TEOREMA [FACOLTATIVO]

$f:(a,b) \longrightarrow \mathbb{R}$, CONVESSA

ALLORA f è CONTINUA in (a,b).

DIMOSTRAZIONE: $x_0 \in (a,b)$

$$a < x_1 < x_0 < x_2 < b$$

$$P_f(x,x_0) = \frac{f(x)-f(x_0)}{x-x_0}, \quad x \in [x_1,x_2], \ x \neq x_0$$

è CRESCENTE in x.

$$\frac{f(x_1)-f(x_0)}{x_1-x_0} \leq \frac{f(x)-f(x_0)}{x-x_0} \leq \frac{f(x_2)-f(x_0)}{x_2-x_0}$$

$$\forall x \in [x_1, x_2].$$

$$L_1 = \frac{f(x_1)-f(x_0)}{x_1-x_0}, \quad L_2 = \frac{f(x_2)-f(x_0)}{x_2-x_0}$$

$$-5 \le h(x) \le -3 \qquad x \in I$$

$$\Longrightarrow \quad |h(x)| \le 5 = \max\{|-3|, |-5|\}$$

$$f(x)-f(x_0) = \frac{f(x)-f(x_0)}{x-x_0} \cdot (x-x_0)$$

$x \in [x_1, x_2]$

$x \ne x_0$

$$|f(x)-f(x_0)| = \left|\frac{f(x)-f(x_0)}{x-x_0}\right| |x-x_0|$$

$$\le \max\{|L_1|, |L_2|\} \, |x-x_0|$$

$$\forall x \in [x_1, x_2] \setminus \{x_0\}$$

PER CONFRONTO

$$\lim_{x \to x_0} |f(x)-f(x_0)| \le$$

$$\lim_{x \to x_0} \max\{|L_1|, |L_2|\} |x - x_0| = 0$$

$$\Longrightarrow \lim_{x \to x_0} f(x) = f(x_0)$$

$\blacksquare$

TEOREMA $f: I \to \mathbb{R}$, I un INTERVALLO. Se f è STRETTAMENTE CONVESSA in I, allora ammette ALPIÙ un punto di minimo in I

$$f(x) = e^x \quad x \in \mathbb{R}$$

$$0 = \inf_{\mathbb{R}} f, \nexists \min_{\mathbb{R}} f$$

PER ASSURDO

$\exists\, x_1 \in I,\ x_2 \in I :\ x_1 \neq x_2$ e

$$f(x_1) = \min_I f = f(x_2)$$

$$f(t x_1 + (1-t) x_2) < t f(x_1) + (1-t) f(x_2)$$

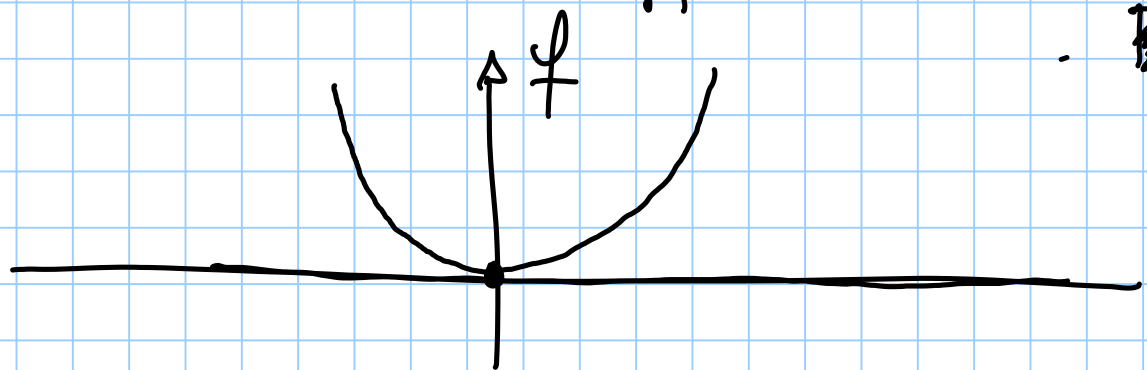
$t \in (0,1)$

$$= t \min_I f + (1-t) \min_I f = \min_I f$$

$$\Rightarrow \forall\, y \in (x_1, x_2) : f(y) < \min_I f$$

ASSURDO

$\blacksquare$



Il cambiamento del segno della $f'(x)$ in x_0 implica

SE f è continua in x_0

che x_0 è un punto di ESTREMO LOCALE.

