f CONTINUA in $[a,b]$

$$F_a(x) = \int_a^x f(t)\,dt$$

$$\Longrightarrow \quad \frac{dF_a}{dx}(x) = f(x), \ \forall x \in [a,b]$$

F_a è una PRIMITIVA di f

TEOREMA [FORMULA FONDAMENTALE DEL CALCOLO INTEGRALE]

(i) SE F e G sono PRIMITIVE di f in $[a,b]$, allora $\exists\, c \in \mathbb{R} : F(x) = G(x) + c$

(ii) Se f è CONTINUA in $[a,b]$ e F è una primitiva di f in $[a,b]$, allora

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$F(b) - F(a) = \Big[F(x)\Big]_a^b$$

$$= F(x)\Big|_a^b$$

DIMOSTRAZIONE

(i) $H(x) = F(x) - G(x)$, $x \in [a,b]$

Vogliamo dimostrare che $\exists\, c \in \mathbb{R}$:

$$H(x) = c \quad \forall x \in [a,b]$$

PER IPOTESI H è DERIVABILE in $[a,b]$

$\Longrightarrow$ H è continua in $[a,b]$

$$H'(x) = F'(x) - G'(x) =$$

$$= f(x) - f(x) = 0 \quad \forall x \in (a,b)$$

T. di LAGRANGE, $\quad x \in (a,b]$

$$H(x) - H(a) = H'(y)(x-a) = 0$$

$y \in (a,x)$, $\quad H(x) = H(a)$

$$\forall x \in [a,b]$$

SEGUE CHE $H(x) = C = H(a)$ in $[a,b]$

ESEMPIO: $f(x) = x^2$, $[a,b] = [0,4]$

$$F(x) = \frac{x^3}{3} \qquad G(x) = \frac{x^3}{3} + 1$$

$$H(x) = H(a) = H(0) = -1.$$

(ii) F è una PRIMITIVA di f in $[a,b]$,

Sia $G_a(x) = \int_a^x f(t)\,dt$, $x \in [a,b]$

Dal T. FONDAMENTALE DEL CALCOLO INTEGRALE

$$G_a'(x) = f(x), \quad x \in [a,b]$$

Da (i) $F(x) = G_a(x) + c$

$$F(a) = G_a(a) + c = c$$

$$F(x) = G_a(x) + F(a)$$

$$F(b) = G_a(b) + F(a)$$

$$G_a(b) = \int_a^b f(t)\,d\tau = F(b) - F(a)$$

$$\int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 =$$

$$\frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

Sia $f \in \mathcal{R}(a,b)$,

L'INSIEME di TUTTE LE PRIMITIVE di f in $[a,b]$ si indica con

$$\int f(x)\,dx$$

ed è PER DEFINIZIONE L'INTEGRALE INDEFINITO di f in $[a,b]$

$$\int x^2\,dx = \frac{x^3}{3} + c \quad , \quad c \in \mathbb{R}$$

$$\int f'(x)\,dx = f(x) + c$$

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

$$\int x^{\alpha}\,dx = \frac{x^{\alpha+1}}{\alpha+1} + c, \quad \alpha \neq -1$$

$$\int \frac{1}{x}\,dx = \log|x| + c,$$

ATTENZIONE AL MODULO

$$\int e^{\alpha x}\,dx = \frac{1}{\alpha} e^{\alpha x} + c, \quad \alpha \neq 0$$

$$\int \sin(x)\,dx = -\cos(x) + c,$$

$$\int \cos(x)\,dx = \sin(x) + c,$$

$$\int \frac{dx}{1+x^2} = \operatorname{arctg}(x) + c$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + c$$

$$\int \frac{dx}{\sqrt{1+x^2}} = \log\left(x + \sqrt{1+x^2}\right) + c$$

$$= \operatorname{SeTTsinh}(x) + c$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \begin{cases} \log\left(x+\sqrt{x^2-1}\right)+c, & x \geq 1 \\ -\log\left(-x+\sqrt{x^2-1}\right)+c, & x \leq -1 \end{cases}$$

$$= \begin{cases} \operatorname{Sett}\cosh(x)+c, & x \geq 1 \\ -\operatorname{Sett}\cosh(-x)+c, & x \leq -1 \end{cases}$$

