$$\int \sqrt{1-x^2}\,dx\,;$$

$$\int R\left(x, \sqrt{c+bx-ax^2}\right)dx$$

$$a>0, \quad \Delta = b^2-4ac>0$$

$$\int \sqrt{x^2-1}\,dx\,; \int \sqrt{x^2+1}\,dx$$

$$\int R\left(x, \sqrt{ax^2+bx+c}\right)dx$$

$$a>0$$

$$\Delta = b^2-4ac>0, \quad \Delta = b^2-4ac<0$$

$$\int R\left(x, \sqrt{1-x^2}\right) dx =$$

$$x = \sin(t) \quad , \sqrt{1-x^2} = |\cos(t)|$$

$$\int \tilde{R}\left(\sin(t), \cos(t)\right) dt$$

$$y = \operatorname{Tg}\left(\tfrac{t}{2}\right)$$

$$\int R\left(x, \sqrt{c+bx-ax^2}\right) dx =$$

$$a > 0$$

$$\Delta = b^2 - 4ac > 0$$

$$-ax^2 + bx + c = a\left(y_1^2 - (x-x_1)^2\right)$$

$$x - x_1 = t$$

$$= \int R\left(t, \sqrt{a\left(y_1^2 - t^2\right)}\right) dt =$$

$$= \int R\left(t, \sqrt{a y_1^2} \sqrt{1 - t^2/y_1^2}\right) dt =$$

$$s = t/4,$$

$$= \int \tilde{\hat{R}}\left(s, \sqrt{1-s^2}\right) ds$$

$$J = \int_0^1 \sqrt{6 - 4x - 2x^2}\, dx =$$

$$6 - 4x - 2x^2 = -2\left(x^2 + 2x - 3\right) =$$

$$= -2\left((x+1)^2 - 4\right) = 2\left(4 - (x+1)^2\right)$$

$$J = \int_0^1 \sqrt{2\left(4 - (x+1)^2\right)}\, dx = \left\{ x+1 = t \right\}$$

$$= \sqrt{2} \int_1^2 \sqrt{4 - t^2}\, dt = \left\{ t = 2\sin y \right\}$$

$$= \dots$$ Ex

$$\int R\left(x, \sqrt{1-x^2}\right) dx$$

$$\int R\left(x, \sqrt{c+bx-ax^2}\right) dx$$

$$\Delta = b^2-4ac>0, \quad a>0$$

Siano $x_1 < x_2$ le radici del polinomio

$$t = \sqrt{\frac{x_2 - x}{x - x_1}}$$

ATTENZIONE

CAMBIO VARIABILE SINGOLARE PER $x \to x_1^+$

$$x_1 < x < x_2$$

$$J = \int_0^{\frac{1}{2}} \frac{\sqrt{6-4x-2x^2}}{1-x} \, dx$$

$x_1 = -3$

$x_2 = 1$

$$t = \sqrt{\frac{1-x}{x+3}} \quad , \quad t \geq 0$$

$$6 - 4x - 2x^2 = -2(x-1)(x+3)$$

$$\boxed{= 2\,(1-x)(x+3)}$$

$$t^2 = \frac{1-x}{x+3} = \dots\dots \quad x = \frac{1-3t^2}{1+t^2}$$

$$dx = \dots\dots = (-8)\,\frac{t}{(1+t^2)^2}\,dt$$

$$\sqrt{6-4x-2x^2} = \sqrt{2(1-x)(x+3)} =$$

$$\left\{\begin{array}{l} 1 - x = 1 - \frac{1-3t^2}{1+t^2} = \dots = \frac{4t^2}{1+t^2} \\ x+3 = \dots\dots = \dots = \frac{4}{1+t^2} \end{array}\right\}$$

$$= \sqrt{2 \cdot \frac{4t^2}{1+t^2} \cdot \frac{4}{1+t^2}} = 4\sqrt{2}\,\frac{t}{1+t^2}$$