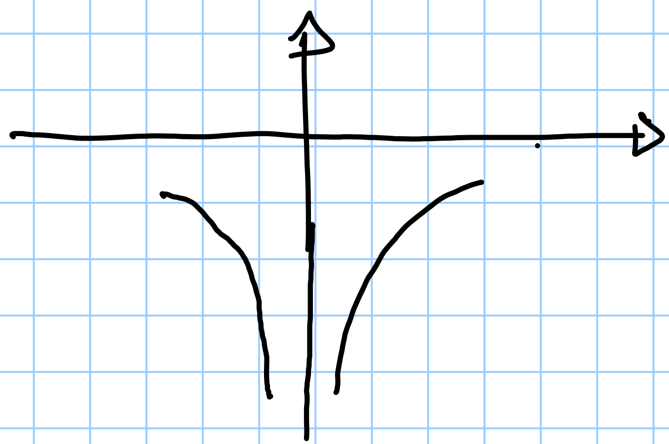
Senza la continuità non è VERO.

$$f(x) = -\frac{1}{x^2}, \quad x \neq 0$$

f è definitivamente $\downarrow$ $x < x_0$,

f è definitivamente $\uparrow$ $x > x_0$

$\nexists \text{ Min } f$



$$g(x) = \begin{cases} -x & x < 0 \\ 1 + x & x \geq 0 \end{cases}$$

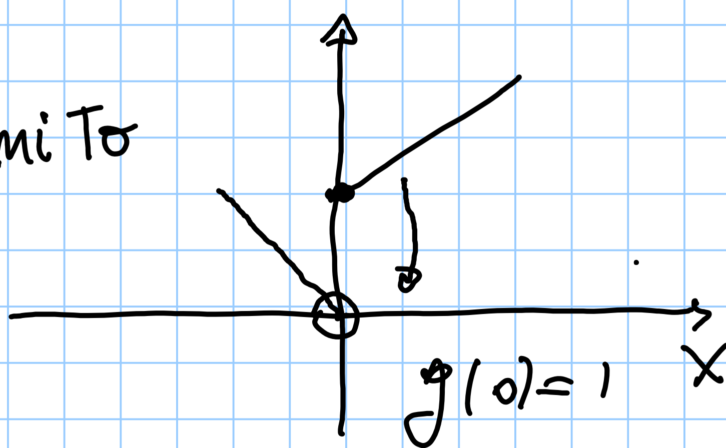
$g \downarrow \quad x < 0$

$g \uparrow \quad x > 0$

$\exists \lim_{x \to 0^-} g(x)$ finito

$\exists \lim_{x \to 0^+} g(x)$ finito

$\nexists \text{ Min } g$



Analogamente il cambiamento del segno di $f''(x)$ in x_0 implica

SE $\exists$ il $\lim\limits_{x \to x_0} f'(x)$

CHE x_0 è un punto di FLESSO

DEFINIZIONE $f:(a,b)\longrightarrow \mathbb{R}$

SI DICE CHE IN $x_0 \in (a,b)$

SI HA UN PUNTO DI FLESSO

SE:

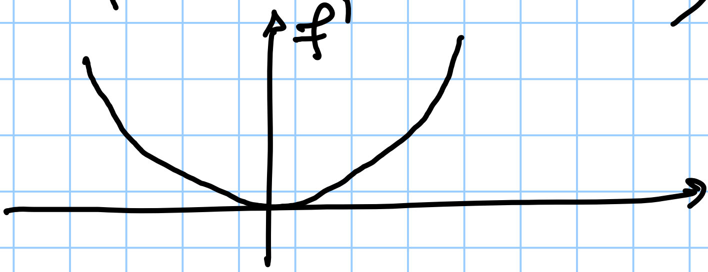
(i) $\exists$ (finito o infinito) $\lim_{x\to x_0} f'(x)$

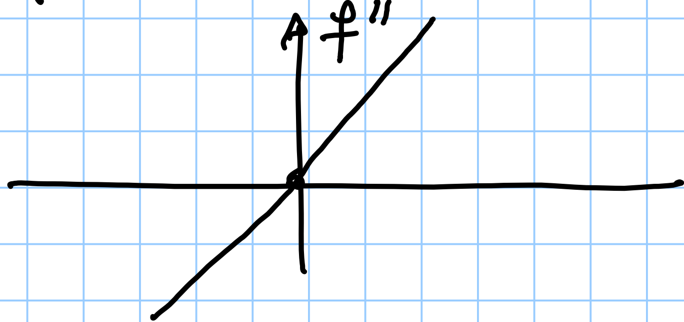
(ii) $f''(x) > 0$ (<0) definitivamente per $x \to x_0^+$

