N intervalli $I_j = [x_{j-1}, x_j]$

$j = 1, \dots, N$ .

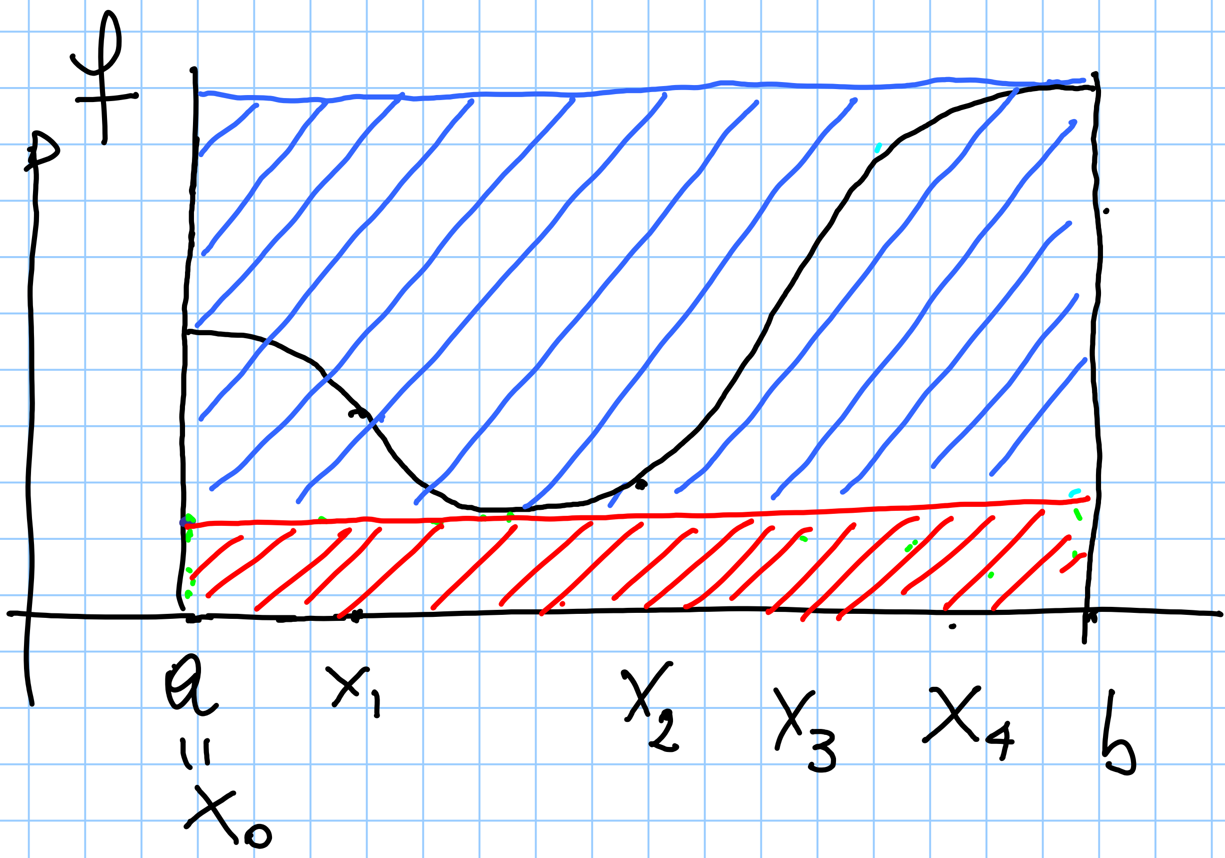
$$[a,b] = \bigcup_{j=1}^{N} I_j$$

Se $N = 1$ $\exists$ un unico Intervallo

$$I_1 = [a, b] = [x_0, x_1]$$

e $\boxed{s(\{a,b\}, f)} = \inf_{[a,b]} f (b-a)$

$$\boxed{S(\{a,b\}, f)} = \sup_{[a,b]} f (b-a)$$




IN PARTICOLARE ABBIAMO VISTO CHE

$$s(D,f) \leq S(P,f) \quad \forall D, \forall P$$

e che

$$(1) \quad \sup_D s(D,f) \leq \inf_P S(P,f)$$

TEOREMA

$f : [a,b] \longrightarrow \mathbb{R}$ MONOTONA.

