$\exists$ un UNICO POLINOMIO di 2° GRADO,

$$T_2(x) = 1 + x + \frac{x^2}{2!} \quad :$$

$$e^x = 1 + x + \frac{x^2}{2!} + o(x^2), \quad x \to 0$$

? $\exists \cdot a_3 \in \mathbb{R}$:

$$e^x = 1 + x + \frac{x^2}{2!} + a_3 x^3 + o(x^3), \quad x \to 0 \ ?$$

? $\exists\, a_3 \in \mathbb{R}$:

$$\lim_{x\to 0} \frac{e^x - \left(1 + x + \frac{x^2}{2!} + a_3 x^3\right)}{x^3} = 0$$

$$0 = \lim_{x\to 0} \frac{e^x - \left(1 + x + \frac{x^2}{2!} + a_3 x^3\right)}{x^3} \overset{H}{=}$$

$$= \lim_{x \to 0} \frac{e^x - (1 + x + 3a_3x^2)}{3x^2} \overset{H}{=}$$

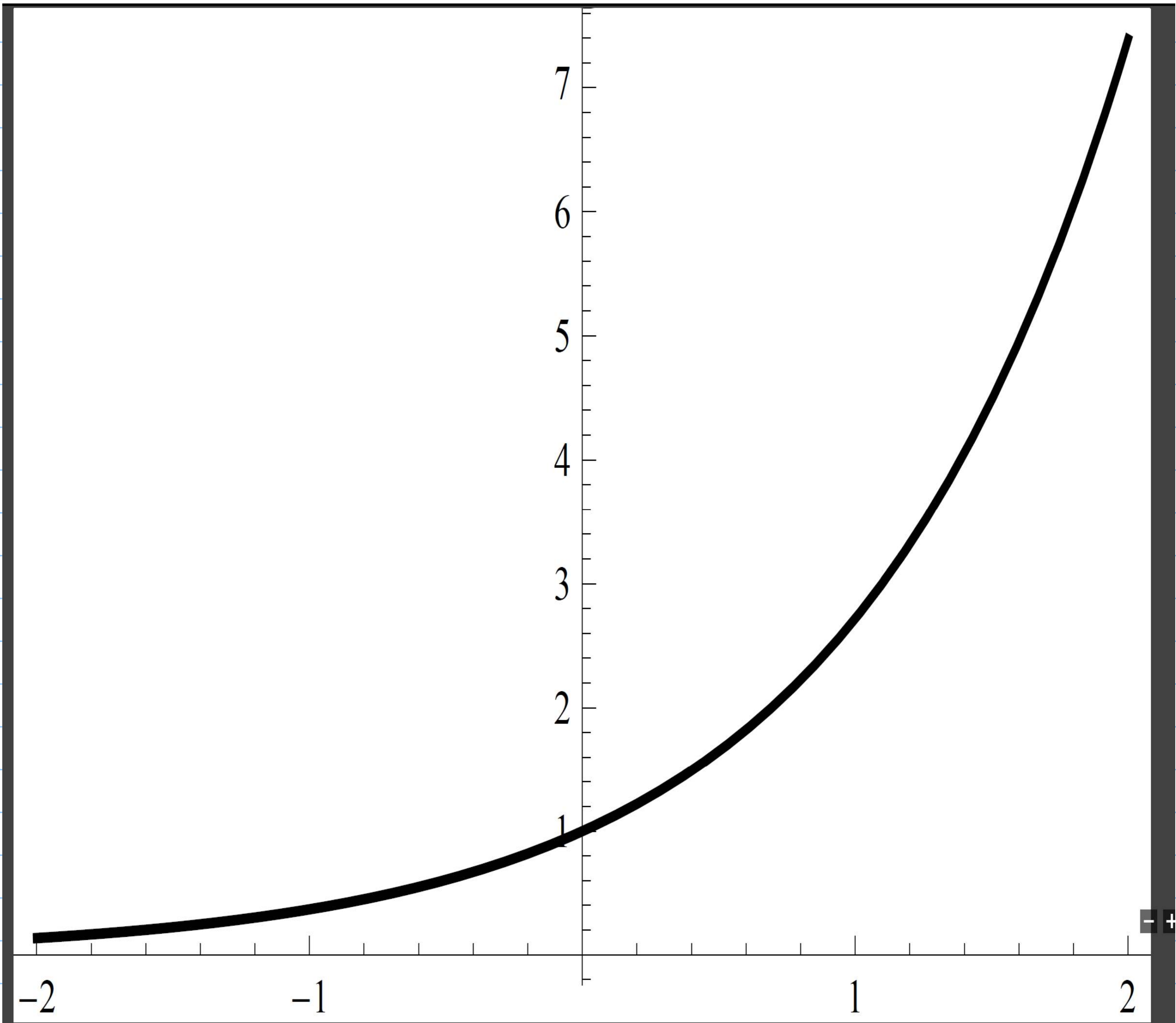
$$= \lim_{x \to 0} \frac{e^x - (1 + 3 \cdot 2a_3 x)}{3 \cdot 2 \cdot x} \overset{H}{=}$$