$f''(x) < 0$ (>0) definitivamente per $x \to x_0^-$

$f(x) = x^3, \quad x \in \mathbb{R}, \quad x_0 = 0$

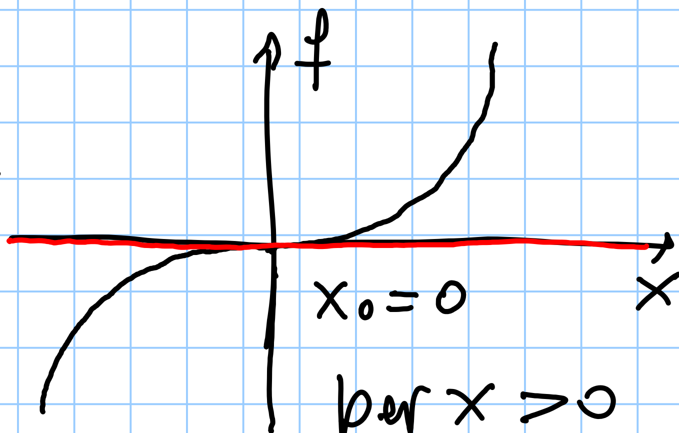
$f'(x) = 3x^2, \quad f''(x) = 6x$





RETTA TANGENTE in $x_0 = 0$

$y = 0$



FLESSO in $x_0 = 0$

per $x > 0$ f è CONVESSA si trova SOPRA LA RETTA TANGENTE

$y = 0$

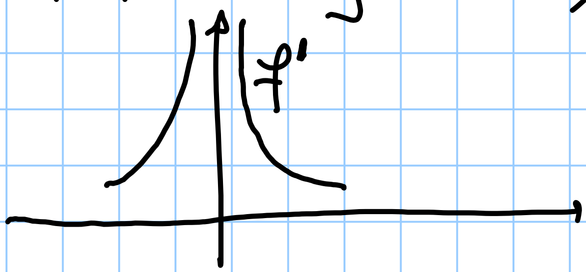
per $x < 0$ f è CONCAVA

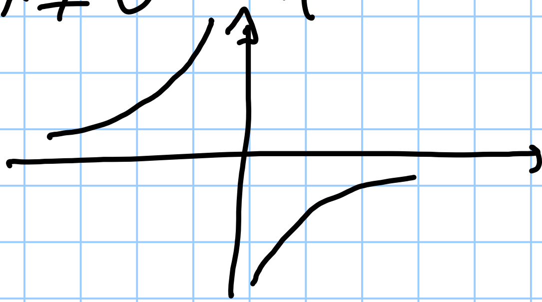
si trova SOTTO LA RETTA TANGENTE

$f(x) = x^{\frac{1}{3}}$, $x \in \mathbb{R}$, $x_0 = 0$

$f'(x) = \frac{1}{3} x^{-\frac{2}{3}}$, $x \neq 0$

$f''(x) = -\frac{2}{9} x^{-\frac{5}{3}}$, $x \neq 0$

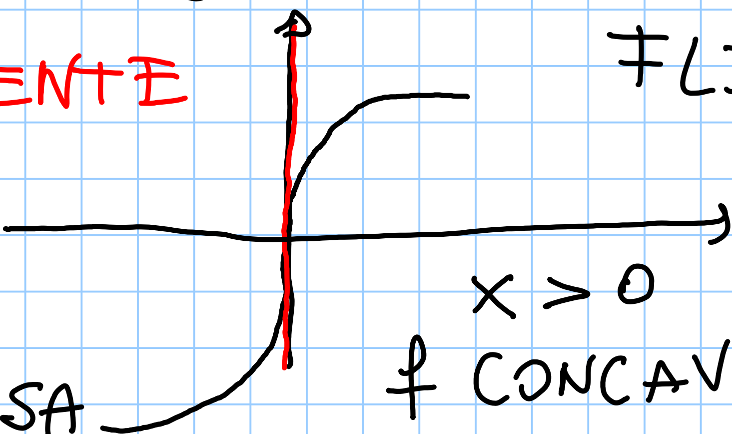




$$\lim_{x \to 0} f'(x) = \frac{1}{3 \cdot 0^+} = +\infty$$

$x_0 = 0$ FLESSO

RETTA TANGENTE "$x = 0$"