$$t \geq 0 \Rightarrow \sqrt{t^2} = t$$

$$J = \int_{\frac{\sqrt{3}}{3}}^{\frac{\sqrt{7}}{7}} \frac{4\sqrt{2}\,\frac{t}{1+t^2}}{\frac{4t^2}{1+t^2}} \cdot (-8) \cdot \frac{t}{(1+t^2)^2}\,dt =$$

$$x = 0 \;, \; t = \frac{\sqrt{3}}{3}$$

$$x = \frac{1}{2} \;, \; t = \frac{\sqrt{7}}{7}$$

$$8\sqrt{2} \int_{\frac{\sqrt{7}}{7}}^{\frac{\sqrt{3}}{3}} \frac{1}{(1+t^2)^2}\,dt =$$

$$8\sqrt{2} \left[\frac{1}{2} \operatorname{arcTg}(t) + \frac{1}{2} \frac{t}{1+t^2} \right]_{\frac{\sqrt{7}}{7}}^{\frac{\sqrt{3}}{3}} =$$

$$8\sqrt{2}\left[\frac{1}{2}\operatorname{arctg}\left(\frac{\sqrt{3}}{3}\right)-\frac{1}{2}\operatorname{arctg}\left(\frac{\sqrt{7}}{7}\right)\right.$$

$$\left.+\frac{1}{2}\,\frac{\sqrt{3}/3}{1+\frac{1}{3}}-\frac{1}{2}\,\frac{\sqrt{7}/7}{1+\frac{1}{7}}\right]=$$

$$=4\sqrt{2}\left[\frac{\pi}{6}-\operatorname{arctg}\left(\frac{\sqrt{7}}{7}\right)+\frac{\sqrt{3}}{4}-\frac{\sqrt{7}}{8}\right]$$

$$\int\sqrt{x^2-1}\,dx \quad ?$$

$$\int \sqrt{x^2-1}\,dx = \int (x)' \sqrt{x^2-1}\,dx =$$

$$= x\sqrt{x^2-1} - \int \frac{x \cdot 2x}{2\sqrt{x^2-1}}\,dx =$$

$$= x\sqrt{x^2-1} - \int \frac{x^2}{\sqrt{x^2-1}}\,dx =$$

$$= x\sqrt{x^2-1} - \int \frac{x^2-1+1}{\sqrt{x^2-1}}\,dx =$$

$$= x\sqrt{x^2-1} - \int \frac{1}{\sqrt{x^2-1}}\,dx - \int \sqrt{x^2-1}\,dx$$

$$\int \sqrt{x^2-1} = \frac{1}{2} x\sqrt{x^2-1} - \frac{1}{2}\int \frac{1}{\sqrt{x^2-1}}\,dx$$

$$\int \sqrt{x^2-1} = -\frac{1}{2}\log\left(x+\sqrt{x^2-1}\right)+\frac{1}{2}x\sqrt{x^2-1}+c$$

$$= \begin{cases} -\frac{1}{2}\log\left(x+\sqrt{x^2-1}\right)+\frac{x}{2}\sqrt{x^2-1}+c, & x\geq 1 \\ \frac{1}{2}\log\left(-x+\sqrt{x^2-1}\right)+\frac{x}{2}\sqrt{x^2-1}+c, & x\leq -1 \end{cases}$$

$$\left(\log\left(x+\sqrt{x^2-1}\right)\right)' =$$

$$\frac{1}{x+\sqrt{x^2-1}}\cdot\left(1+\frac{x}{\sqrt{x^2-1}}\right) =$$

$$\frac{1}{\cancel{x+\sqrt{x^2-1}}}\cdot\frac{\cancel{\sqrt{x^2-1}+x}}{\sqrt{x^2-1}} = \frac{1}{\sqrt{x^2-1}}$$