$$= \lim_{x \to 0} \frac{e^x - 3 \cdot 2 \cdot a_3}{3 \cdot 2} = \frac{1 - 3!\, a_3}{3!}$$

$$0 = \frac{1 - 3!\, a_3}{3!} \iff a_3 = \frac{1}{3!}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + o(x^3)$$

$$= T_3(x) + \underbrace{o(x^3)}_{\text{RESTO}}, \quad x \to 0$$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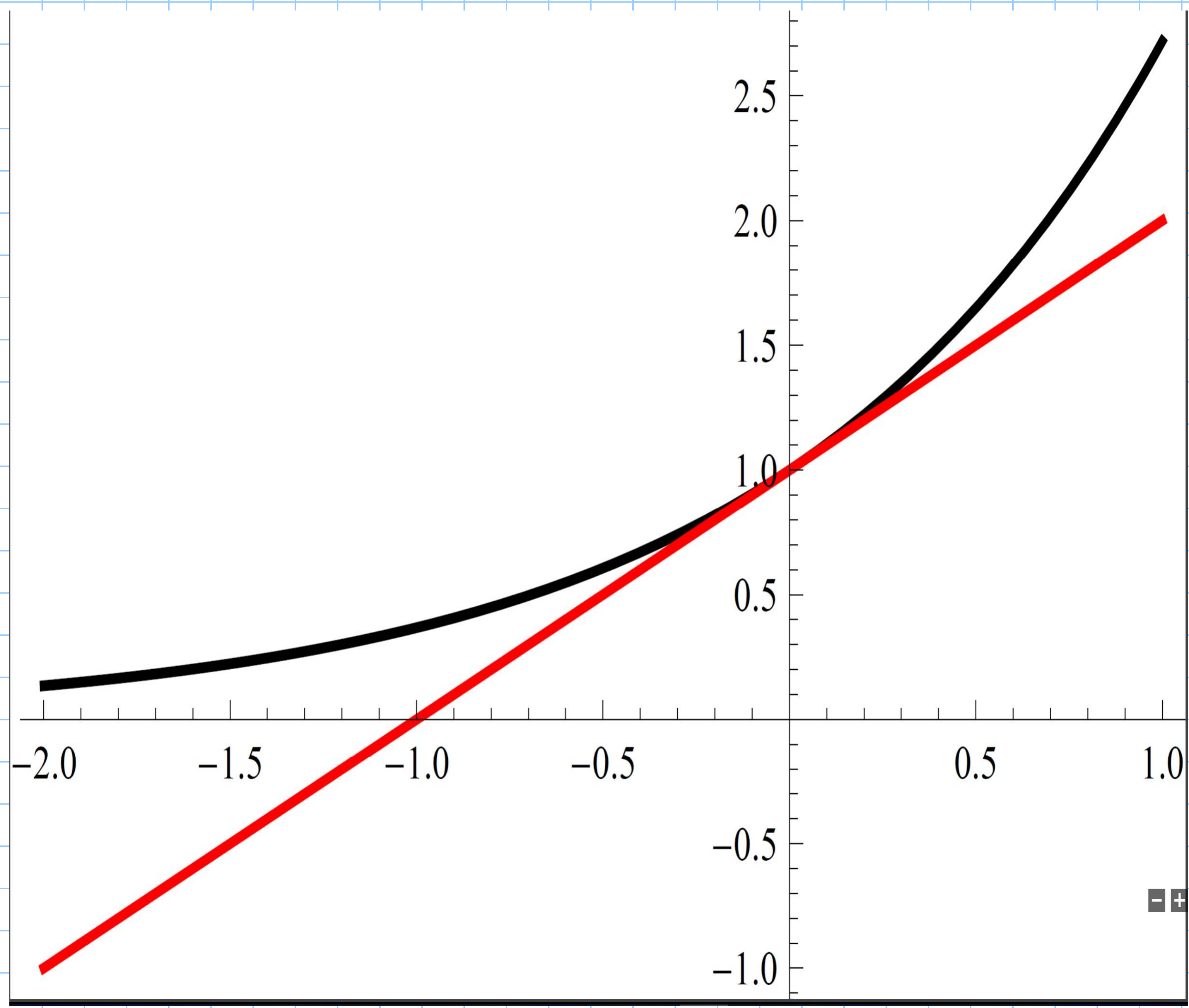
6
5
4
3
2
1
−2
−1
1
2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2.0
1.5
1.0
0.5
−0.5
−1.0
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−1.5
−1.0
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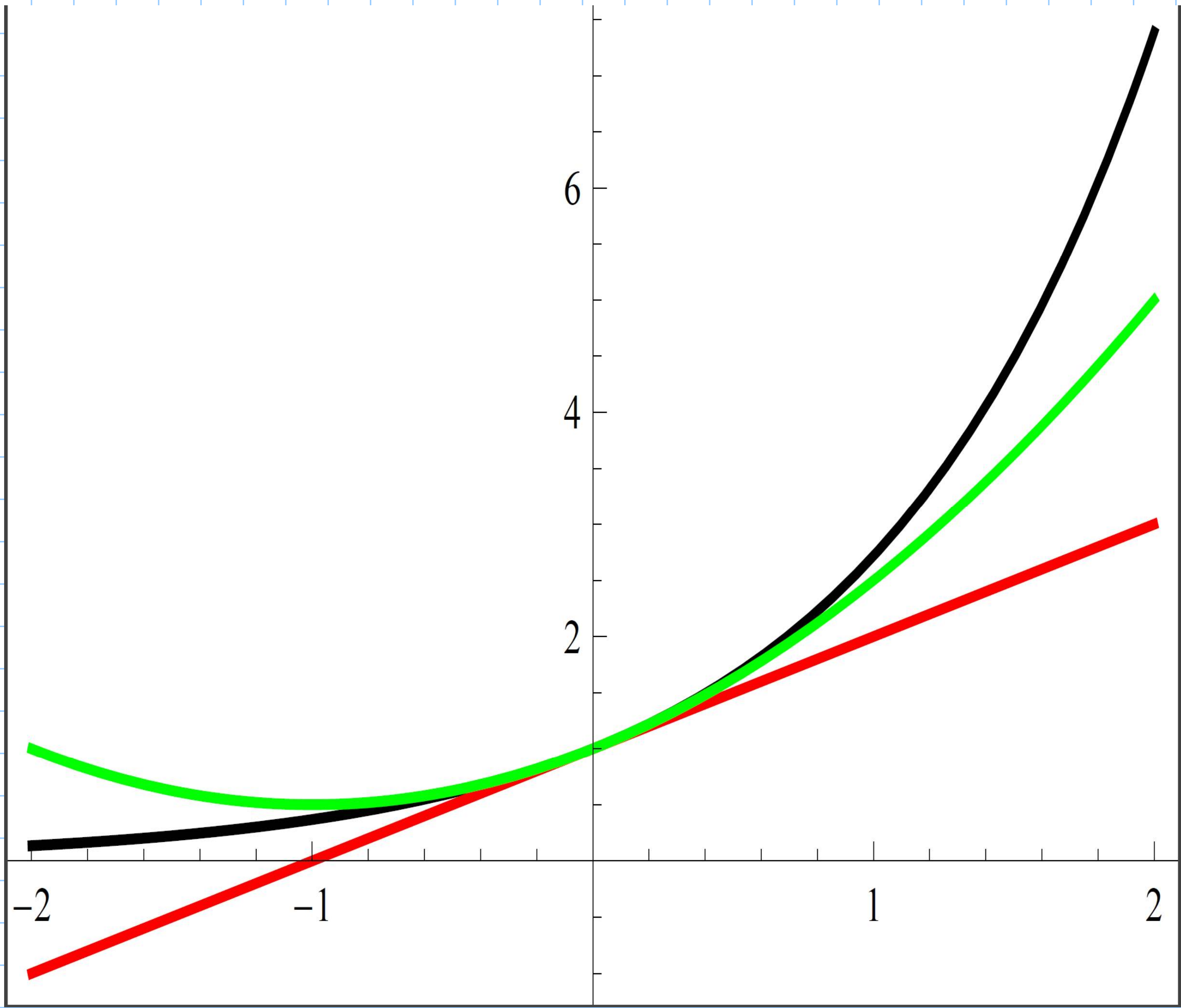
0.5
1.0