FORMULE DI INTEGRAZIONE

PER SOSTITUZIONE

E PER PARTI

$$\int \cos(\log(x))\,\frac{1}{x}\,dx =$$

$$g(x) = \log(x) \longrightarrow$$

$$\int \cos(g(x)) \cdot g'(x)\,dx =$$

$$\int \left(\sin(g(x))\right)'\,dx =$$

$$\sin(g(x)) + c =$$

$$\sin(\log(x)) + c$$

$$\int e^{\sin(x)} \cos(x)\,dx = \int e^{\sin(x)} (\sin(x))'\,dx$$

$$= \int \left(e^{\sin(x)}\right)'\,dx =$$

$$e^{\sin(x)} + c$$

FORMULA di INTEGRAZIONE PER SOSTITUZIONE

Se F è DERIVABILE, $F' = f$ e SE g è DERIVABILE

Allora

$$\int f(g(x))\, g'(x)\, dx =$$

$$= F(g(x)) + c$$

DIMOSTRAZIONE

$$(F(g(x)) + c)' =$$

$$F'(g(x)) \cdot g'(x) = f(g(x))\, g'(x)$$

$\blacksquare$

$$\int (g(x))^{\alpha} g'(x)\,dx = \frac{g(x)^{\alpha+1}}{\alpha+1} + c, \quad \alpha \neq -1$$

$$\int \frac{g'(x)}{g(x)}\,dx = \log|g(x)| + c,$$

$$\int \sin(g(x))\, g'(x)\,dx = -\cos(g(x)) + c,$$

$$\int \cos(g(x))\, g'(x)\,dx = \sin(g(x)) + c,$$

$$\int \frac{g'(x)\,dx}{1+g^2(x)} = \arctg(g(x)) + c$$

$$\int e^{g(x)} g'(x)\, dx = e^{g(x)} + c$$

$$\int \frac{g'(x)\, dx}{\sqrt{1 - g^2(x)}} = \arcsin(g(x)) + c$$

$$\int \frac{g'(x)}{\sqrt{1 + g^2(x)}} = \log\left(g(x) + \sqrt{1 + g^2(x)}\right) + c$$

$$= \text{settsinh}(g(x)) + c$$

COME CAMBIA dx quando SOSTITUIAMO $g(x)=t$?

$$\int (\sin(x))^2 \cos(x)\,dx =$$

$$\boxed{\begin{gathered}\sin(x)=t \\ \left(\frac{d}{dx}(\sin(x))dx = \frac{d}{dt}(t)\,dt\right) \\ \cos(x)\,dx = dt\end{gathered}}$$

$$\int t^2\,dt = \frac{t^3}{3} + c =$$

$$= \frac{1}{3}\sin^3(x) + c$$

$$\int e^{6x^5} x^4\,dx =$$

$$6x^5 = t,$$

$$\frac{d}{dx}(6x^5)\,dx = \frac{d}{dt}(t)\,dt$$

$$30x^4\,dx = dt,$$

$$x^4\,dx = \frac{dt}{30}$$

$$= \int e^t \frac{dt}{30} = \frac{e^t}{30} + c = \frac{e^{6x^5}}{30} + c$$

$$\int \frac{x^2}{1+x^3}\,dx = \int \frac{1}{3}\frac{dt}{1+t} =$$

$$x^3 = t$$

$$3x^2\,dx = dt\ ;\quad x^2\,dx = \frac{dt}{3}.$$

$$s = 1 + t \quad , \quad ds = dt$$

$$= \frac{1}{3} \int \frac{ds}{s} = \frac{1}{3} \log|s| + c$$

$$= \frac{1}{3} \log|1+t| + c = \frac{1}{3} \log|1+x^3| + c$$

$$\int \frac{x\, e^{x^2} dx}{1 + e^{2x^2}} =$$

$$e^{x^2} = t$$

$$2x\, e^{x^2} dx = dt$$

$$x\, e^{x^2} dx = \frac{dt}{2}$$

$$\frac{1}{2} \int \frac{dt}{1+t^2} = \frac{1}{2} \operatorname{arcTg}(t) + c$$

$$= \frac{1}{2} \operatorname{arctg}\left(e^{x^2}\right) + c$$

$$\int \frac{\left(1 + 4\log(x)\right)^{\frac{1}{3}}}{x}\, dx =$$

$\log(x) = t$

$\frac{dx}{x} = dt$

$$= \int (1 + 4t)^{\frac{1}{3}}\, dt =$$

$1 + 4t = s$

$4\,dt = ds$

$$= \int s^{\frac{1}{3}} \frac{ds}{4} = \frac{1}{4} \frac{s^{\frac{1}{3}+1}}{\frac{1}{3}+1} + c$$