$x < 0$

f CONVESSA

$x > 0$

f CONCAVA

$$f(x) = |x| + x^3 \quad , \quad x \in \mathbb{R}$$

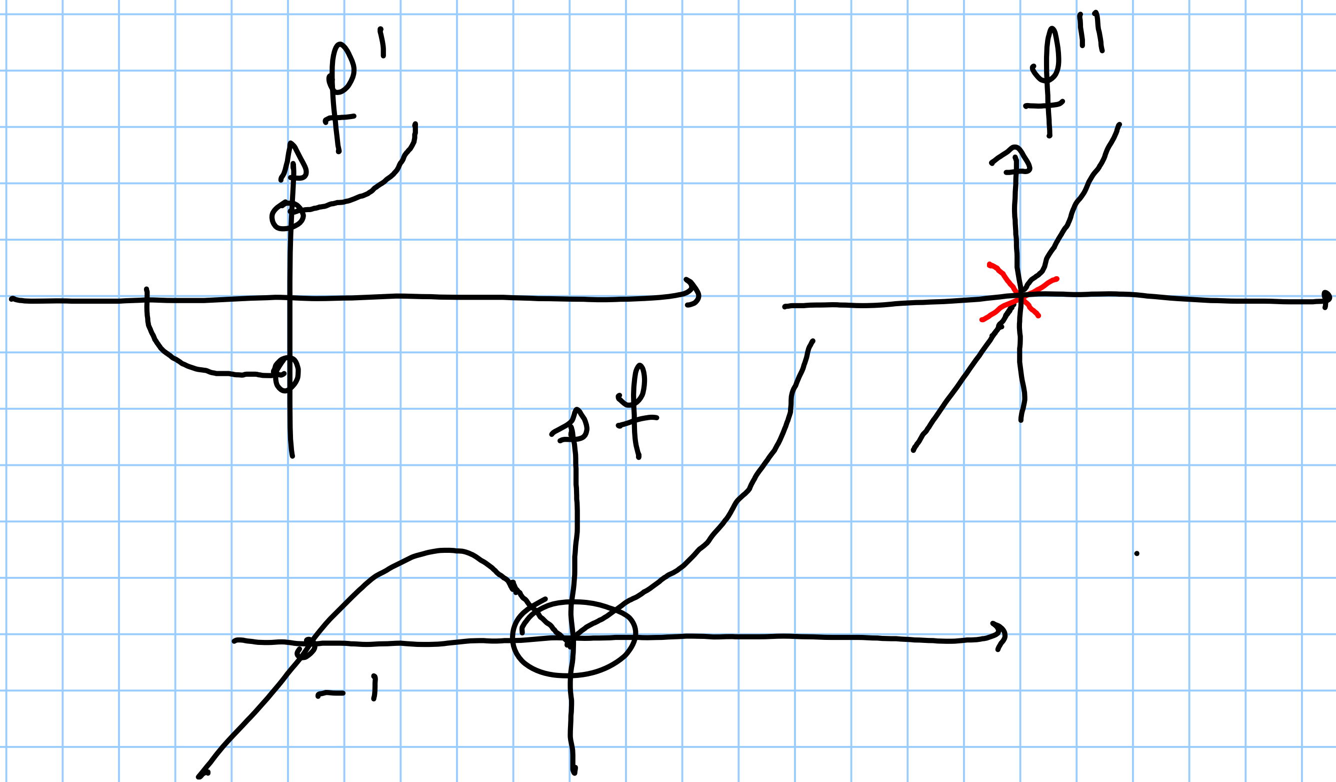
$$f(x) \xrightarrow[x \to +\infty]{} +\infty \quad , \quad f(x) \xrightarrow[x \to -\infty]{} -\infty$$

$$f'(x) = \begin{cases} 1 + 3x^2 & , x > 0 \\ -1 + 3x^2 & , x < 0 \end{cases}$$

$$\lim_{x \to 0^+} f'(x) = 1 \quad , \quad \lim_{x \to 0^-} f'(x) = -1$$

$$f''(x) = \begin{cases} 6x, & x > 0 \\ 6x, & x < 0 \end{cases}$$

f' ha una discontinuità di SALTO in $x_0 = 0$



$x_0 = 0$ NON E' FLESSO

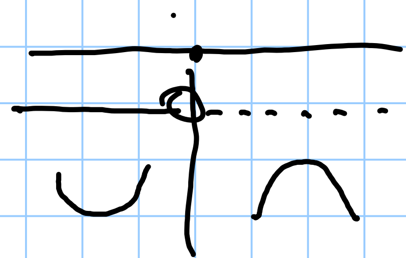
Ex: PERCHE' questo Esempio non contraddice il T. di continuità di f'?

