^b f(x)\,dx$ è CONVERGENTE

"DIM" e $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

$\boxed{f = f^+ - f^-}$, $f^+ = \max\{0, f\}$

$f^- = \max\{0, -f\}$

$f(x) = x$



$f(x) = x^+ - x^-$

$x^+ = \begin{cases} x & \text{se } x \ge 0 \\ 0 & \text{se } x < 0 \end{cases}$

$x^- = \begin{cases} 0 & \text{se } x > 0 \\ |x| & \text{se } x \le 0 \end{cases}$

$0 \le x^+ \le |x|$

$0 \le x^- \le |x|$

$0 \le f^+(x) \le |f(x)|$, $0 \le f^-(x) \le |f(x)|$

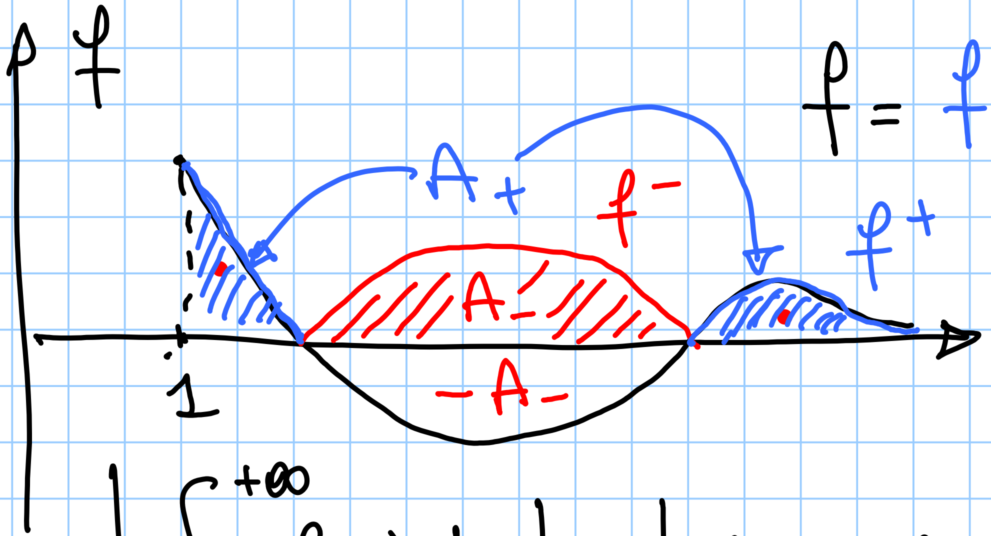
T. di CONFRONTO: PER IPOTESI $\exists$ FINITO

$$0 \le \int_a^b f^+(x)\,dx \le \int_a^b |f(x)|\,dx$$

$$0 \leq \int_a^b f^-(x)\,dx \leq \int_a^b |f(x)|\,dx$$

$$\Longrightarrow \quad \int_a^b f(x)\,dx = \int_a^b f^+(x)\,dx - \int_a^b f^-(x)\,dx \qquad \blacksquare$$

$$f = f^+ - f^-$$



$$\left| \int_1^{+\infty} f(x)\,dx \right| = \left| A_+ - A_- \right| \leq$$

$$\leq \int_1^{+\infty} |f(x)|\,dx = A_+ + A_-$$

$$\int_0^1 \frac{(-\cos(x))}{\sqrt[4]{x}}\,dx \quad ; \quad f(x) = -\frac{\cos(x)}{\sqrt[4]{x}}$$

$$x \in (0,1]$$

$$|f(x)| = \frac{\cos(x)}{\sqrt[4]{x}} \quad , \; x \in (0,1]$$

$$|f(x)| \leq \frac{1}{\sqrt[4]{x}} = g(x) \quad , \; x \in (0,1]$$

$$\int_0^1 g(x)\,dx \text{ CONVERGE} \Rightarrow$$

$$\int_0^1 |f(x)|\,dx \text{ CONVERGE}$$

cioè $\int_0^1 f(x)\,dx$ CONVERGE

ASSOLUTAMENTE e in
PARTICOLARE CONVERGE.