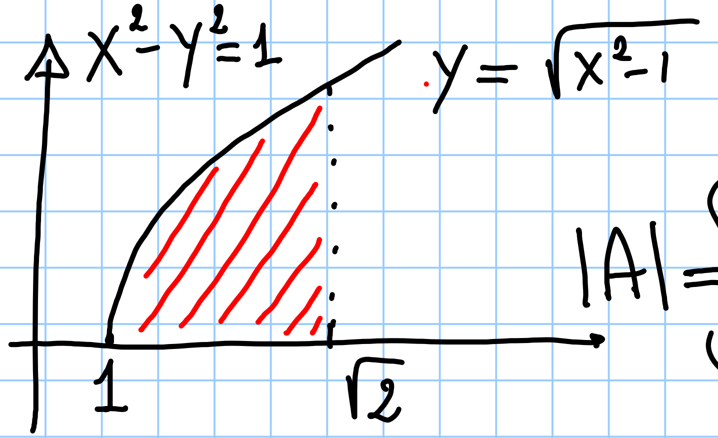
Ex

$$\int \sqrt{x^2+1}\,dx = \frac{1}{2}\,\text{Settsinh}(x) + \frac{1}{2}x\sqrt{x^2+1} + c$$

$$= \frac{1}{2}\log\left(x+\sqrt{x^2+1}\right) + \frac{1}{2}x\sqrt{x^2+1} + c$$

AREA DEL SETTORE IPERBOLICO

$$A = \left\{ (x,y) : \begin{array}{l} x \in [1, \sqrt{2}] \\ 0 \le y \le \sqrt{x^2-1} \end{array} \right\}$$



$$|A| = \int_1^{\sqrt{2}} \sqrt{x^2-1}\, dx =$$

$$-\frac{1}{2}\log\left(1+\sqrt{2}\right) + \frac{\sqrt{2}}{2}$$

ALTRO METODO
(FACOLTATIVO)

$$\int \sqrt{x^2-1}\, dx \, ; \quad \cosh^2(t) - \sinh^2(t) = 1$$

$$\cosh^2(t) - 1 = \sinh^2(t)$$

$$x = \cosh(t) \, , \quad t \geq 0 \, , \quad x \geq 1$$

$$dx = \sinh(t)\, dt$$

$$\sqrt{\cosh^2(t)-1} = \sqrt{\sinh^2(t)} =$$

$$\{t \geq 0\} = \sinh(t)$$

$$\int \sqrt{x^2 - 1}\, dx = \int \sinh^2(t)\, dt =$$

$$= \frac{1}{4}\int \left(e^{2t} - 2 + e^{-2t}\right) dt =$$

$$= \frac{1}{4}\left(\frac{e^{2t}}{2} - 2t - \frac{e^{-2t}}{2}\right) + c =$$

$$= -\frac{t}{2} + \frac{1}{4}\sinh(2t) =$$

$$\left\{ \sinh(2t) = \frac{e^{2t} - e^{-2t}}{2} = 2\,\frac{e^{t} - e^{-t}}{2} \cdot \frac{e^{t} + e^{-t}}{2} = \right.$$

$$= 2\sinh(t)\cosh(t)$$

$$\int \sinh^2(t)\,dt = -\frac{t}{2} + \frac{1}{2}\sinh(t)\cosh(t) + c$$

$$x = \cosh(t) \quad , \quad \sqrt{x^2-1} = \sinh(t)$$

$$t = \operatorname{settcosh}(x) = \log\left(x+\sqrt{x^2-1}\right)$$

$$x \geq 1$$

$$\int \sqrt{x^2-1} = \int \sinh^2(t)\,dt =$$

$$= -\frac{1}{2}\log\left(x+\sqrt{x^2-1}\right) + \frac{1}{2}x\sqrt{x^2-1} + c$$

$$x \geq 1$$

$$\int \frac{x+1}{\sqrt{x^2+2x+2}}\,dx = \int \frac{x+1}{\sqrt{(x+1)^2+1}}\,dx =$$