ALLORA $f \in \mathcal{R}(a,b)$

DIMOSTRAZIONE

S.P.D.G. $f \nearrow$ in $[a,b] \Longrightarrow$

$$f(a) \leq f(x) \leq f(b) \quad \forall x \in [a,b]$$

Se $f(a) = f(b) \Longrightarrow$

$$f(x) = c \in \mathbb{R} \quad \forall x \in [a,b]$$

$$\Rightarrow s(D,f) = c(b-a) = S(D,f) \quad \forall D$$

Quindi $f \in \mathcal{R}(a,b)$

$$\int_a^b f(x)\,dx = c(b-a)$$

S.P.D.G. $f(a) < f(b)$

SI DICE

AMPIEZZA DELLA PARTIZIONE

$$D = \{x_0 < x_1 < \dots < x_N\}$$

$$\varepsilon_D := \max_{j=1 \dots N} (x_j - x_{j-1})$$

Quindi $\varepsilon_D$ è la ampiezza dell'intervallo più grande

$$I_j = [x_{j-1}, x_j], \; j = 1, \dots, N$$

DIMOSTRIAMO CHE

$\forall \varepsilon > 0$ $\exists$ una partizione di AMPIEZZA $\bar{\varepsilon}$, $D_{\bar{\varepsilon}}$:

$$(2) \quad S(D_{\bar{\varepsilon}}, f) < s(D_{\bar{\varepsilon}}, f) + \varepsilon$$

Se (2) è VERA, ALLORA da (1) si ha

$$0 \leq \inf_{D} S(D, f) - \sup_{D} s(D, f)$$

$$\leq S(D_{\bar{\varepsilon}}, f) - s(D_{\bar{\varepsilon}}, f) \leq \varepsilon$$

$$\forall \varepsilon > 0$$

Quindi ⊛

$$0 \le \inf_D S(D,f) - \sup_D s(D,f) \le 0$$

OVVERO

$$\sup_D s(D,f) = \inf_D S(D,f)$$

Dimostriamo (2)

Sia $D_{\bar{\varepsilon}}$ una partizione di AMPIEZZA $\bar{\varepsilon} = \dfrac{\varepsilon}{2(f(b)-f(a))}$

$$D_{\bar{\varepsilon}} = \{ x_0 < x_1 < \dots < x_{\bar{N}} \}$$

$$I_j = [x_{j-1}, x_j], \quad j = 1, \dots, \bar{N}$$

Si ha

$$S(D_{\bar{\varepsilon}}, f) - s(D_{\bar{\varepsilon}}, f) = \sum_{j=1}^{\bar{N}} \sup_{I_j} f \, (x_j - x_{j-1}) - \sum_{j=1}^{\bar{N}} \inf_{I_j} f \, (x_j - x_{j-1}) =$$

$$= \sum_{j=1}^{\bar{N}} \underbrace{\left[ \sup_{I_j} f - \inf_{I_j} f \right]}_{\geq 0} \underbrace{(x_j - x_{j-1})}_{\leq \bar{\varepsilon}} \leq$$

$$\leq \bar{\varepsilon} \sum_{j=1}^{\bar{N}} \left[ \sup_{I_j} f - \inf_{I_j} f \right]$$

$$\leq \left\{ f(x_{j-1}) \leq f(x) \leq f(x_j) \,;\, f \uparrow \right\}$$

$$\leq \bar{\varepsilon} \sum_{j=1}^{N} \left( f(x_j) - f(x_{j-1}) \right) =$$

$$= \bar{\varepsilon} \Big( f(\cancel{x_1}) - f(x_0) + f(\cancel{x_2}) - f(\cancel{x_1}) + f(\cancel{x_3}) - f(\cancel{x_2}) + \cdots + \cdots + f(\cancel{x_{\bar{N}-1}}) - f(\cancel{x_{\bar{N}-2}}) + f(x_{\bar{N}}) - f(\cancel{x_{\bar{N}-1}}) \Big) =$$

$$= \bar{\varepsilon} \left( f(x_{\bar{N}}) - f(x_0) \right) =$$

$$= \frac{\varepsilon}{2(f(b) - f(a))} \left( f(b) - f(a) \right) = \frac{\varepsilon}{2} < \varepsilon \quad \blacksquare$$