6
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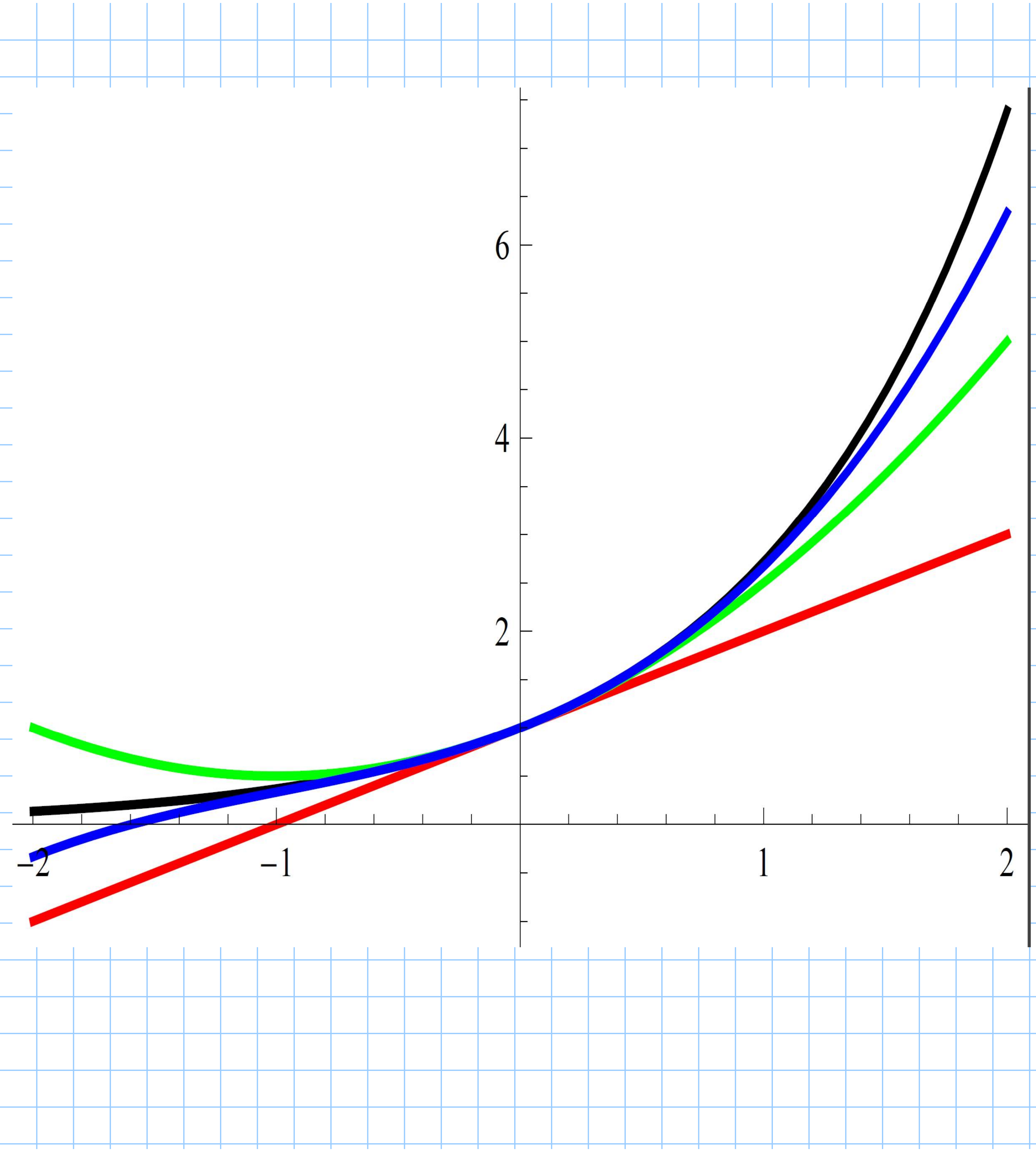

6
4
2
−2
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2

TEOREMA [FORMULA DEL POLINOMIO DI TAYLOR CON RESTO di PEANO]

Sia $f : (a,b) \longrightarrow \mathbb{R}$, $x_0 \in (a,b)$

f DERIVABILE $m-1$ VOLTE in (a,b), $m \geq 1$.