$$f(x) = \operatorname{arcTg}(x) \ , \quad x \in \mathbb{R}$$

$$\lim_{x \to +\infty} f(x) = \frac{\pi}{2}^{-} \ , \quad \lim_{x \to -\infty} f(x) = \left(-\frac{\pi}{2}\right)^{+}$$

$$f'(x) = \frac{1}{1+x^2} > \quad \forall x \in \mathbb{R}$$

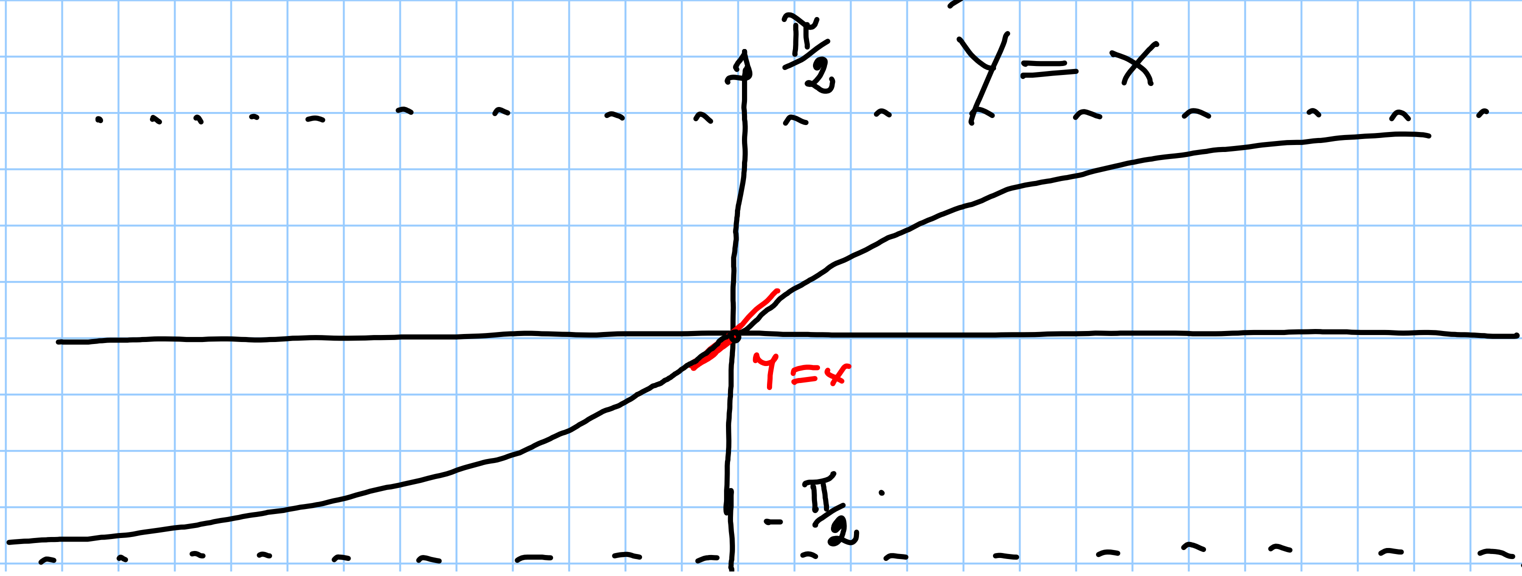
$$f''(x) = -\frac{1}{(1+x^2)^2} \cdot 2x$$