$f$ continua in $[a,b]$

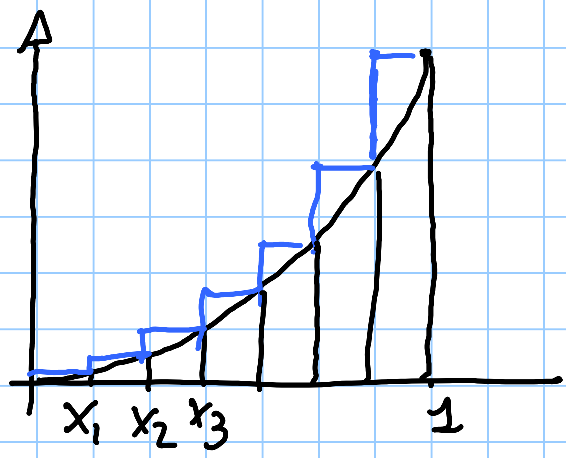
ALLORA $$\int_a^b f(x)\,dx = \lim_{N\to+\infty} \sum_{k=1}^{N} f(x_k)\Delta x_k$$

$$I_k = [x_{k-1}, x_k], \quad a = x_0 < x_1 < \dots < x_N = b$$

$$\Delta x_k = x_k - x_{k-1}; \quad \max_{k\in\{1,\dots,N\}} |\Delta x_k| \to 0, \; N\to+\infty$$

ESEMPIO: $f(x) = x^2$, $x \in [0,1]$

$$\int_0^1 f(x)\,dx = \frac{1}{3}$$




$x_0 = 0$, $x_1 = \frac{1}{N}$, $x_2 = \frac{2}{N}$, …

…, $x_k = \frac{k}{N}$, …, $x_N = 1$

$$\sum_{k=1}^{N} f(x_k)\Delta x_k = \sum_{k=1}^{N} (x_k)^2 \frac{1}{N} =$$

$$= \sum_{K=1}^{N} \left(\frac{K}{N}\right)^2 \frac{1}{N} = \frac{1}{N^3} \sum_{K=1}^{N} k^2 =$$

$$= \frac{1}{N^3} \frac{N(N+1)(2N+1)}{6} \xrightarrow[N \to +\infty]{} \frac{1}{3}.$$

$\blacksquare$

TEOREMA [ della MEDIA INTEGRALE ]

$f \in \mathcal{R}(a,b)$

(i) $$(b-a) \inf_{[a,b]} f \leq \int_a^b f(x)\,dx \leq \sup_{[a,b]} f \,(b-a)$$

(ii) Se $f$ è CONTINUA in $[a,b]$

ALLORA $\exists\, c \in [a,b]$ :

$$\int_a^b f(x)\,dx = f(c)\,(b-a)$$

DIMOSTRAZIONE

(i) $\displaystyle\int_a^b f(x)\,dx = \inf_D S(D,f) \leq$

$\displaystyle\leq S(\{a,b\},f) = \sup_{[a,b]} f \cdot (b-a)$

Analogamente per $\displaystyle\inf_{[a,b]} f\,(b-a)$.

(ii) $f$ continua $\Longrightarrow$ T. DEGLI ZERI

$f$ ASSUME TUTTI I VALORI $y$ TRA

$\displaystyle m = \min_{[a,b]} f$ e $\displaystyle M = \max_{[a,b]} f$

Da (i)

$$m = \inf_{[a,b]} f \leq \frac{\int_a^b f(x)\,dx}{b-a} \leq \sup_{[a,b]} f = M$$

$$\frac{\int_a^b f(x)\,dx}{b-a} = y \in [m, M] \Rightarrow \exists\, c \in [a,b]:$$