$$= \frac{1}{4} \frac{(1+4t)^{4/3}}{4/3} + c = \frac{3}{16}\left(1+4\log(x)\right)^{4/3} + c$$

$$\int_0^2 \frac{x^2}{1+x^6}\,dx =$$

$$x^3 = t$$

$x=0 \longrightarrow t=0$

$3x^2\,dx = dt$

$x=2 \longrightarrow t=8$

$x^6 = (x^3)^2 = t^2$

$$\int_0^8 \frac{1}{3} \frac{dt}{1+t^2} = \frac{1}{3} \operatorname{arctg}(t)\Big|_0^8$$

$$= \frac{1}{3} \operatorname{arctg}(8) \quad ;$$

$$\int \frac{x^2}{1+x^6}\,dx \underset{x^3=t}{=} \int \frac{1}{1+t^2} \frac{dt}{3} =$$

$$= \frac{1}{3} \operatorname{arctg}(t) = \frac{1}{3} \operatorname{arctg}(x^3) + c_1$$

$$\int_0^2 \frac{x^2}{1+x^6} dx = \frac{1}{3} \operatorname{arctg}(x^3) \Big|_0^2 =$$

$$= \frac{1}{3} \operatorname{arcTg}(8)$$

FORMULA DI INTEGRAZIONE PER PARTI

f e g DERIVABILI

$$\int f(x) g'(x) dx = f(x) g(x) - \int f'(x) g(x) dx$$

DIMOSTRAZIONE

$$(f g)' = f' g + f g'$$

$$f(x) g'(x) = \big(f(x) g(x)\big)' - f'(x) g(x)$$

$$\int f(x)\,g'(x)\,dx = \int \big(f(x)g(x)\big)'\,dx - \int f'(x)\,g(x)\,dx$$

$$= f(x)g(x) - \int f'(x)g(x)\,dx \qquad \blacksquare$$

$$\int x e^{x}\,dx = \int x\,(e^{x})'\,dx$$

$$= x e^{x} - \int e^{x}\,dx =$$

$$= x e^{x} - e^{x} + c$$

$$\int x^{2} e^{x}\,dx = \int x^{2}\,(e^{x})'\,dx =$$

$$= x^2 e^x - \int 2x e^x dx =$$

$$= x^2 e^x - 2\int x e^x dx =$$

$$= x^2 e^x - 2\left(x e^x - e^x \right) + c$$

$$= x^2 e^x - 2x e^x + 2 e^x + c$$

$$\int \log(x)\, dx = \int \log(x) \cdot (x)' dx$$

$$= x \log(x) - \int \cancel{\frac{x}{x}}\, dx =$$

$$= x \log(x) - x + c .$$

$$\int x^2 \log(x)\,dx = \dots$$ Ex.

$$\int \operatorname{arctg}(x)\,dx = \int \operatorname{arctg}(x)\cdot(x)'\,dx$$

$$= x\operatorname{arctg}(x) - \int \frac{x}{1+x^2}\,dx$$

$$\int \frac{x}{1+x^2}\,dx \underset{1+x^2=t}{=} \frac{1}{2}\int \frac{dt}{t} =$$

$$2x\,dx = dt$$

$$= \frac{1}{2}\log|t| + c = \frac{1}{2}\log\left(1+x^2\right) + c$$

$$\int \operatorname{arctg}(x)\,dx = x\operatorname{arctg}(x) - \frac{1}{2}\log(1+x^2) + C$$

$$\int x^2 \sin(x)\,dx = \int x^2 \left(-\cos(x)\right)' dx$$

$$= -x^2\cos(x) - \int (x^2)'\left(-\cos(x)\right)dx$$

$$= -x^2\cos(x) + \int 2x\cos(x)\,dx$$

$$= -x^2\cos(x) + 2\int x\left(\sin(x)\right)' dx$$

$$= -x^2\cos(x) + 2\left[x\sin(x) - \int \sin(x)\,dx\right]$$

$$= -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + c$$

$$\int \frac{kx+h}{ax^2+bx+c}\,dx$$

$$\{a, b, c, h, k\} \subset \mathbb{R}$$

$$\Delta = b^2 - 4ac \begin{array}{l} \nearrow > 0 \\ \rightarrow = 0 \\ \searrow < 0 \end{array}$$