SE f è DERIVABILE in x_0 m VOLTE,

ALLORA $\exists$ un UNICO POLINOMIO di GRADO m (POLINOMIO di TAYLOR di CENTRO x_0),

$$T_m(x) = T_m(x; f, x_0):$$

$$f(x) = T_m(x) + o((x-x_0)^m) \quad x \longrightarrow x_0$$

$$T_m(x) = \sum_{k=0}^{m} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

$$T_m(x) = \frac{f^{(0)}(x_0)}{0!}(x-x_0)^0 + \frac{f^{(1)}(x_0)}{1!}(x-x_0)^1$$

$k=0$ $\quad$ $k=1$

$$+ \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(m)}(x_0)}{m!}(x-x_0)^m$$

$k=2$ $\quad$ $k=m$

$$= f(x_0) + f^{(1)}(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(m)}(x_0)}{m!}(x-x_0)^m$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^m}{m!} + o(x^m).$$

$x_0 = 0$ $\quad$ $x \to 0$

DIMOSTRAZIONE.

PER SEMPICITA' CONSIDERIAMO SOLO IL CASO $m=3$.

PER IPOTESI

$\exists\, f^{(2)}$ in (a,b) e $\exists\, f^{(3)}(x_0)$,

E LA TESI È CHE:

$$f(x) - \sum_{k=0}^{3} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = o\left((x-x_0)^3\right) \quad x \to x_0$$

$$0 = \lim_{x \to x_0} \frac{f(x) - \sum_{k=0}^{3} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}{(x-x_0)^3} \overset{H}{=}$$

$$\lim_{x \to x_0} \frac{f^{(1)}(x) - \sum_{\boxed{k=1}} ^{3} \frac{f^{(k)}(x_0)}{k!} \cdot k(x-x_0)^{k-1}}{3(x-x_0)^2} \overset{H}{=}$$

!

• NOTARE CHE LA SOMMA PARTE DA K=1 PERCHÈ IL TERMINE CON K=0 È COSTANTE E SI ANNULLA

DERIVANDO.

.. DATO CHE $\exists f^{(2)}$ in (a,b), la $f^{(1)}$ è DERIVABILE e quindi CONTINUA in (a,b). PERTANTO SI HA $f^{(1)}(x) \longrightarrow f^{(1)}(x_0)$, $x \to x_0$

ABBIAMO ANCORA UNA FORMA INDETERMINATA

$$= \lim_{x \to x_0} \frac{f^{(2)}(x) - \sum_{k=2}^{3} \frac{f^{(k)}(x_0)}{k!} k \cdot (k-1)(x-x_0)^{k-2}}{3 \cdot 2 (x-x_0)} =$$

$$= \lim_{x \to x_0} \frac{f^{(2)}(x) - \frac{f^{(2)}(x_0)}{2!} 2(2-1)(x-x_0)^{2-2} - \frac{f^{(3)}(x_0)}{3!} 3(3-1)(x-x_0)^{3-2}}{3 \cdot 2 (x-x_0)} =$$

$$= \frac{1}{6} \lim_{x \to x_0} \frac{f^{(2)}(x) - f^{(2)}(x_0) - f^{(3)}(x_0)(x-x_0)}{x-x_0} =$$

$$= \frac{1}{6} \lim_{x \to x_0} \left(\frac{f^{(2)}(x) - f^{(2)}(x_0)}{x - x_0} - f^{(3)}(x_0) \right) = 0 \qquad \blacksquare$$

$$f(x) = e^x \; ; \quad D^{(k)}(e^x) = e^x, \; \forall k \in \mathbb{N}$$

$$x_0 = 0 \qquad f^{(k)}(0) = e^x \Big|_{x=0} = 1, \; \forall k \in \mathbb{N}$$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k =$$

$$= \sum_{k=0}^{n} \frac{x^k}{k!} .$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} .$$