$$\int_1^{+\infty} \frac{\sin(x)}{x^4}\,dx \ ; \ f(x) = \frac{\sin(x)}{x^4}$$

$$|f(x)| = \frac{|\sin(x)|}{x^4} \leq \frac{1}{x^4} = g(x)$$

$$\int_1^{+\infty} \frac{1}{x^4}\,dx \quad \text{CONVERGE}$$

$$\int_1^{+\infty} f(x)\,dx \quad \text{CONVERGE}$$

ASSOLUTAMENTE

e in

PARTICOLARE CONVERGE.

$$\int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^2} = \int_{-\infty}^{0} \frac{dx}{(1+x^2)^2} + \int_{0}^{+\infty} \frac{dx}{(1+x^2)^2}$$

$$f(x) = \frac{1}{(1+x^2)^2} \overset{x\to\pm\infty}{=} \frac{1}{x^4(1+o(1))}$$

$$f(x) = \frac{1}{x^4(1+o(1))} = \frac{1}{x^4}(1+o(1))$$

$o(1) \leq \frac{1}{2}$, DEFINITIVAMENTE PER $x \to +\infty$

$$f(x) \leq \frac{1+o(1)}{x^4} \leq \frac{3}{2}\frac{1}{x^4}, \quad x \to +\infty$$

$x \to +\infty$

$$\exists x_0 : f(x) \leq g(x) = \frac{3}{2}\frac{1}{x^4} \quad \forall x > x_0$$

$$\int_1^{+\infty} g(x)\,dx \quad \text{CONVERGE}$$

$$\Downarrow$$

$$\int_1^{+\infty} f(x)\,dx \quad \text{CONVERGE}$$

$$\int_0^{+\infty} f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^{+\infty} f(x)\,dx$$

CONVERGE

ANALOGAMENTE $\int_{-\infty}^{0} f(x)\,dx$

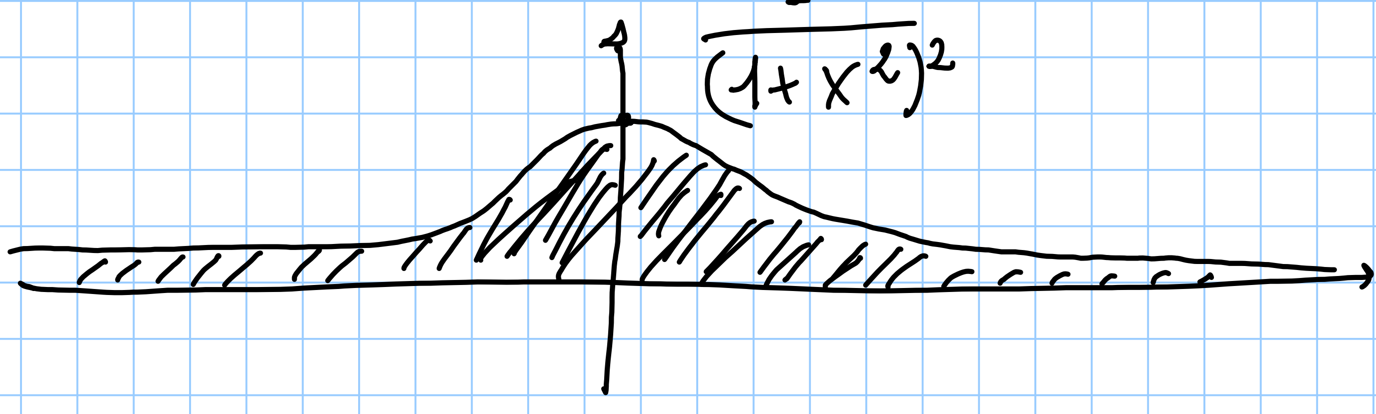
CONVERGE

$$\int_{-\infty}^{+\infty} \frac{1}{(1+x^2)^2}\,dx = \lim_{\substack{t\to+\infty \\ s\to-\infty}} \int_s^t \frac{dx}{(1+x^2)^2} =$$

$$\left\{ \int \frac{dx}{(1+x^2)^2} = \frac{1}{2}\operatorname{arctg}(x) + \frac{1}{2}\,\frac{x}{1+x^2} + C \right\}$$

$$= \lim_{\substack{t \to +\infty \\ s \to -\infty}} \left(\frac{1}{2}\operatorname{arctg}(t) + \frac{1}{2}\,\frac{t}{1+t^2} - \frac{1}{2}\operatorname{arctg}(s) - \frac{1}{2}\,\frac{s}{1+s^2} \right) =$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}.$$



$$\int_{-\infty}^{0} e^{x}\,dx = \lim_{M\to-\infty} \int_{M}^{0} e^{x}\,dx =$$

$$= \lim_{M\to-\infty} e^{x}\Big|_{M}^{0} = \lim_{M\to-\infty} \left(1-e^{M}\right)$$

$$= 1$$. CONVERGE

$$\int_{0}^{+\infty} e^{-x}\,dx$$ CONVERGE

Ex.