$$= \int \frac{t\,dt}{\sqrt{t^2+1}} = \sqrt{t^2+1} + C$$

$$= \sqrt{x^2+2x+2} + C$$

$$\int R\left(x, \sqrt{ax^2+bx+c}\right) dx$$

$a>0$ e $\Delta>0$ o $\Delta<0$

SOSTITUZIONE RAZIONALIZZANTE CASO GENERALE

$$ax^2+bx+c = a(t-x)^2, \quad t>x$$

$$\sqrt{ax^2+bx+c} = \sqrt{a}(t-x)$$

SI PREFERISCE PRIMA SEMPLIFICARE

$$ax^2+bx+c = a\left[(x-x_1)^2 \pm y_1^2\right]$$

$$t = x-x_1$$

$$\int \tilde{R}\left(t, \sqrt{t^2 \pm y_1^2}\right) dt$$

$$\int \sqrt{x^2+\alpha^2}\,dx, \quad \int \sqrt{x^2-\alpha^2}\,dx$$

$$x^2+\alpha^2=(t-x)^2 \qquad\qquad x^2-\alpha^2=(t-x)^2$$

$$\sqrt{x^2+\alpha^2}=t-x \qquad\qquad \sqrt{x^2-\alpha^2}=t-x$$

$$t=x+\sqrt{x^2+\alpha^2} \qquad\qquad t=x+\sqrt{x^2-\alpha^2}$$

FACOLTATIVO

$$\int \sqrt{x^2+1}\,dx \; ; \; \begin{cases} 1+x^2=(t-x)^2 \\ t>x \end{cases}$$

$$\left.\begin{aligned} \sqrt{x^2+1} &= (t-x) \\ t &= x+\sqrt{1+x^2} \end{aligned}\right\} \quad \left(\log(t)=\operatorname{settsinh}(x)\right)$$

$$1+x^2=(t-x)^2, \qquad 1+\cancel{x^2}=t^2-2tx+\cancel{x^2}$$

$$x=\frac{t^2}{2t}-\frac{1}{2t}=\frac{t}{2}-\frac{1}{2t}=\frac{t^2-1}{2t}$$

$$dx = \left(\frac{1}{2} + \frac{1}{2t^2}\right) dt = \frac{t^2+1}{2t^2}\, dt$$

$$\sqrt{x^2+1} = t - x = t - \left(\frac{t}{2} - \frac{1}{2t}\right) =$$

$$= \frac{t}{2} + \frac{1}{2t} = \frac{t^2+1}{2t}$$

$$\int \frac{t^2+1}{2t} \cdot \frac{t^2+1}{2t^2}\, dt =$$

$$\int \frac{t^4+2t^2+1}{4t^3}\, dt = \frac{1}{4}\int \left(t + \frac{2}{t} + \frac{1}{t^3}\right) dt =$$

$$= \frac{1}{4}\left(\frac{t^2}{2} - \frac{1}{2t^2} + 2\log(t)\right) + c$$

$$= \frac{1}{4}\left(\frac{\left(x+\sqrt{x^2+1}\right)^2}{2} - \frac{1}{2\left(x+\sqrt{x^2+1}\right)^2} + 2\log\left(x+\sqrt{x^2+1}\right)\right) + c$$

$$\boxed{\frac{1}{\left(x+\sqrt{x^2+1}\right)^2} = \left(x-\sqrt{x^2+1}\right)^2 \quad \text{Ex}} \quad (1)$$

$$= \frac{1}{4}\left[\frac{1}{2}\left(\left(x+\sqrt{x^2+1}\right)^2 - \left(x-\sqrt{x^2+1}\right)^2\right) + 2\log\left(x+\sqrt{x^2+1}\right)\right] + c$$

$$= \frac{1}{4}\left[2x\sqrt{x^2+1} + 2\log\left(x+\sqrt{x^2+1}\right)\right] + c$$

$$= \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\log\left(x+\sqrt{x^2+1}\right) + c$$