SVILUPPI di TAYLOR di CENTRO $x_0 = 0$ SI DICONO di

Mac LAURIN

POLINOMIO di MacLAURIN = POLINOMIO di TAYLOR CON CENTRO $X_0=0$

$$f(x) = \log(1+x)$$

$x_0 = 0$

$$D^{(k)}(\log(1+x))\Big|_{x=0} = (-1)^{k-1}(k-1)! \qquad k \geq 1$$

$$T_m(x) = \sum_{k=0}^{m} \frac{f^{(k)}(0)}{k!} x^k = \quad (f(0)=0)$$

$$= \sum_{k=1}^{m} \frac{(-1)^{k-1} \cdot (k-1)!}{k!} x^k =$$

$$= \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{(-1)^{m-1}}{m} x^m$$

$f(x) = \sin(x)$,

$x_0 = 0$

ATTENZIONE: POLINOMIO di GRADO $2m+1$

$$T_{2m+1}(x) = \sum_{k=0}^{m} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{(-1)^m x^{2m+1}}{(2m+1)!}$$

$f(x) = \cos(x)$

$x_0 = 0$

$$T_{2m}(x) = \sum_{k=0}^{m} (-1)^k \frac{x^{2k}}{(2k)!} =$$

GRADO $2m$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^m \frac{x^{2m}}{(2m)!}$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3), \quad x \to 0$$

$$T_1(x) = x$$

$$\sin(x) = x + o(x) = x + o(x^2) =$$

$$T_2(x) = x$$

$$= x - \frac{x^3}{3!} + o(x^3) =$$

$$T_3(x) = x - \frac{x^3}{3!}$$

$$= x - \frac{x^3}{3!} + o(x^4)$$

$$T_4(x) = x - \frac{x^3}{3!}$$

$$T_1(x) = 1$$

$$\cos(x) = 1 + o(x) = 1 - \frac{x^2}{2} + o(x^2)$$

$$T_2(x) = 1 - \frac{x^2}{2}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^5)$$

$$T_3(x) = T_2(x) \; ; \; T_5(x) = T_4(x)$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{4!}$$

$$f(x) = (1+x)^\alpha \quad , \alpha \neq 0 \text{ e } x > -1 \text{ se } \alpha < 0.$$

$$f^{(1)}(x) = \alpha(1+x)^{\alpha-1},$$

$$f^{(2)}(x) = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

$$f^{(3)}(x) = \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}$$

$$f^{(1)}(0) = \alpha, \quad f^{(2)}(0) = \alpha(\alpha-1),$$

$$f^{(3)}(0) = \alpha(\alpha-1)(\alpha-2)$$

$$\dots$$

$$f^{(k)}(0) = \alpha(\alpha-1)(\alpha-2)\dots(\alpha-(k-1))$$

$$(1+x)^{\alpha} \underset{x_0=0}{=} 1+\alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + o(x^3), \quad x \to 0$$

$$\frac{1}{(1+x)} = (1+x)^{-1} = 1 - x + \frac{(-1)(-2)}{2}x^2 + o(x^2)$$

$$= 1 - x + x^2 + o(x^2), \quad x \to 0$$

$$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + o(x^2)$$

$$= 1 + x + x^2 + x^3 + \dots + x^m + o(x^m), \quad x \to 0$$

$$\frac{1}{1-x} = \sum_{k=0}^{m} x^k + o(x^m), \quad x \to 0$$

DIM.

$$\frac{1-x^{m+1}}{1-x} = \sum_{k=0}^{m} x^k \Rightarrow \frac{1}{1-x} = \sum_{k=0}^{m} x^k + \frac{x^{m+1}}{1-x} \Rightarrow$$

$$\Rightarrow \frac{1}{1-x} = \sum_{k=0}^{m} x^k + o(x^m), \quad x \to 0$$

DATO CHE IL POLINOMIO di TAYLOR È UNICO e DATO CHE $T_m(x) = \sum_{k=0}^{m} x^k$ VERIFICA

$$\frac{1}{1-x} = T_m(x) + o(x^m), \quad x \to 0$$

ALLORA T_m è il POLINOMIO di TAYLOR di $\frac{1}{1-x}$ CON CENTRO $x_0 = 0$ $\blacksquare$

TEOREMA [CARATTERIZZAZIONE PUNTI DI ESTREMO RELATIVO]

$$f:(a,b)\longrightarrow \mathbb{R}\ ,\quad x_0\in(a,b)$$

f DERIVABILE n VOLTE in (a,b)

$n+1$ VOLTE in x_0

SUPPONIAMO CHE

$$f^{(k)}(x_0)=0\quad ,\ k=1,2,\dots,n$$

e $f^{(n+1)}(x_0) \neq 0$. ALLORA

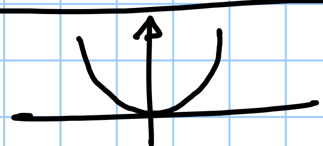
SE $n \in 2\mathbb{N}-1$ $(n+1 \in 2\mathbb{N})$