$$\int_{1}^{+\infty} e^{-x^2}\,dx \quad ?$$

ANCHE SE NON SAPPIAMO CALCOLARE UNA PRIMITIVA

PER DIMOSTRARE LA CONVERGENZA È SUFFICIENTE STABILIRE CHE

(1) $e^{-x^2} \leq \frac{1}{x^2} \quad \forall x \geq x_0$

oppure

(2) $e^{-x^2} \leq e^{-x} \quad \forall x \geq x_0$

LA (2) è OVVIA PER $x \geq 1$

LA (1) ?

$$\lim_{x \to +\infty} \frac{e^{-x^2}}{\frac{1}{x^2}} = \lim_{x \to +\infty} x^2 e^{-x^2} = 0$$

$\Longrightarrow$ PER $\varepsilon = 1 \ \exists M_\varepsilon$:

$0 \leq x^2 e^{-x^2} < \varepsilon \quad \forall x > M_\varepsilon \Longrightarrow$

$$e^{-x^2} < \frac{1}{x^2} \quad \forall x > x_0$$

$$x_0 = M_1$$

Per il T. di CONFRONTO

$$\int_1^{+\infty} e^{-x^2}\,dx \quad \text{CONVERGE}$$

TEOREMA [CONFRONTO ASINTOTICO]

$f : [a,b) \to \mathbb{R}$, $g : [a,b) \to \mathbb{R}$

$b \in (a, +\infty]$

$f \in \mathcal{R}(a,t)$, $g \in \mathcal{R}(a,t)$

$\forall t \in (a,b)$

(1) $f(x) \geq 0$, $g(x) \geq 0$, DEFINITIVAMENTE

per $x \to b^-$

(2) $\lim_{x \to b^-} \frac{f(x)}{g(x)} = L$

ALLORA • $L \in [0, +\infty]$ e si ha:

(i) SE $L \in (0, +\infty) \Longrightarrow$

$\int_a^b f(x)\,dx$ CONVERGE (DIVERGE) SE E SOLO SE $\int_a^b g(x)\,dx$ CONVERGE (DIVERGE)

(ii) SE $L = 0 \Longrightarrow$

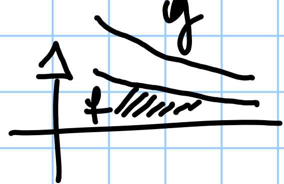
SE $\int_a^b f(x)\,dx$ DIVERGE

ALLORA $\int_a^b g(x)\,dx$ DIVERGE

SE $\int_a^b g(x)\,dx$ CONVERGE

ALLORA $\int_a^b f(x)\,dx$ CONVERGE

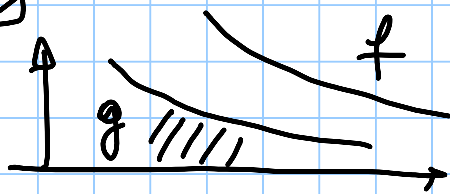
"$f < g$"



$$\frac{f(x)}{g(x)} \longrightarrow 0^+ \qquad x \longrightarrow b^-$$

(iii) SE $L = +\infty$

"$f > g$"



SE $\int_a^b f(x)\,dx$ CONVERGE

ALLORA $\int_a^b g(x)\,dx$ CONVERGE

SE $\int_a^b g(x)\,dx$ DIVERGE

ALLORA $\int_a^b f(x)\,dx$ DIVERGE

DIMOSTRAZIONE, $b=+\infty$

(i) Da (1) $\exists$ (FINITI O INFINITI)

$$\lim_{t\to+\infty} I(t)=\int_a^t f(x)\,dx$$

PERCHE'?
Ex.

$$\lim_{t\to+\infty} J(t)=\int_a^t g(x)\,dx$$

da (2)

$\forall \varepsilon>0 \ \exists x_\varepsilon \geq a$:

- $$L-\varepsilon < \frac{f(x)}{g(x)} < L+\varepsilon \quad \forall x > x_\varepsilon$$

- $$(L-\varepsilon)g(x) \leq f(x) \leq g(x)(L+\varepsilon) \quad \forall x > x_\varepsilon$$

$L>0$, $\varepsilon=\frac{L}{2}>0$

$$\frac{L}{2}g(x) \le f(x) \le \frac{3}{2}L\,g(x)$$

$$\forall\, x > x_\varepsilon$$

DALLA disuguaglianza a DESTRA e dal T. del CONFRONTO si ha: SE $\int_a^b g(x)\,dx$ CONVERGE ALLORA $\int_a^b f(x)\,dx$ CONVERGE

DALLA disuguaglianza a SINISTRA e dal T. del CONFRONTO $\left[\, g(x) \le \frac{2}{L} f(x) \,\right]$: SE $\int_a^b f(x)\,dx$ CONVERGE ALLORA $\int_a^b g(x)\,dx$ CONVERGE

(ii) $L=0$

Da (1)

$$\forall \varepsilon>0, \exists X_\varepsilon \geq a : 0 \leq \frac{f(x)}{g(x)} < \varepsilon \quad \forall x > X_\varepsilon$$

$$0 \leq f(x) \leq \varepsilon g(x) \quad , \forall x > X_\varepsilon$$

LA TESI SEGUE COME SOPRA dal T. di CONFRONTO

(iii) $L=+\infty$

$$\forall M>0 \ \exists X_M \geq a : \frac{f(x)}{g(x)} > M \quad \forall x > X_M$$

$$f(x) \geq M g(x) \quad \forall x > X_M$$

LA TESI SEGUE COME SOPRA dal T. di CONFRONTO

$\blacksquare$

NOTA IMPORTANTE SU COME VERIFICARE (2)

$$\lim_{x \to b^-} \frac{f(x)}{g(x)} = L \iff \underset{x \to b^-}{f(x) = g(x)\,(L + o(1))}$$

$L \in (0, +\infty)$

$$\int_1^{+\infty} \frac{dx}{(1+2x^2)^2} \quad , \quad f(x) = \frac{1}{(1+2x^2)^2} \overset{x \to +\infty}{=} \frac{1}{x^4}\left(\frac{1}{4} + o(1)\right)$$

$g(x) = \frac{1}{x^4}$ T. CONFRONTO ASINTOTICO

$\int_1^{+\infty} g(x)\,dx$ CONVERGE $\Rightarrow$

$\int_1^{+\infty} f(x)\,dx$ CONVERGE

$$\int_1^2 \frac{\operatorname{arcTg}^2(x)}{(x-1)^2}\,dx\,, \quad f(x) = \left(\frac{\operatorname{arcTg}(x)}{x-1}\right)^2$$

$$f(x) \geq 0 \quad \forall \; x \in (1,2)$$

$$f(x) \underset{x \to 1}{=} \frac{\left(\frac{\pi}{4} + o(1)\right)^2}{(x-1)^2} = \frac{\left(\frac{\pi}{4}\right)^2 \left(1 + o(1)\right)}{(x-1)^2}$$

$$g(x) = \left(\frac{\pi}{4}\right)^2 \frac{1}{(x-1)^2}$$

$$\int_1^2 \frac{1}{(x-1)^2}\,dx \quad \text{DIVERGE}$$

del T. del CONFRONTO ASINTOTICO

$$\int_1^2 f(x)\,dx \quad \text{DIVERGE}$$

$$\int_1^2 \frac{\operatorname{arctg}(x)}{(x-1)^{\beta-2}}\,dx \qquad f_\beta(x) = \frac{\operatorname{arctg}(x)}{(x-1)^{\beta-2}}$$

Se $\beta - 2 \leq 0$ $\quad f_\beta \in \mathcal{R}(1,2)$

$\beta - 2 > 0$;

$$f_\beta(x) \leq \frac{\pi}{2} \cdot \frac{1}{(x-1)^{\beta-2}} \quad x \in (1,2)$$

T. di CONFRONTO $\Longrightarrow$ $\displaystyle\int_1^2 \frac{1}{(x-1)^\alpha}\,dx$ CONVERGE SE E SOLO SE $\alpha < 1$

$$\alpha = \beta - 2 < 1$$

L'INTEGRALE CONVERGE SE

$\beta < 3$

CON IL CONFRONTO ASINTOTICO

POSSIAMO DIRE DI PIÙ.