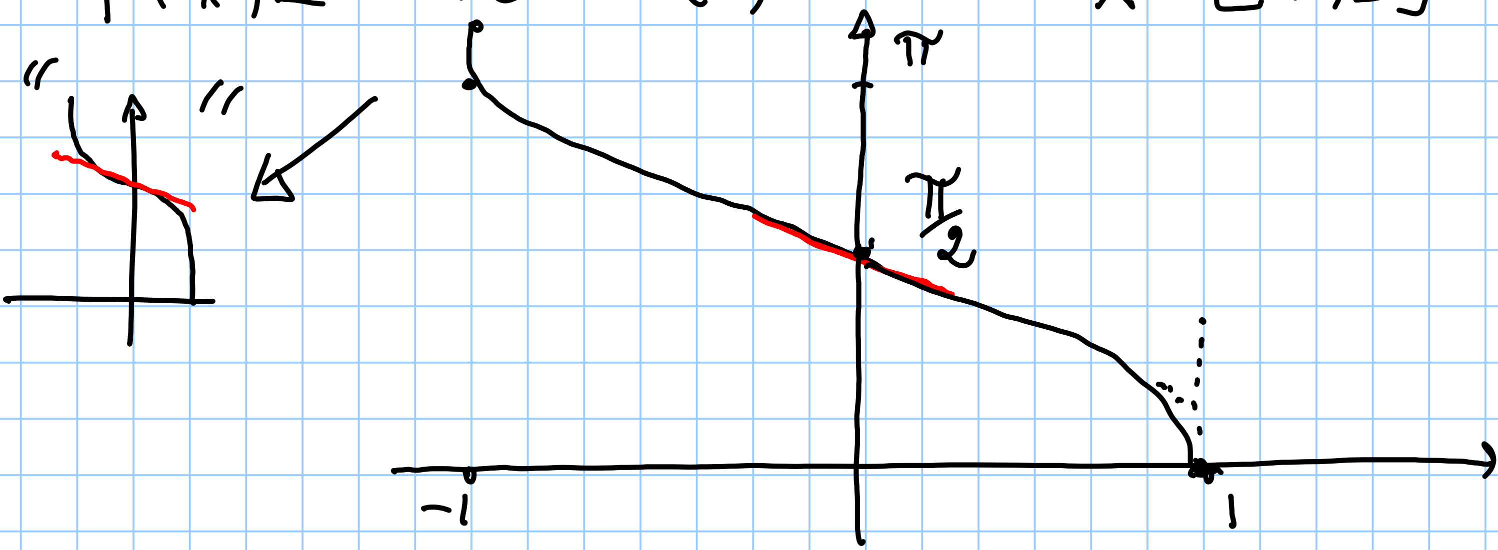
$x < 0$, f CONVESSA

$x > 0$, f CONCAVA

$x_0 = 0$ FLESSO, $f'(0) = 1$



$$f(x) = \arccos(x) \qquad x \in [-1, 1]$$



$$f'(x) = -\frac{1}{\sqrt{1-x^2}}\,, \qquad x \in (-1, 1)$$

$$\lim_{x \to (-1)^+} f'(x) = -\infty = \lim_{x \to (1)^-} f'(x)$$

$$f'(0) = -1$$

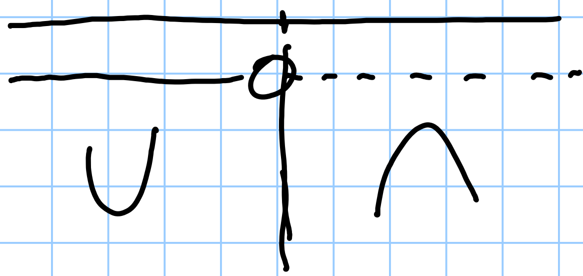
$$f''(x) = \frac{d}{dx}\left(-\left(1-x^2\right)^{-\frac{1}{2}}\right) =$$

$$= (-1)\left(-\frac{1}{2}\right)\left(1-x^2\right)^{-\frac{3}{2}} \cdot (-2x)$$

$$= -\frac{\cancel{2}}{\cancel{2}}\left(1-x^2\right)^{-3/2} \cdot x = -\frac{x}{\left(1-x^2\right)^{3/2}}\,.$$