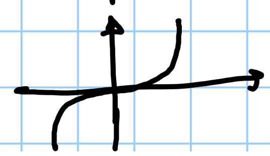
$$\Rightarrow \begin{cases} f^{(n+1)}(x_0) > 0 \Rightarrow & x_0 \text{ è di MINIMO LOCALE} \\ f^{(n+1)}(x_0) < 0 \Rightarrow & x_0 \text{ è di MASSIMO LOCALE} \end{cases}$$

SE $n \in 2\mathbb{N}$

$(n+1 \in 2\mathbb{N}-1) \Rightarrow$

$$\begin{cases} f^{(n+1)}(x_0) > 0 \Rightarrow \begin{cases} f(x) > f(x_0), & x \to x_0^+ \\ f(x) < f(x_0), & x \to x_0^- \end{cases} \text{ definitivamente} \\ f^{(n+1)}(x_0) < 0 \Rightarrow \begin{cases} f(x) < f(x_0), & x \to x_0^+ \\ f(x) > f(x_0), & x \to x_0^- \end{cases} \text{ definitivamente} \end{cases}$$

Es. $n \in 2\mathbb{N}-1$ $f(x) = x^4$ 

$n \in 2\mathbb{N}$ $f(x) = x^3$ 

DIMOSTRAZIONE

CASO $\quad m \in 2\mathbb{N}-1 \Rightarrow m+1 \in 2\mathbb{N}$

$$f(x) = f(x_0) + \sum_{k=1}^{m} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(m+1)}(x_0)}{(m+1)!}(x-x_0)^{m+1} + o\left((x-x_0)^{m+1}\right) =$$

$$\boxed{f^{(k)}(x_0) = 0 \,, \quad \forall k = 1, 2, \dots, m}$$

$$= f(x_0) + \frac{f^{(m+1)}(x_0)}{(m+1)!}(x-x_0)^{m+1} + o\left((x-x_0)^{m+1}\right)$$

$$f(x) - f(x_0) = \frac{f^{(m+1)}(x_0)}{(m+1)!}(x-x_0)^{m+1} + o\left((x-x_0)^{m+1}\right)$$

$$= \frac{f^{(m+1)}(x_0)}{(m+1)!}(x-x_0)^{m+1}\left(1 + o(1)\right), \quad x \to x_0$$

$$\text{Se } f^{(n+1)}(x_0) > 0 \implies \frac{f^{(n+1)}(x_0)}{(n+1)!}(x-x_0)^{n+1} > 0 \quad \text{definitivamente } x \to x_0$$

$$!\ n+1 \in 2\mathbb{N}\ !$$

$$\text{Se } f^{(n+1)}(x_0) < 0 \implies \frac{f^{(n+1)}(x_0)}{(n+1)!}(x-x_0)^{n+1} < 0 \quad \text{definitivamente } x \to x_0$$

$$!\ n+1 \in 2\mathbb{N}\ !$$

$$f^{(n+1)}(x_0) > 0 \implies f(x) - f(x_0) > 0 \quad \text{definitivamente per } x \to x_0$$

$$f^{(n+1)}(x_0) < 0 \implies f(x) - f(x_0) < 0 \quad \text{per } x \to x_0$$

$\blacksquare$

$$\cos(x) = 1 - \frac{x^2}{2!} + o(x^3)$$

$$T_0(x) = 1 = T_1(x)$$

$$T_2(x) = 1 - \frac{x^2}{2!} \quad , \quad x_0 = \text{max locale}$$

NOTA: $T_3(x) = T_2(x)$

$$f(x) = \cos(x^4)$$

$x_0 = 0$ PUNTO di ESTREMO?