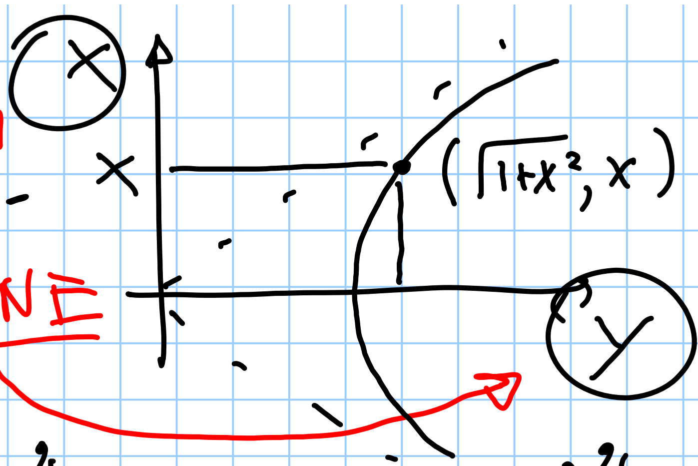
$$= \frac{1}{2}\operatorname{settsinh}(x) + \frac{1}{2}x\sqrt{x^2+1} + c$$

FACOLTATIVO

SPIEGAZIONE GEOMETRICA di (1)

(1)

ATTENZIONE



$y^2 - x^2 = 1$

$y = \sqrt{1+x^2}$

$$y^2 - x^2 = 1, \quad \left(\sqrt{1+x^2}\right)^2 - x^2 = 1$$

$$\left(\sqrt{1+x^2} - x\right)\left(\sqrt{1+x^2} + x\right) = 1 \qquad \mathrm{I}^\circ$$

$$\frac{1}{\left(\sqrt{1+x^2} + x\right)} = \left(\sqrt{1+x^2} - x\right) \Rightarrow (1)$$

II°

$x = \sinh(t)$

$y = \cosh(t)$

(ATTENZIONE $x \longleftrightarrow y$)

$$x = \sinh(t) = \boxed{\frac{e^t - e^{-t}}{2}}$$

$$y = \cosh(t) = \; = \frac{e^t + e^{-t}}{2}$$

$$e^{2t} = \left(x + \sqrt{1+x^2}\right)^2$$

$$e^{-2t} = \left(x + \sqrt{1+x^2}\right)^{-2}$$

$$\left(x + \sqrt{1+x^2}\right)^2 - \frac{1}{\left(x+\sqrt{1+x^2}\right)^2} =$$

$$= e^{2t} - e^{-2t} = \left(e^{t} - e^{-t}\right)\left(e^{t} + e^{-t}\right)$$

$$= 2x \cdot 2y = 2x \cdot 2\sqrt{1+x^2} =$$

$$y^2 - x^2 = 1 \qquad = 4x\sqrt{1+x^2}$$

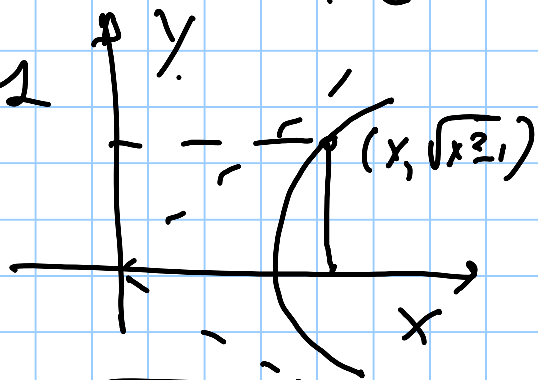
$$\int \sqrt{x^2-1}\,dx = \left\{ \begin{array}{l} x^2-1 = (t-x)^2 \\ t > 1 \end{array} \right.$$