$$g_\beta(x) = \frac{1}{(x-1)^{\beta-2}}, \quad x \in (1,2)$$

$$\lim_{x \to 1^+} \frac{f_\beta(x)}{g_\beta(x)} = \frac{\pi}{4} \quad \text{o} \quad f_\beta(x) = g_\beta(x)\left(\frac{\pi}{4} + o(1)\right) \quad x \to 1^+$$

$\int_1^2 g_\beta(x)\,dx$ CONVERGE SE E SOLO SE $\beta < 3$

e dal T. del CONFRONTO ASINTOTICO

$\int_1^2 f_\beta(x)\,dx$ CONVERGE SE E SOLO SE $\beta < 3$

$$\int_0^1 \frac{(\sin(x))^{\frac{3}{2}}}{\mathrm{Tg}(x^2)}\,dx$$

$$f(x) = \frac{(\sin(x))^{3/2}}{\mathrm{Tg}(x^2)} \underset{x\to 0^+}{=} \frac{(x+o(x))^{3/2}}{x^2+o(x^2)}$$

$$= \frac{x^{3/2}}{x^2}\,\frac{(1+o(1))^{3/2}}{1+o(1)} = \frac{1}{\sqrt{x}}\,(1+o(1))$$

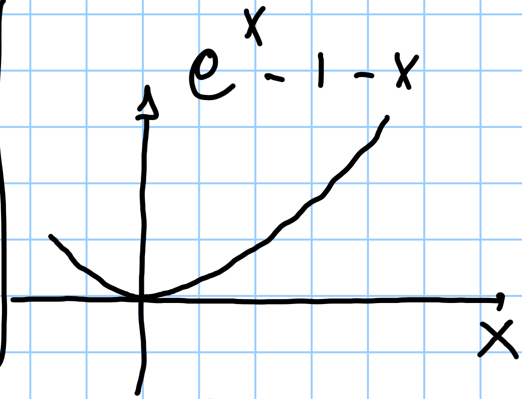
$$g(x) = \frac{1}{\sqrt{x}} \quad , \quad x \in (0,1)$$

$\int_0^1 g(x)\,dx$ CONVERGE

DAL T. del CONFRONTO ASINTOTICO

$\int_0^1 f(x)\,dx$ CONVERGE

$$\int_0^1 \frac{(1-\cos(x))^{\frac{1}{3}}}{|e^x-1-x|^\alpha}\,dx$$



$\alpha \in \mathbb{R}$

$\alpha \le 0 \quad f_\alpha \in \mathcal{R}(0,1)$

$$\boxed{\alpha > 0} \quad f_\alpha(x) \underset{x\to 0^+}{=} \frac{\left(1-\left(1-\frac{1}{2}x^2+o(x^2)\right)\right)^{\frac{1}{3}}}{\left|1+x+\frac{x^2}{2}+o(x^2)-1-x\right|^\alpha}$$

$$= \frac{2^{-\frac{1}{3}}\left(x^2+o(x^2)\right)^{\frac{1}{3}}}{\left|\frac{1}{2}x^2+o(x^2)\right|^\alpha} =$$

$$= 2^{\alpha-\frac{1}{3}}\,\frac{x^{\frac{2}{3}}}{x^{2\alpha}}\,\frac{(1+o(1))^{\frac{1}{3}}}{(1+o(1))^\alpha} =$$

$$= 2^{\alpha-\frac{1}{3}}\,(1+o(1))\,\frac{1}{x^{2\alpha-\frac{2}{3}}}$$

$$\int_0^1 \frac{1}{x^\beta}dx \quad \begin{cases} \text{CONVERGE SE } \beta<1 \\ \text{DIVERGE SE } \beta \geq 1 \end{cases}$$

$$2\alpha - \frac{2}{3} < 1$$

$$2\alpha < 5/3 \qquad \boxed{\alpha < 5/6}$$

CONVERGE SE

E SOLO SE $\alpha < 5/6$

$$\int_3^{+\infty} \frac{(\log(x))^4}{x^2}dx, \quad f(x) = \frac{(\log(x))^4}{x^2}$$