$$\cos(y) = 1 - \frac{y^2}{2} + o(y^3), \quad y \to 0$$

$$y = x^4$$

$$\cos(x^4) = 1 - \frac{x^8}{2} + o(x^{12}), \quad x \to 0$$

$$T_0 = 1 = T_1 = T_2 = \dots = T_7$$

$$a_k = \frac{f^{(k)}(0)}{k!}, \quad f^{(k)}(0) = a_k \cdot k!$$

COEFFICENTE di x^k NEL POLINOMIO

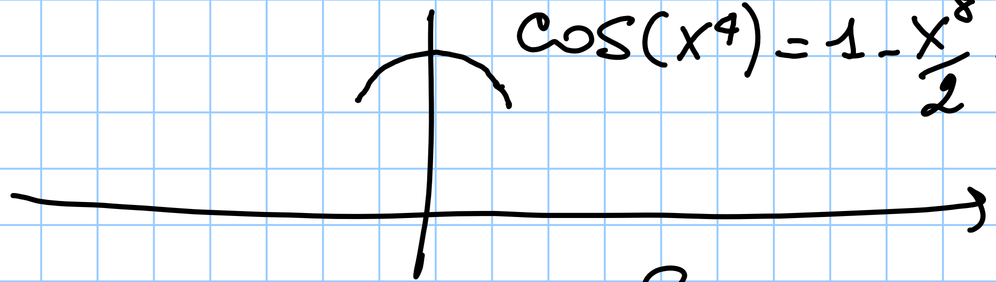
$a_k = 0 \quad , \; k = 1, 2, \dots, 7 \Longrightarrow$

$f^{(k)}(0) = 0 \quad , \; k = 1, 2, \dots, 7$

$a_8 = -\frac{1}{2} \quad , \quad f^{(8)}(0) = a_8 \cdot 8! = -\frac{8!}{2} < 0$

$\Longrightarrow \quad x_0 = 0$ PUNTO DI MASSIMO LOCALE

$\cos(x^4) = 1 - \frac{x^8}{2} + o(x^{12})$



$T_8(x) = 1 - \frac{x^8}{2} = T_9 = T_{10} = T_{11} = T_{12}$

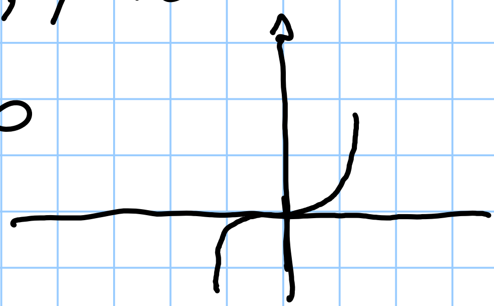
$f^{(k)}(0) = 0 \qquad k = 9, 10, 11, 12$

$f(x) = \sin(x^7) = x^7 + o(x^{14}),$
$x \to 0$

$\sin(y) = y + o(y^2), \; y \to 0$

$T_0 = T_1 = \dots T_6 = 0$

$T_7 = x^7 \quad \dots\dots$



T_6 (Mac LAURIN)

$$f(x) = \log(1+x^3) - \sin(x^2)$$

$$\log(1+y) = y - \frac{y^2}{2} + o(y^2), \quad y \to 0$$

$$\sin(y) = y - \frac{y^3}{3!} + o(y^4), \quad y \to 0$$

$$f(x) = x^3 - \frac{(x^3)^2}{2} + o\left((x^3)^2\right)$$

$$- \left(x^2 - \frac{(x^2)^3}{3!} + o\left((x^2)^4\right) \right)$$

$$= x^3 - \frac{x^6}{2} + o(x^6) - x^2 + \frac{x^6}{6} + o(x^8)$$

$$= -x^2 + x^3 - \frac{1}{2}x^6 + \frac{1}{6}x^6 + o(x^6)$$

$$= \overbrace{-x^2 + x^3 - \frac{x^6}{3}}^{T_6} + o(x^6)$$

$0 = T_0 = T_1 \quad , \quad T_2 = -x^2$

$T_3 = -x^2 + x^3 = T_4 = T_5$

$f^{(k)}(0) = a_k \cdot k! \quad \ldots\ldots$

$$\lim_{x\to 0^+} \frac{\log(1+x^3) - \sin(x^2)}{x^\alpha} = \quad , \alpha \in \mathbb{R}$$

$$= \lim_{x\to 0^+} \frac{-x^2 + o(x^2)}{x^\alpha} = \lim_{x\to 0^+}$$

$\alpha \leq 0 \; ; \; 0$

$\alpha > 0 \quad -\frac{x^2}{x^\alpha}(1+o(1))$

$$= \begin{cases} 0 & \alpha \leq 0 \\ 0^- & 2-\alpha > 0 \; , \; \alpha < 2 \\ -1 & 2-\alpha = 0 \; , \; \alpha = 2 \\ -\infty & 2-\alpha < 0 \; , \; \alpha > 2 \end{cases}$$

$$\lim_{x\to 0^+} \frac{\log(1+x^3) - \sin(x^2) + x^2}{x^\alpha} = \ldots$$