$$\left. \begin{array}{l} \sqrt{x^2-1} = (t-x) \\ t = x + \sqrt{x^2-1} \end{array} \right\}$$

$$= \frac{1}{4}\left(\left(\frac{x+\sqrt{x^2-1}}{2}\right)^2 - \frac{1}{2}\,\frac{1}{\left(x+\sqrt{x^2-1}\right)^2} - 2\log\left(x+\sqrt{x^2-1}\right)\right) + c$$

$$x^2-y^2=1$$

$$y=\sqrt{x^2-1}$$

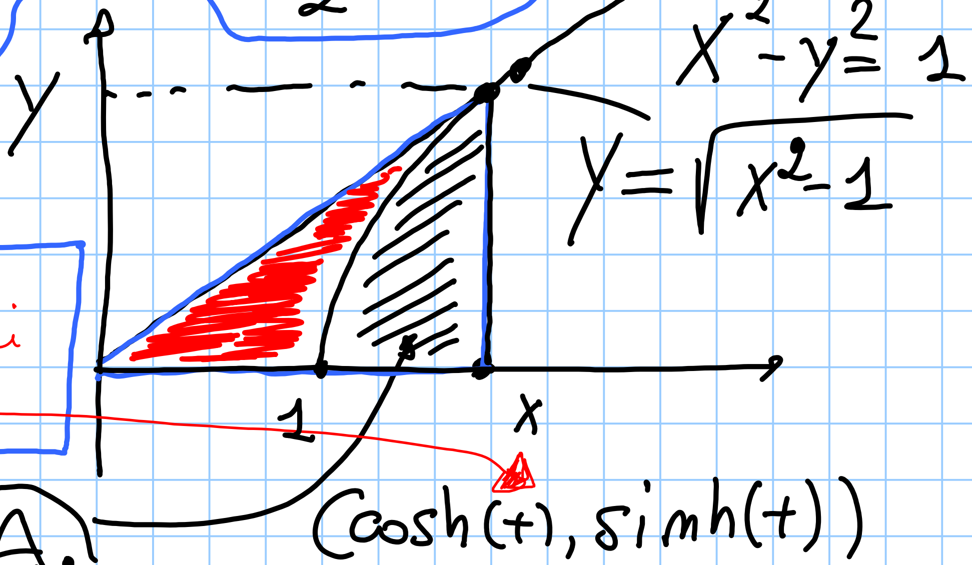


$$= -\frac{1}{2}\log\left(x+\sqrt{x^2-1}\right) + \frac{1}{2}\,x\sqrt{x^2-1} + c$$

$$= -\frac{1}{2}\operatorname{SeTTcosh}(x) + \frac{1}{2}\,x\sqrt{x^2-1} + c$$

$$A = \frac{t}{2}$$

SIGNIFICATO GEOMETRICO di t



$$A = A_1 - A_2$$

$$\begin{cases} x = \cosh(t) = \dfrac{e^{t}+e^{-t}}{2} \\ y = \sinh(t) = \dfrac{e^{t}-e^{-t}}{2} \\ x^2 - y^2 = 1 \end{cases}$$

$$A = A_1 - A_2 = \frac{xy}{2} - \int_1^x \sqrt{x^2-1}\, dx$$

$$= \frac{xy}{2} - \left[-\frac{1}{2}\log\left(x+\sqrt{x^2-1}\right) + \frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2}\log\left(1+\sqrt{0}\right) + \frac{1}{2}\,1\sqrt{0} \right]$$

$$= \cancel{\frac{x\sqrt{x^2-1}}{2}} + \frac{1}{2}\log\left(x+\sqrt{x^2-1}\right) - \cancel{\frac{1}{2}x\sqrt{x^2-1}}$$

$$= \frac{1}{2}\log\left(x+\sqrt{x^2-1}\right) =$$

$$= \frac{1}{2}\log\left(e^{t}\right) = \frac{t}{2} \quad \blacksquare$$