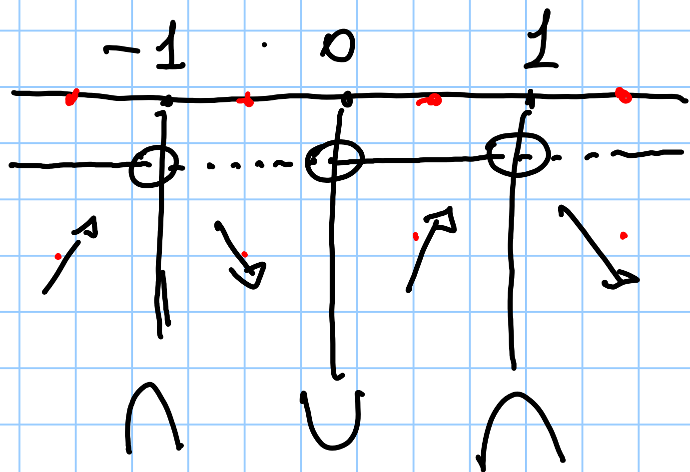
$x_0 = 0$ FLESSO

$$y = -x + \frac{\pi}{2}$$



$$f(x) = x^2 e^{-x^2}, \quad x \in \mathbb{R}$$

$$f'(x) = 2x e^{-x^2}(1-x^2)$$



$$f''(x) = 2e^{-x^2}(1-x^2) + 2x(-2x)e^{-x^2}(1-x^2) +$$

$$+ 2x e^{-x^2}(-2x) =$$

$$= 2e^{-x^2}\left[1 - x^2 - 2x^2(1-x^2) - 2x^2\right]$$

$$= 2e^{-x^2}\left[1 - x^2 - 2x^2 + 2x^4 - 2x^2\right]$$

$$= 2e^{-x^2}\left[2x^4 - 5x^2 + 1\right]$$

$$f''(x) \geq 0 \iff 2x^4 - 5x^2 + 1 \geq 0$$

$$x^2 = t \implies 2t^2 - 5t + 1 \geq 0$$

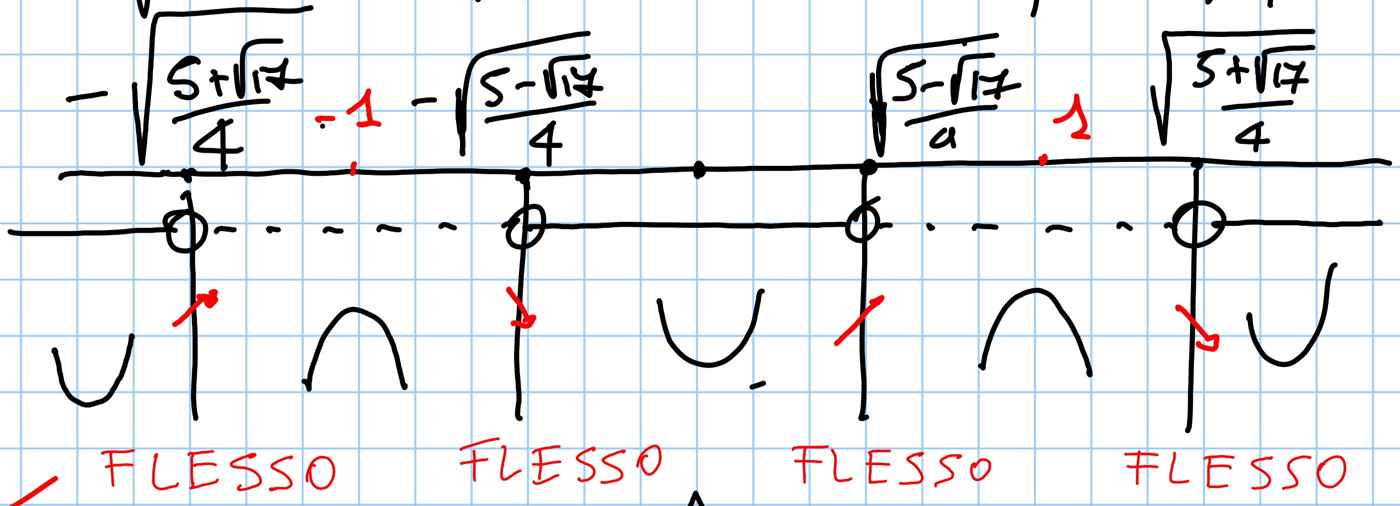
$$t_\pm = \frac{5 \pm \sqrt{25-8}}{4} = \frac{5 \pm \sqrt{17}}{4}$$

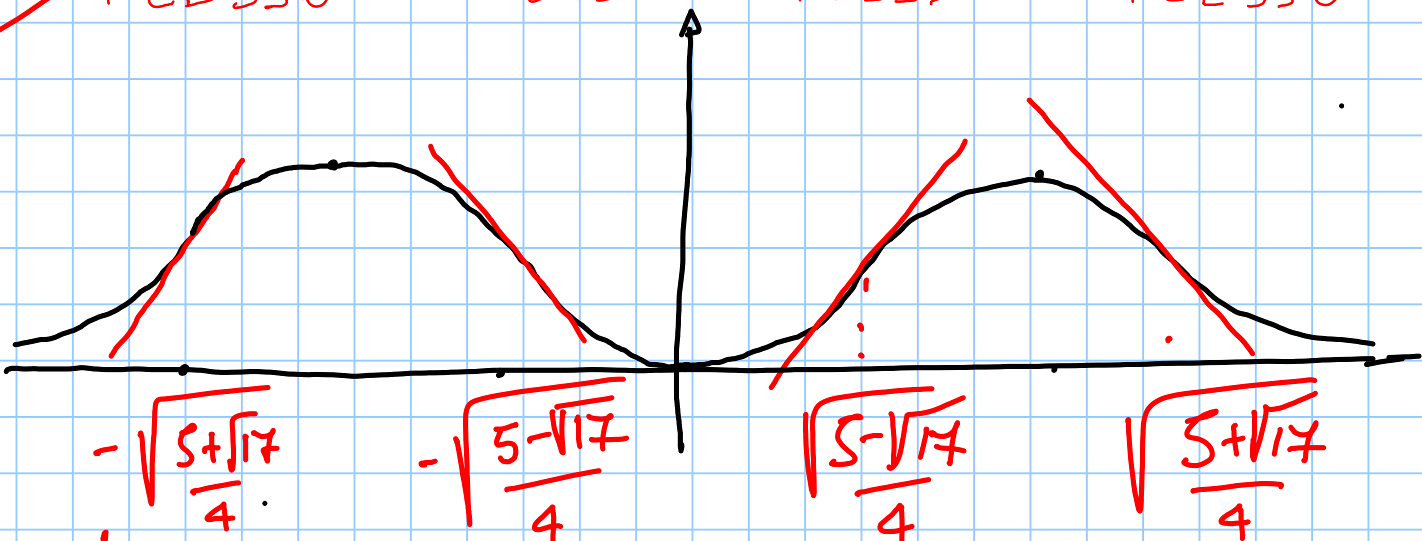
$$t \leq \frac{5-\sqrt{17}}{4} \text{ ή } t \geq \frac{5+\sqrt{17}}{4}$$

