$$\lim_{x \to +\infty} \frac{e^x}{x^3} = +\infty$$

$$\lim_{x \to +\infty} \frac{e^{\sqrt{x}}}{x} \overset{\text{"H"}}{=} \lim_{x \to +\infty} \frac{1}{2\sqrt{x}} e^{\sqrt{x}} = \lim_{x \to +\infty} \frac{e^{\sqrt{x}}}{2\sqrt{x}} \overset{\text{"H"}}{=}$$

$$= \lim_{x \to +\infty} \frac{\cancel{\frac{1}{2\sqrt{x}}}\, e^{\sqrt{x}}}{\cancel{\frac{1}{\sqrt{x}}}} = +\infty$$

$$\lim_{x \to +\infty} \frac{e^{x^\alpha}}{x^\beta} = +\infty \qquad \forall \alpha > 0 \quad \forall \beta > 0$$

$$\lim_{x \to +\infty} \frac{\log(x)}{x} \overset{\text{"H"}}{=} \lim_{x \to +\infty} \frac{\frac{1}{x}}{1} = 0^+$$

$$\lim_{x \to +\infty} \frac{(\log(x))^4}{x} \overset{\text{"H"}}{=} \lim_{x \to +\infty} \frac{4(\log(x))^3}{x} \overset{\text{"H"}}{=}$$

$$\lim_{x \to +\infty} 4 \cdot 3 \frac{(\log(x))^2}{x} \overset{\text{"H"}}{=} \lim_{x \to +\infty} (4!) \frac{\log(x)}{x} = 0^+$$

$$\lim_{x \to +\infty} \frac{(\log(x))^\alpha}{x^\beta} = 0^+ \quad \forall \alpha > 0$$
$$\forall \beta > 0$$

T. di CONTINUITÀ DELLA DERIVATA PRIMA

$f : [a,b] \longrightarrow \mathbb{R}$, $x_0 \in [a,b]$

f CONTINUA in $[a,b]$

DERIVABILE in $[a,b] \setminus \{x_0\}$

SE $\exists \lim_{x \to x_0} f'(x) \Longrightarrow$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} f'(x)$$

DIMOSTRAZIONE

$f(x) - f(x_0) \longrightarrow 0$, $x \longrightarrow x_0$

(CONTINUITÀ di f)

$g(x) = x - x_0 \longrightarrow 0$, $x \longrightarrow x_0$

$g'(x) = 1 \neq 0 \ \forall x \Longrightarrow$

T. di L'HOPITAL

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} f'(x)$$

$\blacksquare$

CONTROESEMPIO

$$f(x) = x^2 \sin\left(\frac{1}{x}\right),$$

CONTINUA in $\mathbb{R}$, $f(x) \to f(0) = 0$,

$$-x^2 \le \underbrace{x^2 \sin\left(\frac{1}{x}\right)}_{\downarrow\, 0} \le x^2 \qquad x \to 0$$

DERIVABILE in $\mathbb{R} \setminus \{0\}$

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) + x^2 \cdot \left(-\frac{1}{x^2}\right) \cos\left(\frac{1}{x}\right)$$

$$= 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), \quad x \neq 0$$

$$\lim_{x\to 0} f'(x) \ \nexists, \quad 2x\sin\left(\frac{1}{x}\right) \to 0,$$

$$\cos\left(\frac{1}{x}\right), \quad , \quad x_n = \frac{1}{2n\pi} \xrightarrow[n\to+\infty]{} 0^+$$

$$\cos\left(\frac{1}{x_n}\right) = 1 \ \forall n$$

$$y_n = \frac{1}{2n\pi + \frac{\pi}{2}} \to 0^+, \quad n \to +\infty$$

$$\cos\left(\frac{1}{y_n}\right) = 0 \ \forall n$$

MA LA FUNZIONE E' DERIVABILE in 0,

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$$

CONTROESEMPI

SE f e g non sono ENTRAMBE INFINITESIME O INFINITE

$$\lim_{x\to 0^+} \frac{\cos(x)}{x} = \frac{1}{0^+} = +\infty \neq \lim_{x\to 0^+} \frac{\sin(x)}{1} = 0$$

SE $\displaystyle\lim_{x\to 0^+} \frac{f'(x)}{g'(x)} \nexists \Rightarrow$ IL TEOREMA NON SEMPRE VALE

$$\lim_{x\to 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{x} = 0,$$ ma

$$\lim_{x\to 0} \frac{2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)}{1} \nexists$$

$$\lim_{x\to 0^+} \overset{0^+}{x} \cdot \overset{-\infty}{\log(x)} = \lim_{y\to +\infty} \frac{\log\left(\frac{1}{y}\right)}{y} =$$