$$g(x) = \frac{1}{x^2} \quad ;$$

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} (\log(x))^4 = +\infty$$

"$f \geq g$" $\quad \int_3^{+\infty} f(x)\,dx \geq \int_3^{+\infty} g(x)\,dx$

Ma $\int_3^{+\infty} g(x)\,dx$ CONVERGE

IL T. del CONFRONTO NON DICE NIENTE IN QUESTO CASO

$$g(x) = \frac{1}{x^{3/2}}, \quad \lim_{x\to+\infty} \frac{f(x)}{g(x)} =$$

$$= \lim_{x\to+\infty} \frac{(\log(x))^4}{\sqrt{x}} = 0 \quad ; \quad \text{"} f \leq g \text{"}$$

$\int_3^{+\infty} g(x)\,dx$ CONVERGE $\Longrightarrow$

T. CONFRONTO ASINTOTICO $\int_3^{+\infty} f(x)\,dx$ CONVERGE

con $L = 0$

$$\int_3^{+\infty} \frac{\log^4(x)}{x}\,dx \quad , \quad f(x) = \frac{\log^4(x)}{x}$$

$$g(x) = \frac{1}{x} \; ;$$

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \log^4(x) = +\infty$$

"$f \geq g$"

$\int_3^{+\infty} g(x)\,dx$ DIVERGE $\Rightarrow$ $\int_3^{+\infty} f(x)\,dx$ DIVERGE

T. CONFRONTO ASINTOTICO

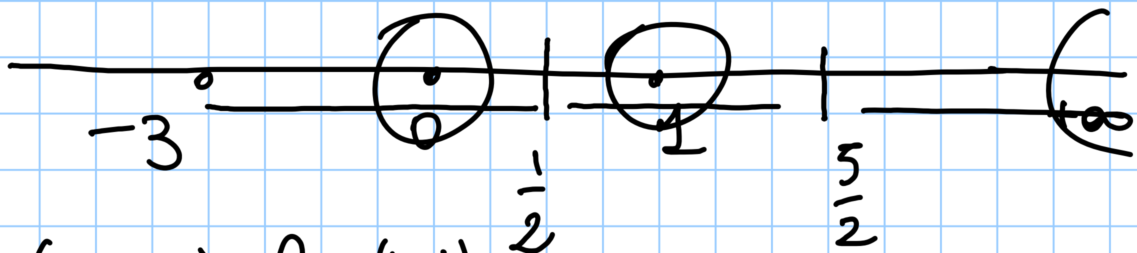
CON $L = +\infty$.

$$\int_{-3}^{+\infty} \frac{e^{-x^2}\log(|x|)}{|x-1|^{\frac{3}{2}}}\,dx$$

$$f(x) = \frac{e^{-x^2}\log(|x|)}{|x-1|^{3/2}}$$

$$\int_{-3}^{+\infty} f(x)\,dx = \int_{-3}^{\frac{1}{2}} f(x)\,dx + \int_{1/2}^{5/2} f(x)\,dx$$

$$+ \int_{5/2}^{+\infty} f(x)\,dx$$



$$f(x) \underset{x \to 0}{=} \frac{(1+o(1))\log(|x|)}{(1+o(1))^{3/2}} = \log|x|\,(1+o(1))$$

$$|f(x)| = \log \frac{1}{|x|}(1+o(1)) \quad , x \to 0$$

$$g(x) = \frac{1}{\sqrt{|x|}} \qquad \lim_{x\to 0} \frac{|f(x)|}{g(x)} =$$

$$= \lim_{x\to 0} \sqrt{|x|} \log\left(\frac{1}{|x|}\right)(1+o(1)) = 0^+ ,$$

"$f \leq g$"

$$y = \frac{1}{|x|} \qquad \lim_{x\to 0} \sqrt{|x|} \log\left(\frac{1}{|x|}\right)$$

$$= \lim_{y\to+\infty} \frac{\log(y)}{\sqrt{y}} = 0^+$$

T. CONFRONTO ASINTOTICO

con $L = 0$, $\int_{-3}^{1/2} g(x)\,dx$ CONVERGE

$$\Longrightarrow \int_{-3}^{1/2} |f(x)|\,dx \text{ CONVERGE}$$

$$\Longrightarrow \int_{-3}^{1/2} f(x)\,dx \text{ CONVERGE}$$