$$x^2 \leq \frac{5-\sqrt{17}}{4} \text{ ή } x^2 \geq \frac{5+\sqrt{17}}{4}$$

$$-\sqrt{\frac{5-\sqrt{17}}{4}} \leq x \leq \sqrt{\frac{5-\sqrt{17}}{4}} \quad \Big| \quad x \leq -\sqrt{\frac{5+\sqrt{17}}{4}},\ x \geq \sqrt{\frac{5+\sqrt{17}}{4}}$$

$$\sqrt{\frac{5-\sqrt{17}}{4}} \simeq \sqrt{\frac{1}{4}} = \frac{1}{2} \quad , \quad \sqrt{\frac{5+\sqrt{17}}{4}} \simeq \sqrt{\frac{9}{4}} = \frac{3}{2}$$





FLESSO ASCENDENTE

$$f'\left(-\sqrt{\frac{5+\sqrt{17}}{4}}\right) > 0$$

$$f(x) = -\frac{1}{2}|x| + \operatorname{arctg}(x)$$

$$f'(x) = \begin{cases} -\frac{1}{2} + \frac{1}{1+x^2}, & x>0 \\ \frac{1}{2} + \frac{1}{1+x^2}, & x<0 \end{cases}$$

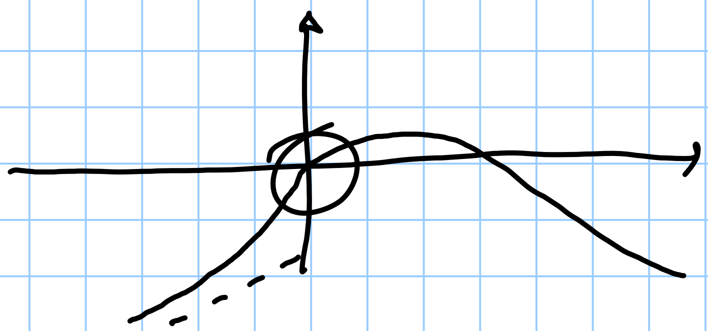
$$f''(x) = \begin{cases} -\frac{2x}{(1+x^2)^2}, & x>0 \\ -\frac{2x}{(1+x^2)^2}, & x<0 \end{cases}$$

$$D^{(+)}f(0) = \frac{1}{2} \neq D^{(-)}f(0) = \frac{3}{2}$$

$x_0 = 0$ NON E' FLESSO

$x_0 = 0$ PUNTO ANGOLOSO

$\lim_{x \to x_0} f' \nexists$



$$f(x) = \left(x^2+1-|3x-1|\right)^{\frac{1}{3}} = \begin{cases} \left(x^2-3x+2\right)^{\frac{1}{3}}, & x \geq \frac{1}{3} \\ \left(x^2+3x\right)^{\frac{1}{3}}, & x < \frac{1}{3} \end{cases}$$

$$f'(x) = \begin{cases} \frac{1}{3} \frac{2x-3}{\left(x^2-3x+2\right)^{2/3}}, & x > \frac{1}{3},\ x \neq 1, 2 \\ \frac{1}{3} \frac{2x+3}{\left(x^2+3x\right)^{2/3}}, & x < \frac{1}{3},\ x \neq -3, 0 \end{cases}$$

$$\lim_{x \to (-3)^{\mp}} f'(x) = -\infty$$

$$\lim_{x \to 0^{\pm}} f'(x) = +\infty, \quad x_0 = \frac{1}{3} \text{ P.TO ANGOLOSO}$$

$$\lim_{x \to (1)^{\pm}} f'(x) = -\infty$$

$$\lim_{x \to (2)^{\pm}} f'(x) = +\infty$$

$$f'(x) \quad \frac{1}{3} \quad (x^2-3x+2)^{-\frac{2}{3}}(2x-3) \quad , x<\frac{1}{3}, x\neq 1,2$$

$$f''(x)=\frac{1}{3}\left(-\frac{2}{3}\right)\left(x^2-3x+2\right)^{-\frac{5}{3}}(2x-3)^2$$

$$+\frac{1}{3}\left(x^2-3x+2\right)^{-\frac{2}{3}}\cdot 2 \quad =$$

$$=\frac{1}{3}\frac{}{(x^2-3x+2)^{\frac{5}{3}}}\left[-\frac{2}{3}(2x-3)^2+2\left(x^2-3x+2\right)\right]$$