$y = \frac{1}{x} \xrightarrow[x\to 0^+]{} +\infty$ $$= -\lim_{y\to +\infty} \frac{\log(y)}{y} = 0^-$$

$$\lim_{x\to 0^+} \frac{\log(x)}{\frac{1}{x}} = -\lim_{x\to 0^+} \frac{-\log(x)}{\frac{1}{x}} \overset{H}{=}$$

$$= -\lim_{x\to 0^+} \frac{-\frac{1}{x}}{-\frac{1}{x^2}} = -\lim_{x\to 0^+} x = 0^-$$

$$\lim_{x\to 0^+} x(\log(x))^4 = 0^+ \quad \left(y=\frac{1}{x} \xrightarrow[x\to 0^+]{} +\infty \dots \right)$$

$$\lim_{x\to 0^+} \frac{(\log(x))^4}{\frac{1}{x}} \overset{H}{=} \dots$$

$$\lim_{x\to 0^+} x^2 e^{\frac{1}{x}} \underset{y=\frac{1}{x}\xrightarrow[x\to 0^+]{} +\infty}{=} \lim_{y\to +\infty} \frac{e^y}{y^2} = +\infty$$

$$\lim_{x\to 0^+} \frac{e^{1/x}}{\frac{1}{x^2}} \overset{H}{=} \lim_{x\to 0^+} \frac{-\frac{1}{x^2} e^{\frac{1}{x}}}{-\frac{2}{x^3}} =$$

$$= \lim_{x\to 0^+} \frac{e^{\frac{1}{x}}}{2/x} \overset{H}{=}$$

$$= \lim_{x\to 0^+} \frac{\cancel{-\frac{1}{x^2}}\, e^{\frac{1}{x}}}{\cancel{-\frac{2}{x^2}}} = +\infty$$

$$x \cdot \log(x) = \frac{x}{\frac{1}{\log(x)}},$$

$$\lim_{x \to 0^+} x \log(x) = \lim_{x \to 0^+} \frac{x}{\frac{1}{\log(x)}} \overset{H}{=}$$

$$= \lim_{x \to 0^+} \frac{1}{-\frac{1}{(\log(x))^2} \cdot \frac{1}{x}} = \lim_{x \to 0^+} -x(\log(x))^2$$

E' PEGGIO
DI PRIMA

DEFINIZIONE [DERIVATE DI ORDINE SUPERIORE]

$$f^{(0)}(x) = f(x)$$

SE PER $n \geq 0$ È DEFINITA LA DERIVATA DI ORDINE n, $f^{(n)}(x)$, IN UN INTORNO APERTO DI x_0, ALLORA DEFINIAMO LA DERIVATA DI ORDINE $n+1$ IN x_0

$$f^{(n+1)}(x_0) = \lim_{x \to x_0} \frac{f^{(n)}(x) - f^{(n)}(x_0)}{x - x_0}$$

SE IL LIMITE ESISTE FINITO.

Es. $f(x) = e^x$, $x \in \mathbb{R}$

$$f^{(n)}(x) = e^x, \quad x \in \mathbb{R} \quad \forall n \in \mathbb{N}.$$

Es. $f(x) = \log(1+x)$, $x > -1$

$$f^{(1)}(x) = \frac{1}{1+x},$$

$$f^{(2)}(x) = \frac{(-1)}{(1+x)^2} = -\frac{1}{(1+x)^2}$$

$$f^{(3)}(x) = \frac{(-1)(-2)}{(1+x)^3} = \frac{2 \cdot 1}{(1+x)^3}$$

$$f^{(4)}(x) = \frac{(-1)(-2)(-3)}{(1+x)^4} = -\frac{3 \cdot 2 \cdot 1}{(1+x)^4}$$

$$(1) \quad f^{(k)}(x) = \frac{(-1)^{k-1} \cdot (k-1)!}{(1+x)^k}, \quad k \geq 1$$

DIMOSTRIAMO LA (1) PER INDUZIONE

(i) $k=1$ $\quad f^{(1)}(x) = \dfrac{(-1)^0 \cdot 0!}{(1+x)^1} = \dfrac{1}{1+x}$

(ii) IPOTESI INDUTTIVA:

$k \geq 1 \quad f^{(k)}(x) = \dfrac{(-1)^{k-1}(k-1)!}{(1+x)^k} \Rightarrow$

$$f^{(k+1)}(x) = \frac{(-1)^k k!}{(1+x)^{k+1}}$$

$$\frac{d}{dx}\left(f^{(k)}(x)\right) = \frac{d}{dx}\left(\frac{(-1)^{k-1}(k-1)!}{(1+x)^k}\right) =$$

$$= \frac{(-1)^{k-1}(-1)\,k\,(k-1)!}{(1+x)^{k+1}} = \frac{(-1)^k k!}{(1+x)^{k+1}}$$

$\blacksquare$

FORMULA di LEIBNIZ

$$D^{(n)}(fg) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$

$$D^{(1)}(fg) = f^{(1)} g + f g^{(1)}$$

$$D^{(2)}(fg) = f^{(2)} g + f^{(1)} g^{(1)} + f^{(1)} g^{(1)} + f g^{(2)}$$

$$= f^{(2)} g + 2 f^{(1)} g^{(1)} + f g^{(2)}$$

$$D^{(3)}(fg) = f^{(3)} g + f^{(2)} g^{(1)} + 2 f^{(2)} g^{(1)} + 2 f^{(1)} g^{(2)} + f^{(1)} g^{(2)} + f g^{(3)}$$

$$= f^{(3)} g + 3 f^{(2)} g^{(1)} + 3 f^{(1)} g^{(2)} + f g^{(3)}$$

di TERMINI CHE SI OTTENGONO

da $D^{(n)}(fg)$, $\sum_{0}^{n} k \binom{n}{k} = \sum_{0}^{n} k \binom{n}{k} 1^k 1^{n-k}$
$= (1+1)^n = 2^n$

$$D^{(4)}\left(e^x \log(1+x)\right) =$$

$$\sum_{k=0}^{4} \binom{4}{k} D^{(k)}\left(\log(1+x)\right) D^{(4-k)}(e^x)$$

$$= \binom{4}{0} \log(1+x)\, D^{(4)}(e^x)$$

$$+ \sum_{k=1}^{4} \binom{4}{k} D^{(k)}\left(\log(1+x)\right) \cdot e^x$$

$$= e^x \log(1+x) + \sum_{k=1}^{4} \binom{4}{k} \frac{(-1)^{k-1}(k-1)!}{(1+x)^k} \cdot e^x$$

$$= e^x \log(1+x) + \binom{4}{1} \frac{(-1)^0 \cdot 0!}{(1+x)} e^x$$

$$+\binom{4}{2}\frac{(-1)^1\cdot 1!\,e^x}{(1+x)^2}+\binom{4}{3}\cdot\frac{2!}{(1+x)^3}e^x$$

$$\boxed{\binom{4}{2}=\frac{4!}{2!\,2!}=3\cdot 2=6}\quad+\binom{4}{4}\frac{(-1)(3!)}{(1+x)^4}e^x$$

$$=e^x\log(1+x)+\frac{4e^x}{1+x}-\frac{6e^x}{(1+x)^2}+$$

$$+\frac{8e^x}{(1+x)^3}-\frac{6e^x}{(1+x)^4}$$

IL POLINOMIO DI TAYLOR

$$\sin(x) = x + o(x), \quad x \longrightarrow 0$$

$$\Updownarrow$$

$$\lim_{x \to 0} \frac{\sin(x) - x}{x} = 0$$

DOMANDA: E' VERO CHE $\exists\, c \in \mathbb{R}$:

$$o(x) = cx^2 + o(x^2)$$

$$\lim_{x\to 0}\frac{\sin(x)-x}{x^2} \overset{H}{=} \lim_{x\to 0}\frac{\cos(x)-1}{2x} =$$

$$\overset{H}{=} \lim_{x\to 0} -\frac{\sin(x)}{2} = 0$$

$$\sin(x) - x = o(x^2)$$

? $o(x) = cx^2 + o(x^2)$? $\Longrightarrow$ Si, com $c=0$

$$\sin(x) = x + 0\cdot x^2 + o(x^2) = x + o(x^2)$$

? $o(x^2) = cx^3 + o(x^3)$?