$$f(c) = \frac{\int_a^b f(x)\,dx}{b-a}$$

$\blacksquare$

DEFINIZIONE

$f \in \mathcal{R}(a,b)$ , $x_0$ $[a,b]$

LA FUNZIONE INTEGRALE di $f$ (relativamente a $x_0$) è

$$F_{x_0}(x) = \int_{x_0}^{x} f(t)\,dt$$

$\forall\, x \in [a,b]$

$F_{x_0}(x_0) = 0$

# TEOREMA FONDAMENTALE DEL CALCOLO INTEGRALE

Sia $f:[a,b]\longrightarrow\mathbb{R}$ continua

e sia $x_0\in[a,b]$

Allora la FUNZIONE INTEGRALE

$$F_{x_0}(x)=\int_{x_0}^{x}f(t)\,dt \quad \text{è}$$

DERIVABILE in $[a,b]$ e si ha

$$F_{x_0}'(x)=f(x),\ \forall x\in[a,b]$$

NOTA: SE $x=a$ $\quad D^{(+)}F_{x_0}(a)=f(a)$

$x=b$ $\quad D^{(-)}F_{x_0}(b)=f(b)$

# DIMOSTRAZIONE

PER SEMPLICITÀ $x \in (a,b)$, $x > x_0$.

$\forall\, h > 0 \;:\; x+h \in (a,b)$, si ha

$$\frac{F_{x_0}(x+h) - F_{x_0}(x)}{h} =$$

$$\left( \int_{x_0}^{x+h} f(t)\,dt - \int_{x_0}^{x} f(t)\,dt \right) \Big/ h =$$

$$\left( \cancel{\int_{x_0}^{x} f(t)\,dt} + \int_{x}^{x+h} f(t)\,dt - \cancel{\int_{x_0}^{x} f(t)\,dt} \right) \Big/ h$$

$$= \frac{\int_{x}^{x+h} f(t)\,dt}{h}$$

T. MEDIA INTEGRALE

$$\frac{\left(\min_{[x,x+h]} f\right)\cdot \cancel{h}}{\cancel{h}} \leq \frac{\int_{x}^{x+h} f(t)\,dt}{h} \leq \frac{\left(\max_{[x,x+h]} f\right)\cdot \cancel{h}}{\cancel{h}}$$

$f$ continua, $\exists\, t_h \in [x, x+h]$ :

$$f(t_h) = \max_{[x,x+h]} f$$

$$x \leq t_h \leq x+h, \quad h \to 0^+$$

$$\Longrightarrow t_h \to x$$

$f$ CONTINUA, $t_h \xrightarrow[h\to 0]{} x \implies f(t_h) \xrightarrow[h\to 0^+]{} f(x)$

$$\lim_{h\to 0} \max_{[x,x+h]} f = f(x)$$

ANALOGAMENTE, $\lim_{h\to 0} \min_{[x,x+h]} f = f(x)$

$$\min_{[x,x+h]} f \le \frac{F_{x_0}(x+h) - F_{x_0}(x)}{h} \le \max_{[x,x+h]} f$$

$\xrightarrow[h\to 0^+]{} f(x)$ $\qquad$ $f(x) \xleftarrow[h\to 0^+]{}$

$$\lim_{h\to 0^+} \frac{F_{x_0}(x+h) - F_{x_0}(x)}{h} = f(x)$$

$\blacksquare$

⊛ $0 \leq x - y$. SE $0 \leq x - y < \varepsilon, \forall \varepsilon > 0,$ ALLORA $x = y$

**DIM**. PER ASSURDO

$x - y = \delta > 0 \quad , \varepsilon = \delta/2$

$0 \leq \delta \leq \delta/2 \quad \Rightarrow 1 < \frac{1}{2}$

ASSURDO ∎

UN ESEMPIO di UNA FUNZIONE $f \notin \mathcal{R}(0,1)$. $f:[0,1] \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \cap [0,1] \\ 1, & x \notin \mathbb{Q} \cap [0,1] \end{cases}$$

$S(D,f) = 1 \ \forall D; \ s(D,f) = 0 \ \forall D$

$\Rightarrow$ $f$ è LIMITATA MA NON E' (Riemann) INTEGRABILE in $[0,1]$