$$\lim_{x\to 0}\frac{\sin(x)-x}{x^3} \overset{H}{=} \lim_{x\to 0}\frac{\cos(x)-1}{3x^2}$$

$$\overset{H}{=} \lim_{x\to 0}\frac{-\sin(x)}{6x} = -\frac{1}{6}$$

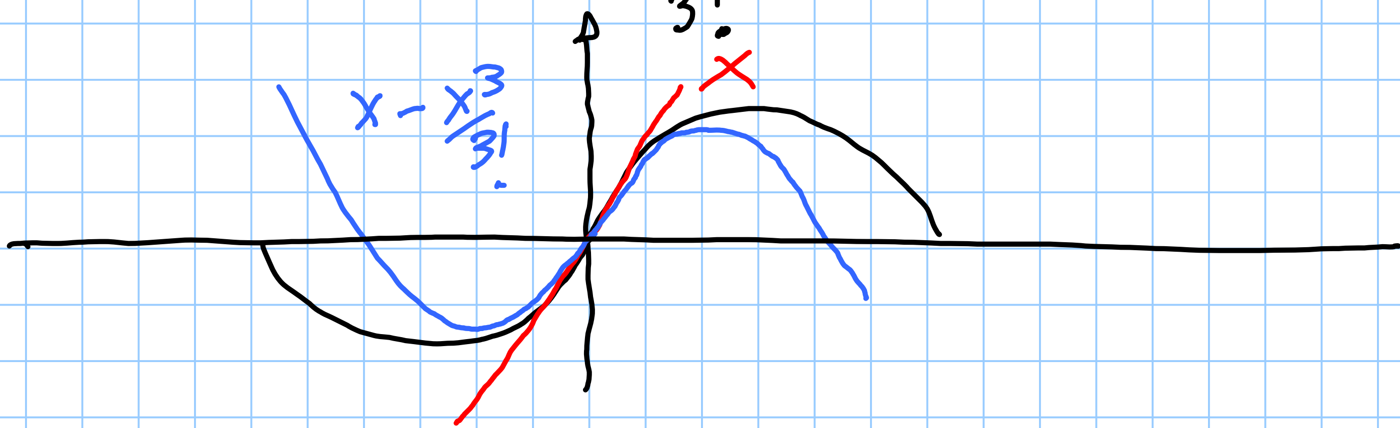
$$\lim_{x\to 0}\frac{\sin(x)-x}{x^3} = -\frac{1}{6}$$

$$\lim_{x\to 0}\frac{\sin(x)-x+\frac{x^3}{6}}{x^3} = 0$$

$$\sin(x) - x + \frac{x^3}{6} = o(x^3)$$

? $o(x^2) = c\, x^3 + o(x^3)$? Si $\to c = -\frac{1}{6}$

$$\sin(x) = x - \frac{x^3}{3!} + o(x^3)$$



$$e^x = 1 + x + o(x), \quad x \to 0$$

? $\exists\, a_2 \in \mathbb{R}$: $e^x = 1 + x + a_2 x^2 + o(x^2)$, $x \to 0$

$$\lim_{x \to 0} \frac{e^x - (1 + x + a_2 x^2)}{x^2} = 0$$

$\boxed{\exists\, a_2 \in \mathbb{R} : \text{IL LIMITE VALE } 0\ ?}$

$$\lim_{x\to 0} \frac{e^x - 1 - x - a_2 x^2}{x^2} \overset{H}{=} \lim_{x\to 0} \frac{e^x - 1 - 2a_2 x}{2x}$$

$$\overset{H}{=} \lim_{x\to 0} \frac{e^x - 2a_2}{2} = \frac{1-2a_2}{2} = 0$$

$\exists$ un UNICO $a_2 \in \mathbb{R}$: IL LIMITE È 0

$$\boxed{a_2 = \frac{1}{2}}$$

$\exists$ un UNICO POLINOMIO di 2° GRADO,

$$T_2(x) = 1 + x + \frac{x^2}{2!} :$$

$$e^x = 1 + x + \frac{x^2}{2!} + o(x^2), \quad x \to 0$$

? $\exists$ $a_3 \in \mathbb{R}$:

$$e^x = 1 + x + \frac{x^2}{2!} + a_3 x^3 + o(x^3),$$
$$x \to 0 \ ?$$

$$\Updownarrow$$

? $\exists\, a_3 \in \mathbb{R}$:

$$\lim_{x \to 0} \frac{e^x - \left(1 + x + \frac{x^2}{2!} + a_3 x^3\right)}{x^3} = 0$$

Ex:

$$a_3 = \frac{1}{3!}$$